Multipoint Padé approximation for parametric model order reduction

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1 Introduction

Parameterized models appear in a wide range of practical applications. For instance during the
design process for microsystems, complex models will be constructed which include geometric
parameters. For the use in interconnect synthesis parameterized interconnect models are
required.

The usual model reduction allows a remarkable acceleration of time- or frequency-dependent
evaluations. However, the reduced-order systems have the disadvantage that they allow only
a small amount of alternatives in model variations. That is, each modification of the physical
model such as a geometrical variation or changing boundary conditions requires a new model
reduction. To overcome these difficulties it is required to generate systems of reduced order
which contain additional parameters.

We present recently developed techniques for parametric model order reduction based on
Krylov subspace methods.

The parameterized systems can be divided into several categories by the following prop-
erties:

- the number of parameters:
  
  - a single parameter linear system is denoted by

\[
E(s)x = bu \\
y = c^T x,
\]

- a multiparameter linear system by

\[
E(s_1, \ldots, s_\nu)x = bu \\
y = c^T x;
\]

- the dependence on the parameter:

  - linear

\[
E(s) = (sE - A)
\]
polynomial
\[ E(s) = E_0 + \sum_{j=1}^{k} s_j E_j \]

- nonlinear.

We will introduce model reduction approaches for some of these system types mainly following [3] (as noted on the last page).

2 Single parameter system, linear dependence on the parameter

As simplest example we consider the frequency response on the Laplace domain of a linear lumped network:

\[ (sE - A)x = bu, \]
\[ y = c^T x. \]

To reduce the dimension we expand the transfer function by a power series at \( s = 0 \)

\[ h(s) = c^T (sE - A)^{-1}b \]
\[ = -c^T (I - sA^{-1}E)^{-1}A^{-1}b \]
\[ = \mu_0 + s\mu_1 + s^2\mu_2 + \ldots \]

with moments \( \mu_j = -c^T (A^{-1}E)^j A^{-1}b \) for \( j \geq 0 \). Moment matching methods use the sequence of moments to generate a Padé approximation to \( h(s) \). The projection matrices \( W, V \in \mathbb{R}^{n \times q} \) are constructed such that \( \text{colsp}(W), \text{colsp}(V) \) contain unions of the following Krylov spaces:

\[ K_q(A^{-1}E, A^{-1}b) = \text{span}\{A^{-1}b, A^{-1}EA^{-1}b, (A^{-1}E)^2 A^{-1}b, \ldots, (A^{-1}E)^q A^{-1}b\} \]
\[ = \text{colspan}\{\bigcup_{j=0}^{q-1} (A^{-1}E)^j A^{-1}b\}, \]
\[ K_q(E^T A^{-T}, c) = \text{span}\{c, E^T A^{-T}c, (E^T A^{-T})^2 c, \ldots, (E^T A^{-T})^q c\} \]
\[ = \text{colspan}\{\bigcup_{j=0}^{q-1} (E^T A^{-T})^j c\}. \]

The transfer function of the reduced-order system

\[ W^T E(s) V \dot{x} = W^T bu, \]
\[ y = c^T V \dot{x} \]

matches the first \( 2q \) moments of the original one.
3 Two-parameter system, linear dependence on the parameters

Consider a two-parameter system with linear dependence on the parameters

\[(s_1E_1 + s_2E_2 - A)x = bu,\]
\[y = c^T x.\]

The transfer function is expanded at the expansion frequencies \(\sigma_1, \sigma_2\) as follows, where we apply a shift of coordinates, the Neumann series, and the binomial theorem

\[h(s_1 + \sigma_1, s_2 + \sigma_2) = c^T(s_1E_1 + s_2E_2 - \underbrace{(A - \sigma_1E_1 - \sigma_2E_2)}_{=: P})^{-1}b\]
\[= c^T[P(s_1P^{-1}E_1 - s_2P^{-1}E_2 - I)]^{-1}b\]
\[= -c^T(I - (s_1P^{-1}E_1 - s_2P^{-1}E_2))^{-1}P^{-1}b\]
\[= -c^T\sum_{j=0}^{\infty}(s_1P^{-1}E_1 - s_2P^{-1}E_2)^jP^{-1}b\]
\[= -c^T\sum_{j=0}^{\infty}\sum_{k=0}^{j} F^j_k(P^{-1}E_1, P^{-1}E_2)s_1^{j-k}s_2^kP^{-1}b,\]

where the scalar values \(-c^TF^j_k(P^{-1}E_1, P^{-1}E_2)P^{-1}b\) are moments of the two-parameter generalization of the transfer function. They are defined as follows:

\[F^j_k(P^{-1}E_1, P^{-1}E_2) = \begin{cases} 
I, & k = j = 0, \\
-P^{-1}E_1F^j_k - P^{-1}E_2F^{j-1}_k, & 0 \leq k \leq j, \\
0, & \text{otherwise}. 
\end{cases}\]

The goal in model reduction based on Krylov subspace methods is to produce projection matrices \(W, V \in \mathbb{R}^{n \times q}\) such that the transfer function of the reduced order system

\[W^T(s_1E_1 + s_2E_2 - A)V\hat{x} = W^Tbu,\]
\[y = c^TV\hat{x}\]

matches some of the moments of the original transfer function. The projection matrices are described as follows:

\[\text{colspan}\{V\} \supset \text{colspan}\left\{\bigcup_{j=0}^{q_b} \left(\sum_{k=0}^{j} F^j_k(P^{-1}E_1, P^{-1}E_2)P^{-1}b\right)\right\}\]  \(=\) \(W_{q_b}(P^{-1}E_1, P^{-1}E_2, P^{-1}b)\);  
\[\text{colspan}\{W\} \supset \text{colspan}\left\{\bigcup_{j=0}^{q_c} \left(\sum_{k=0}^{j} F^j_k(P^{-T}E_1, P^{-T}E_2)P^{-T}c\right)\right\}\]  \(=\) \(W_{q_c}(P^{-T}E_1, P^{-T}E_2, P^{-T}c)\).

**Theorem 3.1.**

The reduced order model constructed with \(V\) and \(W\) given by (1) and (2) is a system where the transfer function is a Padé approximant which matches all the \(q_b + q_c + 1\) moments of the original one.
4 Multiparameter system, linear dependence on the parameters

We extend the approach of the last section to the following multiparameter system:

\[
E(s_1, \ldots, s_\nu)x = bu, \\
y = c^T x.
\]  
(3)

If the dependence on the parameters in nonlinear we generate a power series expansion (for instance use a truncated \(\nu\)-variables Taylor series expansion) to obtain a linear dependence on \(s_1, s_2, \ldots, s_\nu\). Rewrite system (3) as linearly parameterized model

\[
[E_0 + \tilde{s}_1 E_1 + \cdots + \tilde{s}_p E_p]x = bu, \\
y = c^T x.
\]  
(4)

With

\[
M_i = -\bar{E}_0^{-1}\bar{E}_i \quad \text{for} \quad i = 1, \ldots, p, \\
b_M = \bar{E}_0^{-1}b,
\]

we write the system (4) as

\[
[I - (\tilde{s}_1 \bar{M}_1 + \cdots + \tilde{s}_p \bar{M}_p)]x = b_M u, \\
y = c^T x.
\]

Using the Neumann series expansion and the multinomial theorem, the state \(x\) is given by

\[
x = \sum_{j=0}^{\infty} \sum_{k_2=0}^{j-(k_3+\cdots+k_p)} \cdots \sum_{k_p=0}^{j-k_p} [F_{k_2,\ldots,k_p}(M_1,\ldots,M_p)b_Mu]s_1^{j-(k_2+\cdots+k_p)}s_2^{k_2} \cdots s_p^{k_p}.
\]

The coefficients in this formula can be calculated by the following recursive definition

\[
F_{k_2,\ldots,k_p}(M_1,\ldots,M_p) = \begin{cases} 
I, & j = 0, \\
M_1 F_{k_2,\ldots,k_p}^{j-1} + M_2 F_{k_2-1,\ldots,k_p}^{j-1} + \cdots + M_p F_{k_2,\ldots,k_p-1}^{j-1}, & 0 \leq k_2,\ldots,k_p \leq j \text{ or} \\
0, & \text{otherwise}.
\end{cases}
\]

Based on this definition we construct a matrix \(V \in \mathbb{R}^{n \times q}\) such that

\[
\text{colspan}\{V\} = \text{span}\{b_M, M_1 b_M, M_2 b_M, \ldots, M_p b_M, M_1^2 b_M, (M_1 M_2 + M_2 M_1) b_M, \ldots, (M_1 M_p + M_p M_1) b_M, M_2^2 b_M, (M_2 M_3 + M_3 M_2) b_M, \ldots\}
\]

\[
\cup \bigcup_{j=0}^{j_g} \bigcup_{k_2=0}^{j-(k_3+\cdots+k_p)} \cdots \bigcup_{k_p=0}^{j-k_p} F_{k_2,\ldots,k_p}^j(M_1,\ldots,M_p)b_M\}.
\]
holds. Then we compute a reduced-order model by projection with the projection matrix $V$:

$$
[V^T \hat{E}_0 V + \tilde{s}_1 V^T \hat{E}_1 V + \cdots + \tilde{s}_p V^T \hat{E}_p V] \dot{x} = V^T b u,
$$

$$
y = c^T V \dot{x}.
$$

(5)

**Theorem 4.1.** Parameterized Model Order Reduction Moment Matching Theorem

The first $q$ moments (corresponding to the first $j_q$ derivatives in each parameter) of the transfer function for the reduced-order model (5) constructed using $q$ columns of the orthogonal projection matrix $V \in \mathbb{R}^{n \times q}$ match the first $q$ moments of the transfer function of the original system (4).

5 Single parameter system, polynomial dependence on the parameter

We consider a single parameter system with polynomial dependence on the parameter

$$
\sum_{i=0}^{p} s^i E_i x = b u,
$$

$$
y = c^T x.
$$

It is possible to convert the system to a $p$-parameter system (as (4)) with linear dependence on the newly introduced parameters $s_m = s^m$:

$$
E(s) \approx E_0 + s_1 E_1 + s_2 E_2 + \cdots + s_p E_p.
$$

Then the dimension of the system can be reduced by the approach explained in Section 4. Instead of introducing new parameters we include new fictitious states [3]

$$
x_0 = x, \quad x_1 = sx_0, \quad x_2 = sx_1, \ldots
$$

in the transformed system

$$
[I - (s \tilde{M}_1 + \cdots + s^p \tilde{M}_p)] x = b_M u,
$$

$$
y = c^T x,
$$

using the notations

$$
\tilde{M}_i = -E_0^{-1} E_i, \quad \text{for } i = 1, \ldots, p,
$$

$$
b_M = E_0^{-1} b.
$$

The system is rewritten as a single parameter system

$$
\begin{bmatrix}
\begin{pmatrix}
\tilde{M}_1 & \tilde{M}_2 & \cdots & \tilde{M}_p
\end{pmatrix}
\end{bmatrix}
\begin{bmatrix}
\begin{pmatrix}
x_0 \\
x_1 \\
\vdots \\
x_{p-1}
\end{pmatrix}
\end{bmatrix} =
\begin{pmatrix}
b_M \\
0 \\
\vdots \\
0
\end{pmatrix} u.
$$
We reduce the dimension to $q$ by choosing the projection matrices as

$$\text{colspan}\{V\} \supseteq \text{span}\{b_M, \tilde{M}_1 b_M, \tilde{M}_2 b_M + \tilde{M}_1^2 b_M, \ldots, \sum_{k=0}^{q-2} \tilde{M}_{q-1-k} f_k\}. $$

The reduced-order model is computed as follows:

$$\left( \sum_{i=0}^{p} V^T E_i V s^p \right) \dot{x} = V^T b u, $$

$$y = c^T V \dot{x}. $$

References


