Sampling recovery in Bochner spaces and applications to parametric PDEs with random inputs

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Abstract

We proved convergence rates of linear sampling recovery by extended least squares methods of functions in Bochner space satisfying some ℓ_2-summability of their generalized polynomial chaos expansion coefficients. As applications we derive convergence rates of linear collocation approximation of solutions to parametric elliptic PDEs with random inputs, and of infinite dimensional holomorphic functions. These convergence rates significantly improve the known results.

Keywords and Phrases: High dimensional approximation; Sampling recovery; Bochner spaces; Linear collocation approximation; Least squares approximation; Parametric PDEs with random inputs; Infinite dimensional holomorphic function; Convergence rate.

Mathematics Subject Classifications (2010): 65C30, 65N15, 65N35, 41A25.

1 Introduction and main results

In computational uncertainty quantification, the problem of efficient approximation for parametric and stochastic PDEs has been of great interest and achieved significant progress in recent years. Depending on a particular setting, as usual, this problem is reduced to an approximation problem in a Bochner space $L_2(U, X; \mu)$ with an appropriate separable Hilbert space X, an infinite-dimensional domain U and a probability measure μ on U where parametric solutions $u(\boldsymbol{y})$, $\boldsymbol{y} \in U$, to parametric and stochastic PDEs, are treated as elements of $L_2(U, X; \mu)$ and U the parametric domain. There is a vast number of works on this topic to not mention all of them. We point out just some works

[1, 2, 3, 5, 4, 7, 10, 11, 9, 8, 12, 14, 15, 16, 20, 21, 24, 25, 26, 27, 28, 19] which are directly related to the problem setting in our paper.

The key condition which emerged as governing the convergence rates of numerical integration and interpolation methods for a parametric solution $u(\boldsymbol{y})$ in $L_2(U, X; \mu)$ is a sparsity of the coefficients of its GPC expansion. The sparsity is quantified by ℓ_p-summability or weighted ℓ_2-summability of these coefficients which are appropriate for non-linear n-term approximation or linear approximation, respectively. We are interested in the problem of collocation approximation and its convergence rate based on a finite number of particular solvers $u(\boldsymbol{y}_1), ..., u(\boldsymbol{y}_n)$. The problem of adaptive nonlinear collocation approximation was investigated in [10, 11, 1, 8, 9], and of non-adaptive linear collocation approximation in [2, 14, 15, 16, 20, 24, 25, 26, 19, 27, 28]. The last problem naturally leads to the problem of linear sampling recovery in a Bochner space $L_2(U, X; \mu)$. Let us formulate a setting of this problem which will cover the linear collocation approximation problem for a wide class of parametric PDEs with random inputs as well as of infinite dimensional holomorphic functions.

Let (U, Σ, μ) be a probability measure space with Σ being countably generated and let X be a complex separable Hilbert space. Denote by $L_2(U, X; \mu)$ the Bochner space of strongly μ-measurable mappings v from U to X, equipped with the norm

$$\|v\|_{L_2(U,X;\mu)} := \left(\int_U \|v(\boldsymbol{y})\|_X^2 \, \mathrm{d}\mu(\boldsymbol{y}) \right)^{1/2}.$$

Notice that because Σ is countably generated, $L_2(U, \mathbb{C}; \mu)$ is separable by [13, Prop. 3.4.5]. Hence $L_2(U, X; \mu)$ is a separable complex Hilbert space and, moreover, $L_2(U, X; \mu) = L_2(U, \mathbb{C}; \mu) \otimes X$.

Let $(\varphi_s)_{s\in\mathbb{N}}$ be an orthonormal basis of $L_2(U, \mathbb{C}; \mu)$. Then a function $v \in L_2(U, X; \mu)$ can be represented by the expansion

$$v(\boldsymbol{y}) = \sum_{s\in\mathbb{N}} v_s \, \varphi_s(\boldsymbol{y}), \quad v_s \in X, \tag{1.1}$$

with the series convergence $L_2(U, X; \mu)$, where

$$v_s := \int_U v(\boldsymbol{y}) \, \overline{\varphi_s(\boldsymbol{y})} \, \mathrm{d}\mu(\boldsymbol{y}), \quad s \in \mathbb{N}.$$

Moreover, for every $v \in L_2(U, X; \mu)$ represented by the series (1.1), Parseval's identity holds

$$\|v\|_{L_2(U,X;\mu)}^2 = \sum_{s\in U} \|v_s\|_X^2.$$

Assume that v is a function on U taking values in the separable complex Hilbert space X and that $v \in L_2(U, X; \mu)$. Given sample points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_k \in U$ and $h_1, \dots, h_k \in L_2(U, \mathbb{C}; \mu)$, we consider the approximate recovery of v from its values $v(\boldsymbol{y}_1), \dots, v(\boldsymbol{y}_k)$ by the linear sampling algorithm (operator) S_k defined as on U of the form

$$(S_k v)(\boldsymbol{y}) := \sum_{i=1}^{k} v(\boldsymbol{y}_i) h_i(\boldsymbol{y}).$$

For convenience, we assume that some of the sample points $\boldsymbol{y}_i$ may coincide. The approximation error is measured by $\|v - S_k v\|_{L_2(U,X;\mu)}$. Denote by $\mathcal{S}_n^X$ the family of all linear sampling algorithms S_k^X in $L_2(U,X;\mu)$ of the form (1.5) with $k \le n$. To study the optimality of linear sampling algorithms from $\mathcal{S}_n^X$ for a set $F \subset L_2(U,X;\mu)$ and their convergence rates we use the (linear) sampling n-width

$$\varrho_n(F, L_2(U,X;\mu)) := \inf_{S_n^X \in \mathcal{S}_n^X} \sup_{v \in F} \|v - S_n^X v\|_{L_2(U,X;\mu)}.$$

Throughout the present paper, we fix $(\sigma_s)_{s\in\mathbb{N}}$, a non-decreasing sequence of positive numbers such that $\boldsymbol{\sigma}^{-1} := \left(\sigma_s^{-1}\right)_{s\in\mathbb{N}} \in \ell_2(\mathbb{N})$. For given U and μ, denote by $H_{X,\boldsymbol{\sigma}}$ the linear subspace in $L_2(U,X;\mu)$ of all v such that the norm

$$\|v\|_{H_{X,\boldsymbol{\sigma}}} := \left(\sum_{s\in\mathbb{N}} (\sigma_s \|v_s\|_X)^2\right)^{1/2} < \infty.$$

In particular, the space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is the linear subspace in $L_2(U,\mathbb{C};\mu)$ equipped with its own inner product

$$\langle f, g\rangle_{H_{\mathbb{C},\boldsymbol{\sigma}}} := \sum_{s\in\mathbb{N}} \sigma_s^2 \langle f, \varphi_s\rangle_{L_2(U,\mathbb{C};\mu)} \overline{\langle g, \varphi_s\rangle_{L_2(U,\mathbb{C};\mu)}}.$$

The space $H_{\mathbb{C},\boldsymbol{\sigma}}$ is a reproducing kernel Hilbert space with the reproducing kernel

$$K(\cdot, \boldsymbol{y}) := \sum_{s\in\mathbb{N}} \sigma_s^{-2} \varphi_s(\cdot) \overline{\varphi_s(\boldsymbol{y})}$$

with the eigenfunctions $(\varphi_s)_{s\in\mathbb{N}}$ and the eigenvalues $\left(\sigma_s^{-1}\right)_{s\in\mathbb{N}}$. Moreover, $K(\boldsymbol{x},\boldsymbol{y})$ satisfies the finite trace assumption

$$\int_U K(\boldsymbol{x},\boldsymbol{x}) \mathrm{d}\mu(\boldsymbol{x}) \ < \ \infty.$$

The aims of the present paper is to investigate the approximate recovery of functions in the space $H_{X,\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}^{-1} \in \ell_q(\mathbb{N})$ for some $0 < q < 2$ from a finite number of their sample values. We would like to establish convergence rates of the sampling recovery by extensions of several least squares methods which are different with respect to their constructiveness. Obtained results will be applied to linear collocation approximation for parametric PDEs with log-normal or affine inputs as well as for infinite dimensional holomorphic functions.

Let us briefly describe the main results of the present paper.

Let $B_{X,\boldsymbol{\sigma}}$ be the unit ball in the space $H_{X,\boldsymbol{\sigma}}$. Given arbitrary sample points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_k \in U$ and $h_1, \dots, h_k \in L_2(U,\mathbb{C};\mu)$, for the sampling algorithm S_n^X in $L_2(U,X;\mu)$ defined by (1.5), we have

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_n^X v\right\|_{L_2(U,X;\mu)} = \sup_{f\in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\|f - S_n^{\mathbb{C}} f\right\|_{L_2(U,\mathbb{C};\mu)},$$

and, hence,

$$\varrho_n(B_{X,\boldsymbol{\sigma}}, L_2(U,X;\mu)) \ = \ \varrho_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U,\mathbb{C};\mu)), \tag{1.2}$$

which make available bounds on the sampling widths in the classical Lebesgue space $L_2(U,\mathbb{C};\mu)$ applicable to a general Bochner space $L_2(U,X;\mu)$.

For $0<q\le 2$ and $M,N>0$ and $\boldsymbol{\sigma}$ with $\left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{N})}\le N$, denote

$$B^q_{X,\boldsymbol{\sigma}}(M,N) := \left\{ v\in H_{X,\boldsymbol{\sigma}} :\ \|v\|_{H_{X,\boldsymbol{\sigma}}}\le M \right\}.$$

From the equality (1.2) and an inequality between the sampling widths and Kolmogorov widths proven in [18, Theorem 1] we derived that if $0<q<2$, then

$$\varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(U,X;\mu)) \ \ll MNn^{-1/q}. \tag{1.3}$$

In particular, for Bochner space $L_2(\mathbb{D}^\infty,X;\mu)$ with infinite dimensional tensor-product probability,

$$\varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(\mathbb{D}^\infty,X;\mu)) \ \ll MNn^{-1/q}, \tag{1.4}$$

where $\mathbb{D}^\infty$ is $\mathbb{R}^\infty$ or $\mathbb{I}^\infty := [-1,1]^\infty$, μ infinite tensor-product Jacobi probability measure or standard Gaussian measure, respectively. It is worth mentioning that the underlying sampling algorithm performing the convergence rate in (1.3) and (1.4) is an extension to Bochner spaces of a classical least squares approximation with a non-constructive subsampling used in [18]. Moreover, this convergence rate is "quasi-optimal" the sense of the relation

$$MNn^{-1/q}(\log n)^{-\varepsilon} \ \ll \sup_{\|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{N})\le N}} \varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(U,X;\mu)) \ \ll \ MNn^{-1/q}$$

for any fixed $\varepsilon>1/q$. Similar extensions of a pure classical least squares approximation and of a classical least squares approximation with a special constructive subsampling give the bounds $MN(n/\log n)^{-1/q}$ and $MNn^{-1/q}\sqrt{\log n}$, respectively. Thanks to this special constructive subsampling, the cost of computation is significantly reduced for sufficiently large number of sample points (for detail, see [6]).

Under a certain condition the weak parametric solution $u(\boldsymbol{y})$ to a parametric elliptic PDE equation with affine ($\mathbb{D}^\infty=\mathbb{I}^\infty$) or log-normal ($\mathbb{D}^\infty=\mathbb{I}^\infty$) random inputs, satisfies a weighted ℓ_2-summability of the energy norms of the Jacobi or Hermite GPC expansion coefficients, respectively, in terms of the inclusion $u(\boldsymbol{y})\in B_{V,\boldsymbol{\sigma}}(M)$ with $\left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{N})}\le N$ for some $0<q<2$, $M,N>0$ and positive sequence $\boldsymbol{\sigma}$, where $V:=H^1_0(D)$ is the energy space and D is the spatial domain (see Lemmata 3.1 and 3.2 below). This allows us to apply all the above results for abstract Bochner spaces to parametric elliptic PDEs. For example, from (1.4) it follows that there exists a linear sampling algorithm S^V_n in $L_2(\mathbb{D}^\infty,V;\mu)$ of the form

$$S^V_n u(\boldsymbol{y}) := \sum_{i=1}^n u(\boldsymbol{y}_i)h_i(\boldsymbol{y}), \tag{1.5}$$

such that

$$\|u-S^V_n u\|_{L_2(\mathbb{D}^\infty,V;\mu)} \le CMNn^{-1/q},$$

where C is a positive constant independent of M,N,n and u. This means that the convergence rate of linear collocation approximation of the parametric solution $u(\boldsymbol{y})$ by the sampling algorithm S^V_n

is $MNn^{-1/q}$ which for given M, N is in particular, significantly better that the known convergence rate $n^{-(1/q-1/2)}$ of linear collocation approximation (cf. [2, 14, 15, 16, 20, 24, 25, 26, 19, 27, 28]). The same improved convergence rate holds true for linear collocation approximation of infinite dimensional holomorphic functions on $\mathbb{R}^\infty$ (cf. [19, 27, 28]).

The rest of the paper is organized as follows. In Section 2, we investigate sampling recovery in abstract Bochner spaces, in particular, with infinite dimensional measure. Here, we present some least squares methods and their extensions to Bochner spaces. In Section 3 and 4 we apply the results of Section 2 to linear collocation approximation for parametric elliptic PDE equation with affine or log-normal random inputs, for infinite dimensional holomorphic functions on $\mathbb{R}^\infty$, respectively.

Notation As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{N}_0 := \{s \in \mathbb{Z} : s \geq 0\}$. We denote $\mathbb{R}^\infty$ and $\mathbb{I}^\infty := [-1,1]^\infty$ the sets of all sequences $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ with $y_j \in \mathbb{R}$ and $y_j \in [-1,1]$, respectively. Denote by $\mathbb{F}$ the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j\in\mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. If $\boldsymbol{a} = (a_j)_{j\in\mathcal{J}}$ is a set of positive numbers with any index set $\mathcal{J}$, then we use the notation $\boldsymbol{a}^{-1} := (a_j^{-1})_{j\in\mathcal{J}}$. We use letter C to denote general positive constants which may take different values. For the quantities $A_n(f,\boldsymbol{k})$ and $B_n(f,\boldsymbol{k})$ depending on $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$, we write $A_n(f,\boldsymbol{k}) \ll B_n(f,\boldsymbol{k})$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ ($n \in \mathbb{N}$ is specially dropped), if there exists some constant $C > 0$ such that $A_n(f,\boldsymbol{k}) \leq CB_n(f,\boldsymbol{k})$ for all $n \in \mathbb{N}$, $f \in W$, $\boldsymbol{k} \in \mathbb{Z}^d$ (the notation $A_n(f,\boldsymbol{k}) \gg B_n(f,\boldsymbol{k})$ has the obvious opposite meaning), and $A_n(f,\boldsymbol{k}) \asymp B_n(f,\boldsymbol{k})$ if $A_n(f,\boldsymbol{k}) \ll B_n(f,\boldsymbol{k})$ and $B_n(f,\boldsymbol{k}) \ll A_n(f,\boldsymbol{k})$. Denote by $|G|$ the cardinality of the set G.

2 Sampling recovery in Bochner spaces

In this section, we show that the problem of linear sampling recovery of functions in the space $H_{X,\boldsymbol{\sigma}}$ for a general separable Hilbert space X can be reduced to the particular case of the reproducing kernel Hilbert space $H_{\mathbb{C},\boldsymbol{\sigma}}$. This allows, in particular, to extend linear least squares sampling algorithms in $H_{\mathbb{C},\boldsymbol{\sigma}}$ to $H_{X,\boldsymbol{\sigma}}$ with preserving the accuracy of approximation. Hence, we are able to derive convergence rates of various extended linear least squares sampling algorithms for functions in $B^q_{X,\boldsymbol{\sigma}}$ based on some recent results on inequality between sampling widths and Kolmogorov widths of the unit ball $B_{\mathbb{C},\boldsymbol{\sigma}}$ which are fulfilled by the relevant linear least squares sampling algorithms.

2.1 Extension of least squares approximation to Bochner spaces

We will need the following auxiliary result. Let A^X be a general linear operator in $L_2(U, X; \mu)$ defined for $v \in L_2(U, X; \mu)$ by

$$v \mapsto \sum_{k\in\mathbb{N}} \left(\sum_{s\in\mathbb{N}} a_{k,s} v_s \right) \varphi_k,$$

where $(a_{k,s})_{(k,s)\in\mathbb{N}^2}$ is an infinite dimensional matrix.

Lemma 2.1 *There holds the equality*

$$\|A^X\|_{H_{X,\boldsymbol{\sigma}}\to L_2(U,X;\mu)} = \|A^{\mathbb{C}}\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}.$$

Proof. For $f \in H_{\mathbb{C},\boldsymbol{\sigma}}$, we have

$$f = \sum_{s\in\mathbb{N}} f_s\varphi_s \quad \text{with} \quad (\sigma_s|f_s|)_{s\in\mathbb{N}} \in \ell_2,$$

and

$$\|A^{\mathbb{C}}f\|^2_{L_2(U,\mathbb{C};\mu)} \leq \|A^{\mathbb{C}}\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}\|f\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}}.$$

The last inequality is equivalent to inequality

$$\sum_{k\in\mathbb{N}}\left|\sum_{s\in\mathbb{N}} a_{k,s}f_s\right|^2 \leq \|A^{\mathbb{C}}\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}\sum_{s\in\mathbb{N}}\sigma_s^2|f_s|^2 \tag{2.1}$$

for all sequences $(\sigma_s|f_s|)_{s\in\mathbb{N}} \in \ell_2$.

For $v \in H_{X,\boldsymbol{\sigma}}$, we have

$$v = \sum_{s\in\mathbb{N}} v_s\varphi_s \quad \text{with} \quad (\sigma_s\|v_s\|_X)_{s\in\mathbb{N}} \in \ell_2,$$

and

$$\|A^X v\|^2_{L_2(U,X;\mu)} = \sum_{k\in\mathbb{N}}\left\|\sum_{s\in\mathbb{N}} a_{k,s}v_s\right\|^2_X.$$

Let $(\eta_j)_{j\in\mathbb{N}}$ be an orthonormal basis of X and

$$v_s = \sum_{j\in\mathbb{N}}(v_s)_j\eta_j.$$

Then,

$$A^X v = \sum_{k\in\mathbb{N}}\Big(\sum_{s\in\mathbb{N}}\sum_{j\in\mathbb{N}} a_{k,s}(v_s)_j\eta_j\Big)\varphi_k.$$

Since $(\varphi_k\eta_j)_{k,j\in\mathbb{N}}$ is an orthonormal basis of $L_2(U,X;\mu)$, by applying (2.1) to $f_s = (v_s)_j$, we obtain

$$\begin{aligned}
\|A^X v\|^2_{L_2(U,X;\mu)} &= \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}\left|\sum_{s\in\mathbb{N}} a_{k,s}(v_s)_j\right|^2 \\
&\leq \|A^{\mathbb{C}}\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}\sum_{j\in\mathbb{N}}\sum_{s\in\mathbb{N}}\sigma_s^2|(v_s)_j|^2 \\
&\leq \|A^{\mathbb{C}}\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}\sum_{s\in\mathbb{N}}\sigma_s^2\|v_s\|^2_X = \|A^{\mathbb{C}}\|^2_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}\|v\|^2_{H_{X,\boldsymbol{\sigma}}}.
\end{aligned}$$

This proves the inequality

$$\left\|A^X\right\|_{H_{X,\boldsymbol{\sigma}}\to L_2(U,X;\mu)} \leq \left\|A^{\mathbb{C}}\right\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}.$$

In order to prove the inverse inequality, let $(f^n)_{n\in\mathbb{N}} \subset H_{\mathbb{C},\boldsymbol{\sigma}}$ be a sequence such that $\|f^n\|_{H_{\mathbb{C},\boldsymbol{\sigma}}} = 1$ and

$$\lim_{n\to\infty} \left\|A^{\mathbb{C}} f^n\right\|_{L_2(U,\mathbb{C};\mu)} = \left\|A^{\mathbb{C}}\right\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}.$$

Define $v^n := f^n\eta_1$. Then $\|v^n\|_{H_{X,\boldsymbol{\sigma}}} = 1$ and

$$\begin{aligned}\left\|A^X v^n\right\|^2_{L_2(U,X;\mu)} &= \sum_{k\in\mathbb{N}} \left\|\sum_{s\in\mathbb{N}} a_{k,s}\langle f^n,\varphi_s\rangle_{L_2(U,\mathbb{C};\mu)}\eta_1\right\|_X^2 = \sum_{k\in\mathbb{N}} \left|\sum_{s\in\mathbb{N}} a_{k,s}\langle f^n,\varphi_s\rangle_{L_2(U,\mathbb{C};\mu)}\right|^2 \\ &= \left\|A^{\mathbb{C}} f^n\right\|^2_{L_2(U,\mathbb{C};\mu)} \to \left\|A^{\mathbb{C}}\right\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)} \quad \text{as} \quad n\to\infty.\end{aligned}$$

This proves the inequality

$$\left\|A^X\right\|_{H_{X,\boldsymbol{\sigma}}\to L_2(U,X;\mu)} \geq \left\|A^{\mathbb{C}}\right\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}.$$

□

Theorem 2.1 *Given arbitrary sample points* $\boldsymbol{y}_1,\dots,\boldsymbol{y}_k \in U$ *and functions* $h_1,\dots,h_k \in L_2(U,\mathbb{C};\mu)$*, for the sampling algorithm* S_n^X *in* $L_2(U,X;\mu)$ *defined by* (1.5)*, we have*

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_n^X v\right\|_{L_2(U,X;\mu)} = \sup_{f\in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\|f - S_n^{\mathbb{C}} f\right\|_{L_2(U,\mathbb{C};\mu)}.$$

Proof. Denote by I^X the identity operator in $L_2(U,X;\mu)$. Let S_n^X be an arbitrary sampling operator in $L_2(U,X;\mu)$ given for $v \in L_2(U,X;\mu)$ by

$$S_n^X f(\boldsymbol{y}) := \sum_{i=1}^n v(\boldsymbol{y}_i)h_i(\boldsymbol{y}).$$

Applying Lemma 2.1 with $A^X := I^X - S_n^X$, we get

$$\left\|I^X - S_n^X\right\|_{H_{X,\boldsymbol{\sigma}}\to L_2(U,X;\mu)} = \left\|I^{\mathbb{C}} - S_n^{\mathbb{C}}\right\|_{H_{\mathbb{C},\boldsymbol{\sigma}}\to L_2(U,\mathbb{C};\mu)}.$$

Consequently, we obtain

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_n^X v\right\|_{L_2(U,X;\mu)} = \sup_{f\in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\|f - S_n^{\mathbb{C}} f\right\|_{L_2(U,\mathbb{C};\mu)}.$$

□

Let us construct an extension of a least squares approximation in $L_2(U,\mathbb{C};\mu)$ to a space $L_2(U,X;\mu)$. For $c,n,m \in \mathbb{N}$ with $cn \geq m$, let $\boldsymbol{y}_1,\dots,\boldsymbol{y}_{cn} \in U$ be points, $\omega_1,\dots,\omega_{cn} \geq 0$ be weights, and $V_m = \operatorname{span}\{\varphi_s\}_{s=1}^m$ the subspace spanned by the functions φ_s corresponding to the m

smallest σ_s. The weighted least squares approximation $S_{cn}^{\mathbb{C}} f = S_{cn}^{\mathbb{C}}(\boldsymbol{y}_1, \dots, \boldsymbol{y}_{cn}, \omega_1, \dots, \omega_{cn}, V_m) f$ of a function $f\colon U \to \mathbb{C}$ is given by

$$S_{cn}^{\mathbb{C}} f = \arg\min_{g \in V_m} \sum_{i=1}^{cn} \omega_i |f(\boldsymbol{y}_i) - g(\boldsymbol{y}_i)|^2. \tag{2.2}$$

The least squares approximation can be computed using the Moore-Penrose inverse, which gives the solution of smallest error for over-determined systems where no exact solution can be expected. In particular, for $\boldsymbol{L} = [\varphi_s(\boldsymbol{y}_i)]_{i=1,\dots,cn; s=1,\dots,m}$ and $\boldsymbol{W} = \operatorname{diag}(\omega_1, \dots, \omega_n)$ we have

$$S_{cn}^{\mathbb{C}} f = \sum_{s=1}^{m} \hat{g}_s \varphi_s \quad \text{with} \quad (\hat{g}_1, \dots, \hat{g}_m)^\top = (\boldsymbol{L}^* \boldsymbol{W} \boldsymbol{L})^{-1} \boldsymbol{L}^* \boldsymbol{W} (f(\boldsymbol{y}_1), \dots, f(\boldsymbol{y}_{cn}))^\top. \tag{2.3}$$

Notice that $S_{cn}^{\mathbb{C}}$ is a linear sampling algorithm of the form

$$S_{cn}^{\mathbb{C}} f = \sum_{i=1}^{cn} f(\boldsymbol{y}_i) h_i(\boldsymbol{y}). \tag{2.4}$$

Hence we immediately obtain the extension of this least squares algorithm to the Bochner space $L_2(U, X; \mu)$ by replacing $f \in L_2(U, \mathbb{C}; \mu)$ with $v \in L_2(U, X; \mu)$:

$$S_{cn}^{X} v = \sum_{i=1}^{cn} v(\boldsymbol{y}_i) h_i(\boldsymbol{y}). \tag{2.5}$$

As the least squares approximation is a linear operator, worst-case error bounds carry over from the usual Lebesgue space $L_2(U, \mathbb{C}; \mu)$ to the Bochner space $L_2(U, X; \mu)$.

The choice of points $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{cn}$, weights $\omega_1, \dots, \omega_{cn}$, and approximation space V_m is crucial for the error of the least squares approximation. A lot of work has been done in the usual Lebesgue space $L_2(U, \mathbb{C}; \mu)$ of which we present three choices with a trade-off between constructiveness and tightness of the bound and transfer them to the Bochner space $L_2(U, X; \mu)$.

Assumption 2.2 *Let $n \in \mathbb{N}$, $n \geq 90$, $c_1 \geq 1$, $c_2 > 1 + \frac{1}{n}$, and $c_3 \geq 3284$. For $m \in \mathbb{N}$ let the probability measure $\nu = \nu(m)$ be defined by*

$$\mathrm{d}\nu(\boldsymbol{y}) := \varrho(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y}) := \frac{1}{2} \left(\frac{1}{m} \sum_{s=1}^{m} |\varphi_s(\boldsymbol{y})|^2 + \frac{\sum_{s=m+1}^{\infty} |\sigma_s^{-1} \varphi_s(\boldsymbol{y})|^2}{\sum_{s=m+1}^{\infty} \sigma_s^{-2}} \right) \mathrm{d}\mu(\boldsymbol{y}).$$

(i) *Let $m := \lfloor n/(20 \log n) \rfloor$. Let further $\boldsymbol{y}_1, \dots \boldsymbol{y}_{c_1 n} \in U$ be points drawn i.i.d. with respect to ν and $\omega_i := (\varrho(\boldsymbol{y}_i))^{-1}$.*

(ii) *Let $m := n$ and $\lceil 20 n \log n \rceil$ points be drawn i.i.d. with respect to ν and subsampled using [6, Algorithm 3] to $c_2 n \asymp m$ points. Denote the resulting points by $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{c_2 n}$ and $\omega_i = \frac{c_2 n}{\lceil 20 n \log n \rceil} (\varrho(\boldsymbol{y}_i))^{-1}$.*

(iii) *Let $m := n$ and $\lceil 20 n \log n \rceil$ points be drawn i.i.d. with respect to ν. Let further $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{c_3 n}$ be the subset of points fulfilling [18, Theorem 1] with $c_3 n \asymp m$ and $\omega_i := \frac{c_3 n}{\lceil 20 n \log n \rceil} (\varrho(\boldsymbol{y}_i))^{-1}$.*

Let $n \in \mathbb{N}$ and E be a normed space and F a central symmetric compact set in E. Then the Kolmogorov n-width of F is defined by

$$d_n(F,E) := \inf_{L_n} \sup_{f\in F} \inf_{g\in L_n} \|f-g\|_E,$$

where the left-most infimum is taken over all subspaces L_n of dimension at most n in E.

We make use of the abbreviation $d_s := d_s(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U,\mathbb{C};\mu))$. In our setting, we know $d_s = \sigma_{s+1}^{-1}$.

Theorem 2.3 *For $c,n,m \in \mathbb{N}$ with $cn \geq m$, let S_{cn}^X be the extension* (2.5) *of the least squares sampling algorithm $S_{cn}^{\mathbb{C}}$ which is defined as in* (2.2)–(2.4)*. There are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have the following.*

(i) *The points from Assumption 2.2*(i) *fulfill with high probability*

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_{c_1 n}^X v\right\|_{L_2(U,X;\mu)} \leq \sqrt{\frac{\log n}{n} \sum_{s\geq n/\log n} d_s^2}.$$

(ii) *The points from Assumption 2.2*(ii) *fulfill with high probability*

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_{c_2 n}^X v\right\|_{L_2(U,X;\mu)} \leq \sqrt{\frac{\log n}{n} \sum_{s\geq n} d_s^2}.$$

(iii) *The points from Assumption 2.2*(iii) *fulfill with high probability*

$$\sup_{v\in B_{X,\boldsymbol{\sigma}}} \left\|v - S_{c_2 n}^X v\right\|_{L_2(U,X;\mu)} \leq \sqrt{\frac{1}{n} \sum_{s\geq n} d_s^2}.$$

Proof. For the particular case when $X = \mathbb{C}$, the claims (i)–(iii) in this theorem have been proven in [23, Theorem 1] (see also [22, Corollary 5.6]), [6, Theorem 6.7] and [18, Theorem 1], respectively. Hence, by using Theorem 2.1 we prove the theorem. □

Regarding the constructiveness of the linear sampling algorithms in Theorem 2.3, the bound Theorem 2.3(i) is the most coarse bound, but the points construction requires only a random draw, which is computationally inexpensive. The sharper bound in Theorem 2.3(ii) uses an additional constructive subsampling step. This was implemented and numerically tested in [6] for up to 1000 basis functions. For larger problem sizes the current algorithm is to slow as its runtime is cubic in the number of basis functions. The sharpest bound in Theorem 2.3(iii) is a pure existence result. So, up to now, the only way to obtain this point set is to brute-force every combination, which is computational infeasible.

2.2 Convergence rates

Theorem 2.4 *There holds the equality*

$$\varrho_n(B_{X,\boldsymbol{\sigma}}, L_2(U,X;\mu)) \ = \ \varrho_n(B_{\mathbb{C},\boldsymbol{\sigma}}, L_2(U,\mathbb{C};\mu)).$$

Proof. Since the correspondence between S_n^X and $S_n^{\mathbb{C}}$ is one-to-one, we use Theorem 2.1 to show

$$\inf_{S_n^X \in \mathcal{S}_n^X} \ \sup_{v \in B_{X,\boldsymbol{\sigma}}} \left\| v - S_n^X v \right\|_{L_2(U,X;\mu)} = \inf_{S_n^{\mathbb{C}} \in \mathcal{S}_n^{\mathbb{C}}} \ \sup_{f \in B_{\mathbb{C},\boldsymbol{\sigma}}} \left\| f - S_n^{\mathbb{C}} f \right\|_{L_2(U,\mathbb{C};\mu)},$$

which proves the corollary. ∎

Lemma 2.2 *Let $0 < q \le 2$. We have*

$$d_n(B^q_{\mathbb{C},\boldsymbol{\sigma}}(M,N), L_2(U,\mathbb{C};\mu)) \ \le 2^{1/q} M N n^{-1/q} \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Proof. For $\xi > 0$, we introduce the set

$$\Lambda(\xi) := \ \left\{ s \in \mathbb{N} : \, \sigma_s^q \le \xi \right\}.$$

For a function $f \in B^q_{\mathbb{C},\boldsymbol{\sigma}}(M,N)$ represented by the series (1.1), we define the truncation

$$S_{\Lambda(\xi)} f := \sum_{s \in \Lambda(\xi)} f_s \varphi_s. \tag{2.7}$$

Applying the Parseval's identity, noting (2.7), we obtain

$$\begin{aligned} \|f - S_{\Lambda(\xi)} f\|^2_{L_2(U,\mathbb{C};\mu)} \ &\le \ \sum_{\sigma_s > \xi^{1/q}} |f_s|^2 \ = \ \sum_{\sigma_{\boldsymbol{s}} > \xi^{1/q}} (\sigma_{\boldsymbol{s}} |f_s|)^2 \sigma_{\boldsymbol{s}}^{-2} \\ &\le \ \xi^{-2/q} \sum_{\boldsymbol{s} \in \mathbb{N}} (\sigma_{\boldsymbol{s}} |f_s|)^2 \ \le \ M^2 \xi^{-2/q}. \end{aligned}$$

The function $S_{\Lambda(\xi)} f$ belongs to the linear subspace $L(\xi) := \operatorname{span}\{\varphi_s : s \in \Lambda(\xi)\}$ in $L_2(U,\mathbb{C};\mu)$ of dimension $|\Lambda(\xi)|$. We have

$$|\Lambda(\xi)| \ \le \ \sum_{\xi \sigma_{\boldsymbol{s}}^{-q} \ge 1} 1 \ \le \ N^q \xi.$$

For a given $n \in \mathbb{N}$, choose ξ_n satisfying the inequalities $N^q \xi_n \le n < 2N^q \xi_n$. With this choice we derive the upper bound in (2.6):

$$d_n(B^q_{\mathbb{C},\boldsymbol{\sigma}}(M,N), L_2(U,\mathbb{C};\mu)) \ \le \ \|f - S_{\Lambda(\xi_n)} f\|_{L_2(U,\mathbb{C};\mu)} \ \le \ M \xi_n^{-1/q} \le 2^{1/q} M N n^{-1/q}.$$

∎

Theorem 2.5 *Let $0 < q < 2$ and $M, N > 0$. For $c, n, m \in \mathbb{N}$ with $cn \ge m$, let S^X_{cn} be the extension (2.5) of the least squares sampling algorithm $S^{\mathbb{C}}_{cn}$ which is defined as in (2.2)–(2.4). There are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have the following.*

(i) *The points from Assumption 2.2*(i) *fulfill with high probability*

$$\sup_{v \in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\| v - S^X_{c_1 n} v \right\|_{L_2(U,X;\mu)} \ll MN \Big(\frac{n}{\log n}\Big)^{-1/q};$$

(ii) *The points from Assumption 2.2*(ii) *fulfill with high probability*

$$\sup_{v \in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\| v - S^X_{c_2 n} v \right\|_{L_2(U,X;\mu)} \ll MN n^{-1/q} \sqrt{\log n};$$

(iii) *The points from Assumption 2.2*(iii) *fulfill with high probability*

$$\varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(U,X;\mu)) \ll \sup_{v \in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\| v - S^X_{c_3 n} v \right\|_{L_2(U,X;\mu)} \ll MN n^{-1/q}.$$

Proof. From the definitions one can see that

$$\sup_{v \in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\| v - S^X_{cn} v \right\|_{L_2(U,X;\mu)} = MN \sup_{v \in B^q_{X,\boldsymbol{\sigma}}(1,1)} \left\| v - S^X_{cn} v \right\|_{L_2(U,X;\mu)}.$$

Hence, the claims (i)–(iii) in this theorem are derived from the claims (i)–(iii) in Theorem 2.3, respectively, and the asymptotical equivalence

$$\le \sqrt{\frac{1}{m} \sum_{k \ge m} k^{-2/q}} \asymp m^{-1/q}, \quad m \in \mathbb{N}.$$

□

Notice that the convergence rate in Theorem 2.5(iii) is "quasi-optimal" the sense of the relation

$$MN n^{-1/q} (\log n)^{-\varepsilon} \ll \sup_{\|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{N})} \le N} \varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(U,X;\mu)) \ll MN n^{-1/q} \tag{2.8}$$

for any fixed $\varepsilon > 1/q$. The upper bound in (2.8) follows from the fact that the bound in Theorem 2.5(iii) is independent of the sequence $\boldsymbol{\sigma}$. To prove the lower bound, one can take $\boldsymbol{\sigma} = (\sigma_s)_{s \in \mathbb{N}}$ with $\sigma_s = s^{1/q}(\log(s+1))^{\varepsilon}$, and prove that $\boldsymbol{\sigma}^{-1} \in \ell_q(\mathbb{N})$ and that by Theorem 2.4,

$$\begin{aligned} \varrho_n(B^q_{X,\boldsymbol{\sigma}}(M,N), L_2(U,X;\mu)) &= \varrho_n(B^q_{\mathbb{C},\boldsymbol{\sigma}}(M,N), L_2(U,\mathbb{C};\mu)) \\ &\ge d_n(B^q_{\mathbb{C},\boldsymbol{\sigma}}(M,N), L_2(U,\mathbb{C};\mu)) \asymp MN n^{-1/q} (\log n)^{-\varepsilon}. \end{aligned}$$

Next, we apply Theorem 2.5 to Bochner spaces with infinite tensor-product probability measure relevant to the applications to solutions to parametric PDEs with random inputs and to holomorphic functions in Sections 3 and 4, respectively.

For given $a, b > -1$, let $\nu_{a,b}$ be the Jacobi probability measure on $\mathbb{I} := [-1, 1]$ with the density

$$\delta_{a,b}(y) := c_{a,b}(1-y)^a(1+y)^b, \quad c_{a,b} := \frac{\Gamma(a+b+2)}{2^{a+b+1}\Gamma(a+1)\Gamma(b+1)}.$$

Let $(J_k)_{k\in\mathbb{N}_0}$ be the sequence of Jacobi polynomials on $\mathbb{I} := [-1, 1]$ normalized with respect to the Jacobi probability measure $\nu_{a,b}$, i.e.,

$$\int_{\mathbb{I}} |J_k(y)|^2 \mathrm{d}\nu_{a,b}(y) = \int_{\mathbb{I}} |J_k(y)|^2 \delta_{a,b}(y) \mathrm{d}y = 1, \quad k \in \mathbb{N}_0.$$

Let γ be the standard Gaussian probability measure on $\mathbb{R}$ with the density

$$g(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Let $(H_k)_{k\in\mathbb{N}_0}$ be the sequence of Hermite polynomials on $\mathbb{R}$ normalized with respect to the measure γ, i.e.,

$$\int_{\mathbb{R}} |H_k(y)|^2 \mathrm{d}\gamma(y) = \int_{\mathbb{I}} |H_k(y)|^2 g(y) \mathrm{d}y = 1, \quad k \in \mathbb{N}_0.$$

Throughout this section, we use the joint notation: $\mathbb{D}$ denotes either $\mathbb{I}$ or $\mathbb{R}$; $\mathbb{D}^\infty$ either $\mathbb{I}^\infty$ or $\mathbb{R}^\infty$;

$$\mu := \begin{cases} \nu_{a,b} & \text{if } \mathbb{D} = \mathbb{I}, \\ \gamma & \text{if } \mathbb{D} = \mathbb{R}; \end{cases}$$

$$\varphi_k := \begin{cases} J_{k-1} & \text{if } \mathbb{D} = \mathbb{I}, \\ H_{k-1} & \text{if } \mathbb{D} = \mathbb{R}. \end{cases}$$

We next recall a concept of probability measure $\mu(\boldsymbol{y})$ on $\mathbb{D}^\infty$ as the infinite tensor product of the measures $\mu(y_i)$:

$$\mu(\boldsymbol{y}) := \bigotimes_{j\in\mathbb{N}} \mu(y_j), \quad \boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{D}^\infty.$$

(In the case $\mathbb{D}^\infty = \mathbb{R}^\infty$ the sigma algebra for $\gamma(\boldsymbol{y})$ is generated by the set of cylinders $A := \prod_{j\in\mathbb{N}} A_j$, where $A_j \subset \mathbb{R}$ are univariate γ-measurable sets and only a finite number of A_i are different from $\mathbb{R}$. For such a set A, we have $\gamma(A) = \prod_{j\in\mathbb{N}} \gamma(A_j)$).

Let X be a separable Hilbert space. Then a function $v \in L_2(\mathbb{D}^\infty, X; \mu)$ can be represented by the generalized polynomial chaos (GPC) expansion

$$v(\boldsymbol{y}) = \sum_{\boldsymbol{s}\in\mathbb{F}} v_{\boldsymbol{s}} \, \varphi_{\boldsymbol{s}}(\boldsymbol{y}), \quad v_{\boldsymbol{s}} \in X, \tag{2.9}$$

with

$$\varphi_{\boldsymbol{s}}(\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} \varphi_{s_j}(y_j), \quad v_{\boldsymbol{s}} := \int_{\mathbb{D}^\infty} v(\boldsymbol{y}) \, \varphi_{\boldsymbol{s}}(\boldsymbol{y}) \, \mathrm{d}\mu(\boldsymbol{y}), \quad \boldsymbol{s} \in \mathbb{F}.$$

Here $\mathbb{F}$ is the set of all sequences of non-negative integers $\boldsymbol{s} = (s_j)_{j\in\mathbb{N}}$ such that their support $\operatorname{supp}(\boldsymbol{s}) := \{j \in \mathbb{N} : s_j > 0\}$ is a finite set. Notice that $(\varphi_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ is an orthonormal basis of

$L_2(\mathbb{D}^\infty, \mathbb{C}; \mu)$. Moreover, for every $v \in L_2(U, X; \mu)$ represented by the series (1.1), Parseval's identity holds

$$\|v\|^2_{L_2(U,X;\mu)} = \sum_{\boldsymbol{s}\in\mathbb{F}} \|v_{\boldsymbol{s}}\|^2_X.$$

Let $0 < q \leq 2$ and $M, N > 0$. For a set $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}} \in \ell_q(\mathbb{N})$ of positive numbers such that $\|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{N})} \leq N$, denote by $B^q_{X,\boldsymbol{\sigma}}(M, N)$ the set of all functions $v \in L_2(\mathbb{D}^\infty, X; \mu))$ represented by the series (2.9) such that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_X)^2\right)^{1/2} \leq M.$$

Notice that if $v \in B^q_X(M, N)$, the series (2.9) converges absolutely and unconditionally in $L_2(U, X; \mu)$ to v (see [17, Lemma 3.1] for the case $\mathbb{D}^\infty = \mathbb{R}^\infty$, the case $\mathbb{D}^\infty = \mathbb{I}^\infty$ can be proven by the same arguments).

Theorem 2.5 for the space $L_2(\mathbb{D}^\infty, X; \mu)$ is read as

Theorem 2.6 *Let $0 < q < 2$ and $M, N > 0$. For $c, n, m \in \mathbb{N}$ with $cn \geq m$, let S^X_{cn} be the extension* (2.5) *of the least squares sampling algorithm $S^{\mathbb{C}}_{cn}$ which is defined as in* (2.2)–(2.4)*. There are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have the following.*

(i) *The points from Assumption 2.2*(i) *fulfill with high probability*

$$\sup_{v\in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\|v - S^X_{c_1 n} v\right\|_{L_2(\mathbb{D}^\infty,X;\mu)} \ll MN\left(\frac{n}{\log n}\right)^{-1/q};$$

(ii) *The points from Assumption 2.2*(ii) *fulfill with high probability*

$$\sup_{v\in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\|v - S^X_{c_2 n} v\right\|_{L_2(\mathbb{D}^\infty,X;\mu)} \ll MNn^{-1/q}\sqrt{\log n};$$

(iii) *The points from Assumption 2.2*(iii) *fulfill with high probability*

$$\varrho_n(B^q_{X,\boldsymbol{\sigma}}(M, N), L_2(\mathbb{D}^\infty, X; \mu)) \ll \sup_{v\in B^q_{X,\boldsymbol{\sigma}}(M,N)} \left\|v - S^X_{c_3 n} v\right\|_{L_2(\mathbb{D}^\infty,X;\mu)} \ll MNn^{-1/q}.$$

3 Applications to parametric elliptic PDEs with random inputs

3.1 Introducing remarks

One of basic problems in Uncertainty Quantification is approximation for parametric and stochastic PDEs. Since the number of parametric variables may be very large or even infinite, they are treated as high-dimensional or infinite-dimensional approximation problems. As a model we consider parametric divergence-form elliptic PDEs with random inputs.

Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Consider the diffusion elliptic equation

$$- \operatorname{div}(a\nabla u) \; = \; f \quad \text{in} \quad D, \quad u|_{\partial D} \; = \; 0, \tag{3.1}$$

for a given fixed right-hand side f and a spatially variable scalar diffusion coefficient a. Denote by $V := H_0^1(D)$ the energy space and $H^{-1}(D)$ the dual space of V. Assume that $f \in H^{-1}(D)$ (in what follows this preliminary assumption always holds without mention). If $a \in L_\infty(D)$ satisfies the ellipticity assumption

$$0 < a_{\min} \le a \le a_{\max} < \infty,$$

by the well-known Lax-Milgram lemma, there exists a unique solution $u \in V$ to the equation (3.1) in the weak form

$$\int_D a\nabla u \cdot \nabla v \, \mathrm{d}\boldsymbol{x} \; = \; \langle f, v\rangle, \quad \forall v \in V.$$

PDEs with parametric and stochastic inputs are a common model used in science and engineering. Depending on the nature of the modeled object, the parameters involved in them may be either deterministic or random. Random nature reflects the uncertainty in various parameters presented in the physical phenomenon modeled by the equation. For the equation (3.1), we consider the diffusion coefficients having a parametric form $a = a(\boldsymbol{y})$, where $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}}$ is a sequence of real-valued parameters ranging in the set $\mathbb{U}^\infty$ which is either $\mathbb{R}^\infty$ or $\mathbb{I}^\infty$. Denote by $u(\boldsymbol{y})$ the solution to the parametric diffusion elliptic equation

$$-\operatorname{div}(a(\boldsymbol{y})\nabla u(\boldsymbol{y})) \; = \; f \quad \text{in} \quad D, \quad u(\boldsymbol{y})|_{\partial D} \; = \; 0.$$

The resulting solution operator maps $\boldsymbol{y} \in \mathbb{U}^\infty$ to $u(\boldsymbol{y}) \in V$. The objective is to achieve a numerical approximation of this complex map by a small number of parameters with a guaranteed error in a given norm.

In the present paper, we consider both the lognormal case when $\mathbb{U}^\infty = \mathbb{R}^\infty$ and the diffusion coefficient a is of the form

$$a(\boldsymbol{y}) = \exp(b(\boldsymbol{y})), \quad \text{with} \;\; b(\boldsymbol{y}) = \sum_{j=1}^\infty y_j \psi_j, \tag{3.2}$$

and y_j are i.i.d. standard Gaussian random variables, and the affine case when $\mathbb{U}^\infty = \mathbb{I}^\infty$ and the diffusion coefficient a is of the form

$$a(\boldsymbol{y}) = \bar{a} + \sum_{j=1}^\infty y_j \psi_j, \tag{3.3}$$

and y_j are i.i.d. standard Jacobi random variables. Here $\bar{a} \in L_\infty(D)$ and $\psi_j \in L_\infty(D)$ for both the cases.

An approach to studying summability that takes into account the support properties has been recently proposed in [5] for the affine parametric case, in [4] for the log-normal, parametric case, and in [3] for extension of both cases to higher-order Sobolev norms of the corresponding generalized PC expansion coefficients. This approach leads to significant improvements on the results on ℓ_p-summability and weighted ℓ_2-summability of GPC expansion coefficients, and therefore, on best

n-term semi-discrete and fully discrete approximations when the component functions ψ_j have limited overlap, such as splines, finite elements or compactly supported wavelet bases. In this section, we will employ the results of the previous section to receive convergence rates of sampling recovery of solutions to parametric elliptic PDEs with random inputs, which are derived results on weighted ℓ_2-summability in [5, 3].

3.2 Convergence rates

We present first some known weighted ℓ_2-summability results for solutions of parametric elliptic PDEs with random inputs. For the log-normal case, the following lemma combines [4, Theorems 3.3 and 4.2] and [15, Lemma 5.3].

Lemma 3.1 *Let $0 < q < \infty$ and $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers such that $(\rho_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$. Assume further that*

$$\left\| \sum_{j\in\mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \infty .$$

Then we have that for any $\eta \in \mathbb{N}$,

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_V)^2 \right)^{1/2} \le M < \infty \quad \text{with} \quad \|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{N})} \le N < \infty,$$

where with $|\boldsymbol{s}'|_\infty := \sup_{j\in\mathbb{N}} s'_j$ we denote

$$\sigma_{\boldsymbol{s}}^2 := \sum_{|\boldsymbol{s}'|_\infty \le \eta} \binom{\boldsymbol{s}}{\boldsymbol{s}'} \prod_{j\in\mathbb{N}} \rho_j^{2s'_j}. \tag{3.4}$$

For the affine case, the following lemma has been proven in [5].

Lemma 3.2 *Let $\operatorname{ess\,inf} \bar{a} > 0$. Let $0 < q < \infty$ and $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers such that $(\rho_j^{-1})_{j\in\mathbb{N}}$ belongs to $\ell_q(\mathbb{N})$. Assume further that*

$$\left\| \frac{\sum_{j\in\mathbb{N}} \rho_j |\psi_j|}{\bar{a}} \right\|_{L^\infty(D)} < 1.$$

Then we have that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_V)^2 \right)^{1/2} \le M < \infty \quad \text{with} \quad \|\boldsymbol{\sigma}^{-1}\|_{\ell_q(\mathbb{N})} \le N < \infty, \tag{3.5}$$

where

$$\sigma_{\boldsymbol{s}} := \prod_{j\in\mathbb{N}} c_{s_j}^{a,b} \rho_j^{s_j}.$$

By applying Theorem 2.6, from Lemmata 3.1 and 3.2 we obtain

Theorem 3.1 *Let the assumptions of Lemma 3.1 or of Lemma 3.2 with* $0 < q < 2$ *hold for the log-normal case* (3.2) ($\mathbb{D}^\infty = \mathbb{R}^\infty$) *or for the affine case* (3.3) ($\mathbb{D}^\infty = \mathbb{I}^\infty$)*, respectively. For* $c, n, m \in \mathbb{N}$ *with* $cn \geq m$*, let* S_{cn}^X *be the extension* (2.5) *of the least squares sampling algorithm* $S_{cn}^{\mathbb{C}}$ *which is defined as in* (2.2)–(2.4)*. There are universal constants* $c_1, c_2, c_3 \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *we have the following.*

(i) *The points from Assumption 2.2*(i) *fulfill with high probability*

$$\left\| u - S_{c_1 n}^V u \right\|_{L_2(\mathbb{D}^\infty, V; \mu)} \leq CMN \left(\frac{n}{\log n} \right)^{-1/q};$$

(ii) *The points from Assumption 2.2*(ii) *fulfill with high probability*

$$\left\| u - S_{c_2 n}^V u \right\|_{L_2(\mathbb{D}^\infty, V; \mu)} \leq CMN n^{-1/q} \sqrt{\log n};$$

(iii) *The points from Assumption 2.2*(iii) *fulfill with high probability*

$$\left\| u - S_{c_3 n}^V u \right\|_{L_2(\mathbb{D}^\infty, V; \mu)} \leq CMN n^{-1/q}.$$

The constants C *in the above inequalities are independent of of* M, N, n *and* u*.*

In the affine case (3.3), the convergence rate $(n/\log n)^{-1/q}$ with respect to the number n of sampling points has been received in [8] for an *adaptive* least squares approximation based on an ℓ_q-summability of the Legendre GPC expansion coefficients of the parametric solution, and on an adaptive choice of sequence of finite dimensional approximation spaces, which is different from the linear least squares approximation in Theorem 3.1(i). Notice also that the result in Theorem 3.1(i) for the affine case could be also proven by a linear modification of the technique used in [8], based on the weighted ℓ_2-summability (3.5).

4 Applications to holomorphic functions

Using real-variable arguments as, e.g., in [4, 3], establishing sparsity of parametric solutions in Sobolev spaces in D of higher smoothness seems to require more involved technical and notational developments, according to [3, 19]. As observed in [12, 19], one advantage of establishing sparsity of Hermite GPC expansion coefficients via holomorphy rather than by successive differentiation is that it allows to derive, in a unified way, summability bounds for the coefficients of Hermite GPC expansion whose size is measured in Sobolev scales in the domain D.

We recall the concept of "$(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic functions" which has been introduced in [19]. For $m \in \mathbb{N}$ and a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$, we put

$$\mathcal{S}(\boldsymbol{\varrho}) := \{ \boldsymbol{z} \in \mathbb{C}^m : |\mathfrak{Im} z_j| < \varrho_j \ \forall j \} \qquad \text{and} \qquad \mathcal{B}(\boldsymbol{\varrho}) := \{ \boldsymbol{z} \in \mathbb{C}^m : |z_j| < \varrho_j \ \forall j \}.$$

Let X be a complex separable Hilbert space, $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ a positive sequence, and $\xi > 0$, $\varepsilon > 0$. For $m \in \mathbb{N}$ we say that a positive sequence $\boldsymbol{\varrho} = (\varrho_j)_{j=1}^m$ is $(\boldsymbol{b}, \xi)$-*admissible* if

$$\sum_{j=1}^{m} b_j \varrho_j \le \xi \, .$$

A function $v \in L_2(\mathbb{R}^\infty, X; \gamma)$ is called $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic if

(i) for every $m \in \mathbb{N}$ there exists $v_m : \mathbb{R}^m \to X$, which, for every $(\boldsymbol{b}, \xi)$-admissible $\boldsymbol{\varrho}$, admits a holomorphic extension (denoted again by v_m) from $\mathcal{S}(\boldsymbol{\varrho}) \to X$; furthermore, for all $m < m'$

$$v_m(y_1, \ldots, y_m) = v_{m'}(y_1, \ldots, y_m, 0, \ldots, 0) \qquad \forall (y_j)_{j=1}^m \in \mathbb{R}^m,$$

(ii) for every $m \in \mathbb{N}$ there exists $\varphi_m : \mathbb{R}^m \to \mathbb{R}_+$ such that $\|\varphi_m\|_{L^2(\mathbb{R}^m;\gamma)} \le \varepsilon$ and

$$\sup_{\boldsymbol{\rho} \text{ is } (\boldsymbol{b},\xi)\text{-adm.}} \ \sup_{\boldsymbol{z}\in\mathcal{B}(\boldsymbol{\varrho})} \|v_m(\boldsymbol{y} + \boldsymbol{z})\|_X \le \varphi_m(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \mathbb{R}^m,$$

(iii) with $\tilde{v}_m : \mathbb{R}^\infty \to X$ defined by $\tilde{v}_m(\boldsymbol{y}) := v_m(y_1, \ldots, y_m)$ for $\boldsymbol{y} \in \mathbb{R}^\infty$ it holds

$$\lim_{m\to\infty} \|v - \tilde{v}_m\|_{\mathcal{L}_2(X)} = 0.$$

We notice some important examples of $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic functions which are solutions to parametric PDEs equations and which were studied in [19].

Formally, replacing $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in \mathbb{R}^\infty$ in the coefficient $a(\boldsymbol{y})$ in (3.2) by $\boldsymbol{z} = (z_j)_{j\in\mathbb{N}} = (y_j + i\xi_j)_{j\in\mathbb{N}} \in \mathbb{C}^\infty$, the real part of $a(\boldsymbol{z})$ is

$$\mathfrak{R}[a(\boldsymbol{z})] = \exp\left(\sum_{j\in\mathbb{N}} y_j \psi_j(\boldsymbol{x})\right) \cos\left(\sum_{j\in\mathbb{N}} \xi_j \psi_j(\boldsymbol{x})\right).$$

We find that $\mathfrak{R}[a(\boldsymbol{z})] > 0$ if

$$\left\| \sum_{j\in\mathbb{N}} \xi_j \psi_j \right\|_{L_\infty(D)} < \frac{\pi}{2}.$$

This observation motivates the study of the analytic continuation of the solution map $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ to $\boldsymbol{z} \mapsto u(\boldsymbol{z})$ for complex parameters $\boldsymbol{z} = (z_j)_{j\in\mathbb{N}}$ where each z_j lies in the strip

$$\mathcal{S}_j(\boldsymbol{\rho}) := \{z_j \in \mathbb{C} : |\mathfrak{Im} z_j| < \rho_j\}$$

and where $\rho_j > 0$ and $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ is any sequence of positive numbers such that

$$\left\| \sum_{j\in\mathbb{N}} \rho_j |\psi_j| \right\|_{L_\infty(D)} < \frac{\pi}{2} \, .$$

For further detail of this continuation we refer to [19, Proposition 3.8].

In general, let $b(\boldsymbol{y})$ be defined as in (3.2) and $\mathcal{V}$ a holomorphic map from an open set in $L_\infty(D)$ to X. Then function compositions of the type

$$v(\boldsymbol{y}) = \mathcal{V}(\exp(b(\boldsymbol{y})))$$

are $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic under certain conditions [19, Proposition 4.11]. This allows us to apply weighted ℓ_2-summability for collocation approximation of solutions $v(\boldsymbol{y}) = \mathcal{V}(\exp(b(\boldsymbol{y})))$ as $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic functions on various function spaces X, to a wide range of parametric and stochastic PDEs with log-normal inputs. Such function spaces X are high-order regularity spaces $H^s(D)$ and corner-weighted Sobolev (Kondrat'ev) spaces $K^s_\varkappa(D)$ ($s \geq 1$) for the parametric elliptic PDEs (3.1) with log-normal inputs (3.2); spaces of solutions to linear parabolic PDEs with log-normal inputs (3.2); spaces of solutions to linear elastics equations with lognormal modulus of elasticity; spaces of solutions to Maxwell equations with lognormal permittivity. For detail, see [19].

The following key result on weighted ℓ_2-summability of $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic functions has been proven in [19, Corollary 4.9].

Lemma 4.1 *Let v be $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic for some $\boldsymbol{b} \in \ell_p(\mathbb{N})$ with $0 < p < 1$. Let $\eta \in \mathbb{N}$ and let the sequence $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ be defined by*

$$\rho_j := b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|\boldsymbol{b}\|_{\ell_p(\mathbb{N})}.$$

Assume that $\boldsymbol{b}$ is a decreasing sequence and that $b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|\boldsymbol{b}\|_{\ell_p(\mathbb{N})} > 1$ for all $j \in \mathbb{N}$. Then there exist a constant M and an increasing sequence $\boldsymbol{\sigma} = (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ of positive numbers such that

$$\left(\sum_{\boldsymbol{s}\in\mathbb{F}} (\sigma_{\boldsymbol{s}} \|v_{\boldsymbol{s}}\|_X)^2\right)^{1/2} \leq M \ < \infty, \quad \text{with} \quad \left\|\boldsymbol{\sigma}^{-1}\right\|_{\ell_q(\mathbb{N})} \leq N < \infty,$$

where $q := p/(1-p)$, $\boldsymbol{\sigma} := (\sigma_{\boldsymbol{s}})_{\boldsymbol{s}\in\mathbb{F}}$ given by (3.4), and $M := \varepsilon C_{\boldsymbol{b},\xi}$ with some positive constant $C_{\boldsymbol{b},\xi}$.

By applying Theorem 2.6, from Lemma 4.1 we obtain

Theorem 4.1 *Let v be $(\boldsymbol{b}, \xi, \varepsilon, X)$-holomorphic for some $\boldsymbol{b} \in \ell_p(\mathbb{N})$ with $0 < p < 2/3$. Assume that $\boldsymbol{b}$ is a decreasing sequence and that $b_j^{p-1} \frac{\xi}{4\sqrt{\eta!}} \|\boldsymbol{b}\|_{\ell_p(\mathbb{N})} > 1$ for all $j \in \mathbb{N}$. For $c, n, m \in \mathbb{N}$ with $cn \geq m$, let S^X_{cn} be the extension (2.5) of the least squares sampling algorithm $S^{\mathbb{C}}_{cn}$ which is defined as in (2.2)–(2.4). There are universal constants $c_1, c_2, c_3 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have the following.*

(i) *The points from Assumption 2.2*(i) *fulfill with high probability*

$$\left\|v - S^X_{c_1 n} v\right\|_{L_2(\mathbb{R}^\infty, X\mu)} \leq CMN\left(\frac{n}{\log n}\right)^{-1/q};$$

(ii) *The points from Assumption 2.2*(ii) *fulfill with high probability*

$$\left\|v - S^X_{c_2 n} v\right\|_{L_2(\mathbb{R}^\infty, X;\mu)} \le CMNn^{-1/q}\sqrt{\log n};$$

(iii) *The points from Assumption 2.2*(iii) *fulfill with high probability*

$$\left\|v - S^X_{c_3 n} v\right\|_{L_2(\mathbb{R}^\infty, X;\mu)} \le CMNn^{-1/q}.$$

The constants C in the above inequalities are independent of of M, N, n and v.

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References

[1] B. Adcock, S. Brigiapaglia, N. Dexter, and S. Moraga. Near-optimal learning of Banach-valued, high-dimensional functions via deep neural networks. *arXiv e-preprint*, arXiv:2211.12633 [math.NA], 2024.

[2] I. Babuška, F. Nobile, R. Tempone, and C. Webster. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Num. Anal.*, 45:1005–1034, 2007.

[3] M. Bachmayr, A. Cohen, D. Dũng, and C. Schwab. Fully discrete approximation of parametric and stochatic elliptic PDEs. *SIAM J. Numer. Anal.*, 55:2151–2186, 2017.

[4] M. Bachmayr, A. Cohen, R. DeVore, and G. Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. Part II: lognormal coefficients. *ESAIM Math. Model. Numer. Anal.*, 51:341–363, 2017.

[5] M. Bachmayr, A. Cohen, and G. Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. Part I: affine coefficients. *ESAIM Math. Model. Numer. Anal.*, 51:321–339, 2017.

[6] F. Bartel, M. Schäfer, and T. Ullrich. Constructive subsampling of finite frames with applications in optimal function recovery. *Applied and Computational Harmonic Analysis*, 65:209–248, July 2023.

[7] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Models Methods Appl. Sci.*, 22:1250023, 2012.

[8] A. Chkifa, A. Cohen, G. Migliorati, F. Nobile, and R. Tempone. Discrete least squares polynomial approximation withrandom evaluations application to parametric andstochastic elliptic PDEs. *ESAIM Math. Model. and Numer. Analysis*, 49:815–837, 2015.

[9] A. Chkifa, A. Cohen, and C. Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs. *Found. Comput. Math.*, 14(4):601–633, 2013.

[10] A. Chkifa, A. Cohen, and C. Schwab. Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs. *J. Math. Pures Appl.*, 103:400–428., 2015.

[11] A. Cohen and R. DeVore. Approximation of high-dimensional parametric PDEs. *Acta Numer.*, 24:1–159, 2015.

[12] A. Cohen, R. DeVore, and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. *Anal. Appl.*, 9:11–47, 2011.

[13] D. L. Cohn. *Measure Theory: Second Edition.* Springer New York, 2013.

[14] D. Dũng. Linear collocation approximation for parametric and stochastic elliptic PDEs. *Mat. Sb.*, 210:103–227, 2019.

[15] D. Dũng. Sparse-grid polynomial interpolation approximation and integration for parametric and stochastic elliptic PDEs with lognormal inputs. *ESAIM Math. Model. Numer. Anal.*, 55:1163–1198, 2021.

[16] D. Dũng. Erratum to: "Sparse-grid polynomial interpolation approximation and integration for parametric and stochastic elliptic PDEs with lognormal inputs", [Erratum to: ESAIM: M2AN 55(2021) 1163–1198]. *ESAIM Math. Model. Numer. Anal.*, 57:893–897, 2023.

[17] D. Dũng. Collocation approximation by deep neural ReLU networks for parametric elliptic PDEs with lognormal inputs. *Mat. Sb.*, 214:38–75, 2023 (see also English version arXiv:2111.05504 [math.NA]).

[18] M. Dolbeault, D. Krieg, and M. Ullrich. A sharp upper bound for sampling numbers in L_2. *Appl. Comput. Harmon. Anal.*, 63:113–134, 2023.

[19] D. Dũng, V. Nguyen, C. Schwab, and J. Zech. *Analyticity and Sparsity in Uncertainty Quantification for PDEs with Gaussian Random Field Inputs.* Lecture Notes in Mathematics vol. 2334, Springer, 2023.

[20] O. G. Ernst, B. Sprungk, and L. Tamellini. Convergence of sparse collocation for functions of countably many Gaussian random variables (with application to elliptic PDEs). *SIAM J. Numer. Anal.*, 56:877–905, 2018.

[21] V. Hoang and C. Schwab. N-term Galerkin Wiener chaos approximation rates for elliptic PDEs with lognormal Gaussian random inputs. *Math. Models Methods Appl. Sci.*, 24:797–826, 2014.

[22] L. Kämmerer, T. Ullrich, and T. Volkmer. Worst-case recovery guarantees for least squares approximation using random samples. *Constructive Approximation*, 54(2):295–352, Aug. 2021.

[23] D. Krieg and M. Ullrich. Function values are enough for L_2-approximation: Part II. *Journal of Complexity*, 66:101569, Oct. 2021.

[24] G. Migliorati, F. Nobile, E. von Schwerin, and R. Tempone. Analysis of discrete L^2 projection on polynomial spaces with random evaluations. *Found. Comput. Math.*, 14:419–456, 2014.

[25] F. Nobile, R. Tempone, and C. Webster. A sparse grid stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Num. Anal.*, 46:2309–2345, 2008.

[26] F. Nobile, R. Tempone, and C. Webster. An anisotropic sparse grid stochastic collocation method for elliptic partial differentialequations with random input data. *SIAM J. Num. Anal.*, 46:2411–2442, 2008.

[27] J. Zech, D. Dũng, and C. Schwab. Multilevel approximation of parametric and stochastic PDES. *Math. Models Methods Appl. Sci.*, 29:1753–1817, 2019.

[28] J. Zech and C. Schwab. Convergence rates of high dimensional smolyak quadrature. *ESAIM Math. Model. Numer. Anal.*, 54:1259–307, 2020.