Sampling recovery from random partial quadrature nodes

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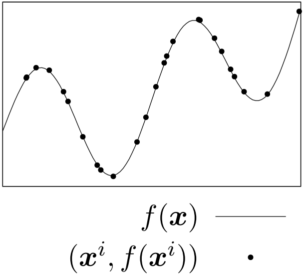
Mathematik!
TU Chemnitz

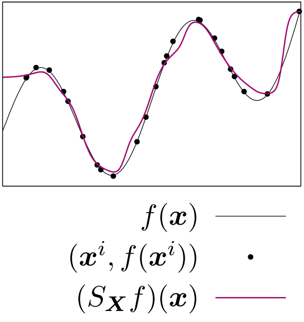
Recovery of functions from RKHS

- **Given:**
  - reproducing kernel Hilbert space $\mathrm{H}(K) \hookrightarrow \mathrm{L}_2$
  - samples $(f(\boldsymbol{x}^1), \dots, f(\boldsymbol{x}^n))^{\mathsf{T}} \in \mathbb{C}^n$ from $f \in \mathrm{H}(K)$



- **Given:**
  - reproducing kernel Hilbert space $\mathrm{H}(K) \hookrightarrow \mathrm{L}_2$
  - samples $(f(\boldsymbol{x}^1), \ldots, f(\boldsymbol{x}^n))^\mathsf{T} \in \mathbb{C}^n$ from $f \in \mathrm{H}(K)$
- **Goal:**
  - recover every $f \in \mathrm{H}(K)$ from the samples in $\boldsymbol{X} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n)$
  - control **worst-case error** for sampling recovery operator $S_{\boldsymbol{X}} \colon \mathrm{H}(K) \to \mathrm{L}_2$

$$\sup_{\|f\|_{\mathrm{H}(K)} \le 1} \|f - S_{\boldsymbol{X}} f\|_{\mathrm{L}_2}$$



Reproducing kernel Hilbert space

- assume the finite trace condition $\int_D K(\boldsymbol{x},\boldsymbol{x})\,\mathrm{d}\nu(\boldsymbol{x}) < \infty$
- embedding $\mathrm{Id}_{K,\nu}\colon \mathrm{H}(K) \to \mathrm{L}_2$ has the representation

$$\mathrm{Id}_{K,\nu}(f) = \sum_{k=1}^{\infty} \sigma_k \langle f, e_k\rangle_{\mathrm{H}(K)} \eta_k$$

  with

  - **singular values** $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ of $\mathrm{Id}_{K,\nu}$
  - **right singular functions** $\{e_1, e_2, \dots\}$ forming an ONS in $\mathrm{H}(K)$
  - **left singular functions** $\{\eta_1, \eta_2, \dots\}$ forming an ONS in L_2

- since $\mathrm{Id}_{K,\nu}$ is the identity on functions

$$e_k = \sigma_k \cdot \eta_k$$

Example: Sobolev spaces with mixed smoothness on $\mathbb{T}^d$

- for $s > 1/2$ we define $\mathrm{H}^s_{\text{mix}}(\mathbb{T}^d) = \{f \in \mathrm{L}_2 : \|f\|_{\mathrm{H}^s_{\text{mix}}} < \infty\}$ with

$$\langle f, g\rangle_{\mathrm{H}^s_{\text{mix}}} := \sum_{\boldsymbol{j}\in\{0,s\}^d} \langle D^{(\boldsymbol{j})} f, D^{(\boldsymbol{j})} g\rangle_{\mathrm{L}_2}$$

- kernel:

$$K^1_s(x,y) = \sum_{k\in\mathbb{Z}} \frac{\exp(2\pi \mathrm{i} k(y-x))}{w_s(k)^2} \quad \text{with} \quad w_s(k) = (1 + (2\pi|k|)^{2s})^{1/2}$$

$$K^d_s(\boldsymbol{x},\boldsymbol{y}) = K^1_s(x_1,y_1) \otimes \cdots \otimes K^1_s(x_d,y_d)$$

- singular numbers: $\sigma_n = (1/w_s(\boldsymbol{k}_n))_n$ (non-increasing rearrangement)
- singular functions: $e_n(\boldsymbol{x}) = \sigma_n \eta_n(\boldsymbol{x}) = \sigma_n \exp(2\pi \mathrm{i} \boldsymbol{k}_n \cdot \boldsymbol{x})$

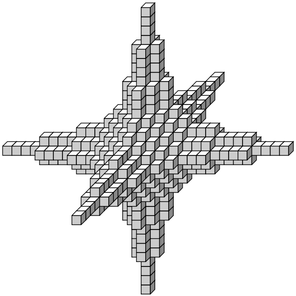
Example: Sobolev spaces with mixed smoothness on $\mathbb{T}^d$

- first $\eta_1, \dots, \eta_{m-1}$ are most important
- frequencies belong to hyperbolic cross

$$I_R^{hc} = \left\{ \boldsymbol{k} \in \mathbb{Z}^d : \prod_{j=1}^{d} (1 + (2\pi|k_j|)^{2s})^{1/2} \leq R \right\}$$

- approximation

$$f(\boldsymbol{x}) \approx \sum_{\boldsymbol{k} \in I_R^{hc}} c_{\boldsymbol{k}} \exp(2\pi \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x})$$



I_R^{hc}

Least squares

- assume $n > m$
- seek for $\boldsymbol{c} = (c_1, \dots, c_{m-1})^\mathsf{T}$ solving

$$\begin{pmatrix} \eta_1(\boldsymbol{x}^1) & \cdots & \eta_{m-1}(\boldsymbol{x}^1) \\ \vdots & \ddots & \vdots \\ \eta_1(\boldsymbol{x}^n) & \cdots & \eta_{m-1}(\boldsymbol{x}^n) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{m-1} \end{pmatrix} = \begin{pmatrix} f(\boldsymbol{x}^1) \\ \vdots \\ f(\boldsymbol{x}^n) \end{pmatrix}$$

Least squares

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for a weight matrix $\boldsymbol{W} = \operatorname{diag}(\omega_1, \ldots, \omega_n)$

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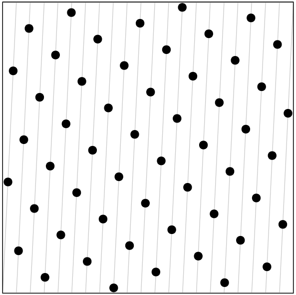
  for a weight matrix $\boldsymbol{W} = \operatorname{diag}(\omega_1, \ldots, \omega_n)$
- the solution is given by

$$\boldsymbol{c} = (\boldsymbol{L}^*\boldsymbol{W}\boldsymbol{L})^{-1}\boldsymbol{L}^*\boldsymbol{W} \cdot \boldsymbol{f} \quad \text{and} \quad (S_{\boldsymbol{X},\boldsymbol{W}} f)(\boldsymbol{x}) = \sum_{k=1}^{m-1} c_k \eta_k(\boldsymbol{x})$$

Deterministic nodes: rank-1 lattices

- $\boldsymbol{X} = \Lambda(\boldsymbol{z}, M)$ with $\boldsymbol{x}^i = \frac{1}{M}(i\boldsymbol{z} \bmod M\mathbf{1})$ for $i = 0, \ldots, M-1$
- [Kämmerer, Potts, Volkmer '15]:
  - given I, there exist algorithms that find $\boldsymbol{z}$ and M such that $\boldsymbol{L}^*\boldsymbol{W}\boldsymbol{L} = \boldsymbol{I}$
  - multiplication with $\boldsymbol{L}$ can be carried out with the LFFT in $\mathcal{O}(M \log M)$



$z = (1, 21)^\mathsf{T}, M = 55$

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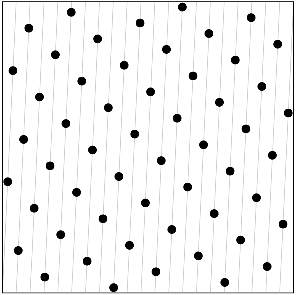


$z = (1, 21)^\mathsf{T}, M = 55$

[Byrenheid, Kämmerer, T. Ullrich, Volkmer '17]:

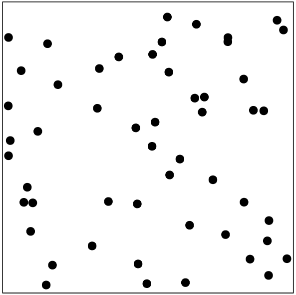
$$M^{-s/2} \lesssim \sup_{\|f\|_{\mathrm{H}(K)} \le 1} \|f - S_{\boldsymbol{X},\boldsymbol{W}} f\|_{\mathrm{L}_2} \lesssim M^{-s/2} (\log M)^{\frac{d-2}{2}s + \frac{d-1}{2}}$$

Random nodes on $\mathbb{T}^d$

- $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that

$$|I| \lesssim \frac{n}{\log n}$$

- no fast matrix-vector multiplication due to lack of structure



random nodes

[M. Ullrich, Krieg '19], [Kämmerer, T. Ullrich, Volkmer '19]:

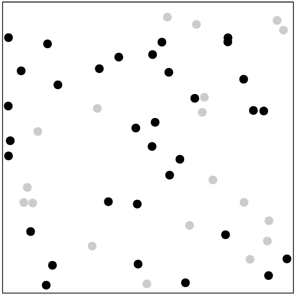
$$\sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \|f - S_{\boldsymbol{X},\boldsymbol{W}} f\|_{\mathrm{L}_2} \lesssim n^{-s} (\log n)^{(d-1)s+s}$$

Weaver subsampled random nodes

- $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\}$ drawn randomly i.i.d. w.r.t. Lebesgue measure such that $|I| \leq n/(10r\log n)$
- there exist $\boldsymbol{X}' = \{\boldsymbol{x}^{i_1}, \dots, \boldsymbol{x}^{i_{n'}}\} \subset \boldsymbol{X}$ subsampled nodes suitable for reconstruction with

$$n \in \mathcal{O}(|I|)$$

- no fast matrix-vector multiplication due to lack of structure



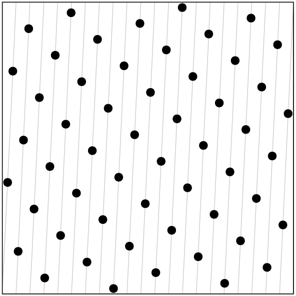
subsampled random nodes

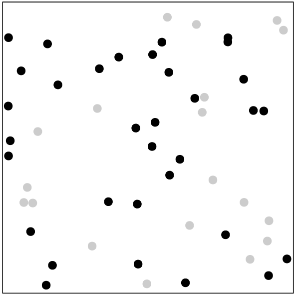
[Nagel, Schäfer, T. Ullrich '21]:

$$\sup_{\|f\|_{\mathrm{H}(K)}\leq 1} \|f - S_{\boldsymbol{X},\boldsymbol{W}} f\|_{\mathrm{L}_2} \lesssim n^{-s}(\log n)^{(d-1)s+1/2}$$

Goal

We want to combine both advantages:





- structure for fast algorithms
- awesome rates

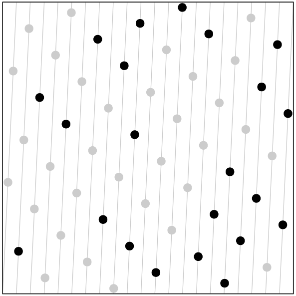
Approach (motivated by [Kunis, Rauhut '08])

1. $\boldsymbol{X}_M$: good deterministic set of M nodes
2. $\boldsymbol{X}$: randomly subsample n nodes from $\boldsymbol{X}_M$ such that

$$|I| \lesssim \frac{n}{\log n}$$

3. $\boldsymbol{X}'$: Weaver subsample n' nodes from $\boldsymbol{X}$ such that

$$n' \in \mathcal{O}(|I|)$$



Starting node set

- nodes $\boldsymbol{X}_M$ and weights $\omega_1, \dots, \omega_M$ fulfill a L_2 **Marcinkiewicz-Zygmund inequality** iff

$$c_{\mathsf{MZ}} \|f\|_{\mathrm{L}_2}^2 \leq \sum_{i=1}^{M} \omega_i |f(\boldsymbol{x}^i)|^2 \leq C_{\mathsf{MZ}} \|f\|_{\mathrm{L}_2}^2$$

  for $f \in \mathrm{span}\{\eta_1, \dots, \eta_{m-1}\}$.
- the full system matrix is well-conditioned

$$\mathrm{spec}(\boldsymbol{W}^{1/2} \boldsymbol{L}_{\boldsymbol{X}_M}) \in [\sqrt{c_{\mathsf{MZ}}}, \sqrt{C_{\mathsf{MZ}}}]$$

Starting node set

- MZ-inequalities are widly available: [Mhaskar, Narcowich, Ward '01], [Keiner, Kunis, Potts '07], [Filbir, Mhaskar '11], [Müller-Gronbach, Novak, Ritter '12], [Temlyakov '18], [Gröchenig '20], [Hielscher, Jahn, Ullrich '21], ...

Starting node set

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Lemma

The MZ-inequality for $c_{\mathsf{MZ}} = C_{\mathsf{MZ}}$ is equivalent to the exact integration

$$\sum_{i=1}^{M} \omega_i g(\boldsymbol{x}^i)\overline{h(\boldsymbol{x}^i)} = \int_D g(\boldsymbol{x})\overline{h(\boldsymbol{x})}\,\mathrm{d}\nu(\boldsymbol{x})$$

for $g, h \in \operatorname{span}\{\eta_1, \ldots, \eta_{m-1}\}$.

- exact integration is widly available: [Nuyens, Cools '06], [Kämmerer, Potts, Volkmer '15], [Trefethen '19], ...

Fast algorithm

- big node set $\boldsymbol{X}_M = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^M\}$ with $\mathcal{O}(M \log M)$ algorithm
- small node set $\boldsymbol{X} = \{\boldsymbol{x}^{j_1}, \dots, \boldsymbol{x}^{j_n}\}$
- we may use the algorithm for the big node set

$$\boldsymbol{L}_{\boldsymbol{X}} = \boldsymbol{P}\boldsymbol{L}_{\boldsymbol{X}_M} \quad \text{where} \quad \boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}$$

$$\begin{matrix} j_1 & & j_2 & & & j_n \end{matrix}$$

Fast algorithm

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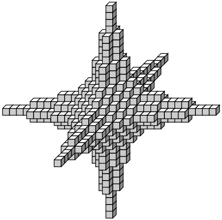
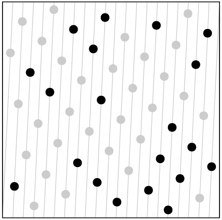
$$\qquad\qquad j_1 \qquad j_2 \qquad\qquad j_n$$

- with $|I| \sim n/\log n$ and $M \sim |I|^2$ we obtain the same complexity as for naive matrix-vector multiplication

$$\mathcal{O}(|I| \cdot n) = \mathcal{O}(M \log M)$$

- $d = 2$, $\boldsymbol{X}_M$ Fibonacci lattice
- hyperbolic cross frequencies





$\lvert I \rvert$	M	n	store matrix	column-wise	$1d$ FFT
6 089	196 418	67 729	17 + 0.176 s 7 GB	16 s	0.047 s
13 581	832 040	162 991	99 + 1.377 s 35 GB	93 s	0.119 s
29 977	3 524 578	385 579	490 + 10 s 185 GB	480 s	0.744 s

Assumptions 1 of 2

Let

- $\mathrm{H}(K)$ RKHS with $\sup_{\boldsymbol{x}\in D} K(\boldsymbol{x},\boldsymbol{x}) < \infty$,
- $I \subset I_M$ frequency index sets,
- $\boldsymbol{X}_M$ nodes and $\boldsymbol{W}$ weights fulfilling an L_2-Marcinkiewicz-Zygmund inequality with constants c_{MZ} and C_{MZ} for $V = \mathrm{span}\{\eta_k\}_{k\in I_M}$,
- n smallest such that

$$|I| \le \frac{c_{\mathsf{MZ}} n}{30 C_{\mathsf{MZ}} r \log n},$$

Assumptions 2 of 2

- $\boldsymbol{X} = (\boldsymbol{x}^i)_{i\in J}$, $|J| = n$ nodes drawn i.i.d. from $\boldsymbol{X}_M$ w.r.t. the **discrete density weights** $\varrho_i = \omega_i \varrho(\boldsymbol{x}^i)$ for $1 \le i \le M$ with

$$\varrho(\boldsymbol{x}^i) = \frac{\sum_{k\in I} |\eta_k(\boldsymbol{x}^i)|^2}{3\sum_{j=1}^{M} \omega_j \sum_{k\in I} |\eta_k(\boldsymbol{x}^j)|^2} + \frac{\sum_{k\in I_M\setminus I} |e_k(\boldsymbol{x}^i)|^2}{3\sum_{j=1}^{M} \omega_j \sum_{k\in I_M\setminus I} |e_k(\boldsymbol{x}^j)|^2} + \frac{\sum_{k\notin I_M} |e_k(\boldsymbol{x}^i)|^2}{3\sum_{j=1}^{M} \omega_j \sum_{k\notin I_M} |e_k(\boldsymbol{x}^j)|^2}$$

- uses spectral properties of the embedding
- $\varrho(\boldsymbol{x}^i)$ neglectable for BOS
- discrete version of [M. Ullrich, Krieg '19], [Kämmerer, T. Ullrich, Volkmer '19]

Central theorem

Theorem

Given the assumptions we have with probability larger than $1-5n^{1-r}$

$$\sup_{\|f\|_{\mathrm{H}(K)}\le 1}\left\|f-S^{I}_{\boldsymbol{X},\tilde{\boldsymbol{W}}}f\right\|_2^2 \le \frac{6C_{\mathsf{MZ}}}{c_{\mathsf{MZ}}}\Bigg(\sup_{k\notin I}\sigma_k^2+\frac{1}{|I|}\sum_{k\in I_M\setminus I}\sigma_k^2$$
$$+\frac{1}{|I|}\sup_{\|f\|_{\mathrm{H}(K)}\le 1}\|f-P_{I_M}f\|_\infty^2+\sup_{\|f\|_{\mathrm{H}(K)}\le 1}\left\|f-S^{I_M}_{\boldsymbol{X}_M,\boldsymbol{W}}f\right\|_{\mathrm{L}_2}^2\Bigg).$$

Theorem

Given the same assumptions but with

$$n' \in \mathcal{O}(|I|)$$

we have the existence of $\boldsymbol{X}' = \{\boldsymbol{x}^{i_1}, \dots, \boldsymbol{x}^{i_{n'}}\}$ such that

$$\sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \left\| f - S^{I}_{\boldsymbol{X}', \tilde{\boldsymbol{W}}} f \right\|_2^2 \lesssim \log|I| \Bigg(\sup_{k \notin I} \sigma_k^2 + \frac{1}{|I|} \sum_{k \in I_M \setminus I} \sigma_k^2$$
$$+ \frac{1}{|I|} \sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \|f - P_{I_M} f\|_\infty^2 + \sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \left\| f - S^{I_M}_{\boldsymbol{X}_M, \boldsymbol{W}} f \right\|_{\mathrm{L}_2}^2 \Bigg).$$

Theorem on $\mathbb{T}^d$ for $\mathrm{H}^s_{\text{mix}}$

Theorem

Let

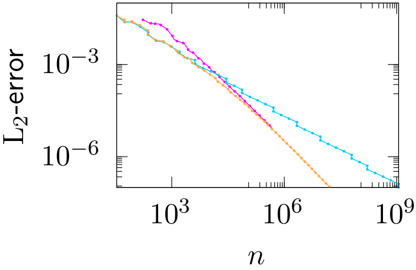
- $I \subset I_M$ frequency index sets,
- $\boldsymbol{X}_M$ and $\boldsymbol{W}$ exact for trigonometric polynomials supported on $\mathcal{D}(I_M)$
- n smallest such that $|I| \leq \frac{n}{30r \log n}$,
- $\boldsymbol{X} = \{\boldsymbol{x}^1, \dots, \boldsymbol{x}^n\}$ drawn w.r.t. the discrete density weights $\boldsymbol{W}$.

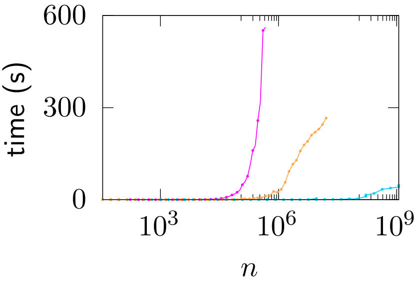
Then we have with probability larger than $1 - 5n^{-r}$

$$\sup_{\|f\|_{\mathrm{H}^s_{\text{mix}}} \leq 1} \left\| f - S^I_{\boldsymbol{X}, \tilde{\boldsymbol{W}}} f \right\|_2^2 \leq \sup_{\boldsymbol{k} \notin I} \frac{5}{w(\boldsymbol{k})^2} + \frac{6}{|I|} \sum_{\boldsymbol{k} \notin I} \frac{1}{w(\boldsymbol{k})^2} + 4 \sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \left\| f - S^{I_M}_{\boldsymbol{X}_M, \boldsymbol{W}} f \right\|_{\mathrm{L}_2}^2 .$$

Numerics on $\mathbb{T}^d$

- $d = 2$, $\boldsymbol{X}_M$ Fibonacci lattice
- hyperbolic cross frequencies





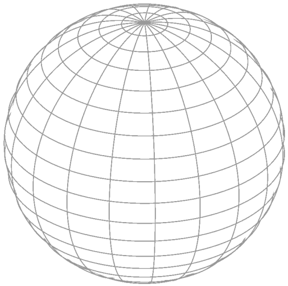
magenta: random nodes
cyan: full rank-1 lattice
orange: subsampled rank-1 lattice

The two-dimensional sphere $\mathbb{S}^2$

- **domain:** $D = \mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\|_2 = 1\}$
- **basis functions** for the spherical harmonics

$$Y_{k,\ell}(\theta, \varphi) = \sqrt{\frac{2k+1}{4\pi}} P_{|\ell|}^{k}(\cos\theta) \exp(\mathrm{i}\ell\varphi)$$

for $k = 0, 1, \ldots$ and $-k \leq \ell \leq k$



Theorem

Let

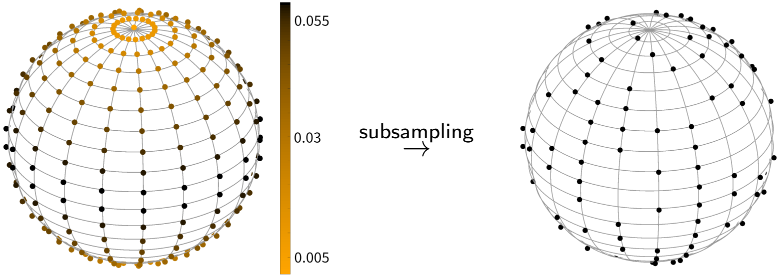
- $m \leq m_M$ polynomial degrees
- $\boldsymbol{X}_M$ and $\boldsymbol{W}$ exact for polynomials upto degre $2m_M$
- n smallest such that $(m+1)^2 \leq \frac{n}{30r\log n}$
- $\boldsymbol{X} = \{\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^n\}$ drawn w.r.t. the discrete density weights $\boldsymbol{W}$.

Then we have with probability larger than $1 - 5n^{-r}$

$$\sup_{\|f\|_{\mathrm{H}^s_{\text{mix}}} \leq 1} \left\| f - S^I_{\boldsymbol{X},\tilde{\boldsymbol{W}}} f \right\|_2^2 \leq \sup_{\boldsymbol{k}\notin I} \frac{5}{w(\boldsymbol{k})^2} + \frac{6}{(m+1)^2} \sum_{\boldsymbol{k}\notin I} \frac{2k+1}{4\pi w(\boldsymbol{k})^2}$$
$$+ 4 \sup_{\|f\|_{\mathrm{H}(K)} \leq 1} \left\| f - S^{I_M}_{\boldsymbol{X}_M,\boldsymbol{W}} f \right\|_{\mathrm{L}_2}^2 .$$

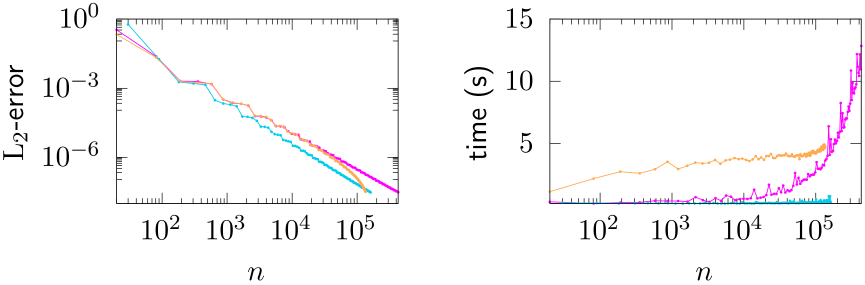
Example: quadrature nodes on $\mathbb{S}^2$

- $\boldsymbol{X}$ tensor product of equispaced nodes and Chebyshev nodes
- $\boldsymbol{W}$ Chebyshev weights



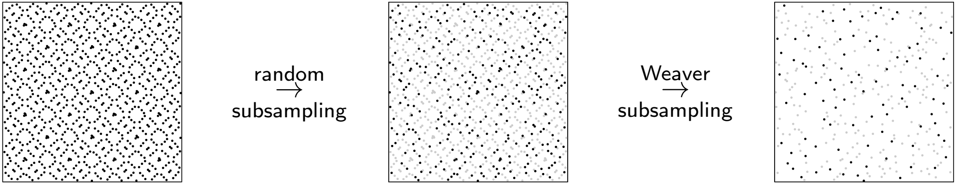
- exact upto polynomial degree m with $n = 2m(m+2)$ nodes
- $2d$ FFT for matrix-vector product

- $\boldsymbol{X}_M$ tensor product of equispaced nodes and Chebyshev nodes



magenta: random nodes
cyan: full tensor product grid
orange: subsampled tensor product grid

Conclusion



- we proposed a two step subsampling procedure
    - making it possible to utilize fast algorithms ✓
    - achieving the optimal convergence rates known so far ✓