Krylov Subspace Methods for Model Order Reduction of Bilinear Discrete-Time Control Systems

Peter Benner†,*, Tobias Breiten‡,**, and Tobias Damm§,***

† Mathematik in Industrie und Technik, Fakultät für Mathematik, TU Chemnitz
‡ Technomathematics Group, Department of Mathematics, TU Kaiserslautern

In this paper, we propose a projection technique for model order reduction of discrete-time bilinear control systems based on the concept of so-called multimoments. We will make use of an explicit solution formula of the system and consider its Z-transform which allows us to characterize the system output by generalized transfer functions.

Copyright line will be provided by the publisher

1 Discrete-time bilinear systems

Many industrial and biological processes can be described by means of systems of ordinary differential equations. Since complex dynamics naturally come along with large-scale system dimensions, one is interested in finding reduction order systems that still accurately describe the original input-output behaviours. While this problem of model order reduction is well-studied for the linear case, theory is certainly less developed for nonlinear dynamics. Here, as a special class of nonlinear systems, we consider discrete-time bilinear systems, which are given by

\[ \Sigma : \begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{m} N_i x(k) u_i(k) + Bu(k), \\ y(k) = C x(k), \quad x(0) = x_0, \end{cases} \]

where \( A, N_i \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, u(k) \in \mathbb{R}^m, y(k) \in \mathbb{R}^p \). For simplicity, throughout this paper we stick to the SISO case (\( m = p = 1 \)) and impose a zero initial condition on the system, i.e. \( x_0 = 0 \). Generalizations to MIMO or \( x_0 \neq 0 \) can be obtained analogously. The following lemma from [1] provides an explicit expression for the solution of \( \Sigma \).

Lemma 1.1 The solution of a SISO discrete-time bilinear control system \( \Sigma \) can be expressed as

\[
x(k) = \sum_{p=1}^{k} \sum_{i_p=0}^{k-p} \sum_{j_1=0}^{i_p-1} \sum_{j_2=0}^{j_1-1} \ldots \sum_{j_r=0}^{j_{r-1}-1} A^{k-i_p-1} N A^{i_p-1} \cdots N A^{i_2-1} N A^{i_1-1} B u(i_1) \cdots u(i_p).
\]

Obviously, multiplying the solution vector with \( C \) from the left leads to an expression of the system output \( y \) as well. Thus, we obtain a representation of the system output which is determined by a sum of what is called degree-\( p \) kernels \( C A^{k-i_p-1} N A^{i_p-1} \cdots NA^{j_1-1} B \). Similarly to the continuous-time case, one might rewrite the last term by a substitution \( j_p := k - i_p - 1, j_r = i_{r+1} - i_r - 1, r < p \), leading to

\[
y(k) = \sum_{p=1}^{k} \sum_{j_p=0}^{k-p-j_p} \sum_{j_1=0}^{j_p} \ldots \sum_{j_r=0}^{j_{r-1}} \left( \frac{CA^{j_p}NA^{j_p-1} \cdots NA^{j_1-1}B u(k - j_p - \cdots - j_1 - p)}{h(j_1, \ldots, j_p)} \right) \cdots u(k - j_p - 1).
\]

Finally, the result of a multivariable Z-transform of \( h(j_1, \ldots, j_p) \) can be interpreted as the \( p \)-th transfer function of the bilinear system \( \Sigma \), i.e.

\[
H(z_1, \ldots, z_p) = C (z_p I - A)^{-1} N (z_{p-1} I - A)^{-1} \cdots N (z I - A)^{-1} B.
\]

2 Multimoment-matching and generalized rational interpolation

A common approach to characterize these transfer functions is based on the Neumann series and its expansion in a multivariable Taylor series.

* E-mail: benner@mathematik.tu-chemnitz.de, Phone: +49 371 531 22540, Fax: +49 371 531 22509
** Corresponding author E-mail: tobias.breiten@mathematik.tu-chemnitz.de, Phone: +49 371 531 38584, Fax: +49 371 531 22509
*** E-mail: dann@mathematik.uni-kl.de, Phone: +49 631 205 4489, Fax: +49 631 205 4986

Copyright line will be provided by the publisher
Depending on the specifically chosen expansion points $\sigma_j$, we obtain the following expressions

$$
\sigma_j = \infty: \ H(z_1, \ldots, z_k) = \sum_{l_1=1}^{\infty} \cdots \sum_{l_k=1}^{\infty} \frac{T A^{l_1-1} N \cdots A^{l_k-1} b z_1^{-l_1} \cdots z_k^{-l_k}}{m_{\infty}(l_1, \ldots, l_k)}
$$

$$
\sigma_j \neq \infty: \ H(z_1, \ldots, z_k) = \sum_{l_1=1}^{\infty} \cdots \sum_{l_k=1}^{\infty} \frac{(-1)^k c^T (A - \sigma_k I)^{-l_1} N \cdots (A - \sigma_k I)^{-l_k} (A - \sigma_k I)^{-l_1} b z_1^{l_1-1} \cdots z_k^{l_k-1}}{m_{\infty}(l_1, \ldots, l_k)},
$$

where $m(l_1, \ldots, l_k)$ denote the so-called multimoments of the $k$-th transfer function. Let $A_k := (A - \sigma I)^{-1}$. Using a sequence of Krylov subspaces, we can construct a reduced model that matches a specified number of multimoments, e.g. [2]:

**Theorem 2.1** Given $\Sigma$, Construct $\hat{\Sigma}$ by projection $P = VW^T$, with $V$ as basis of the union of the Krylov subspaces

$$
\sigma_j \neq \infty: \ \text{span}\{V^{(1)}\} = K_q (A_{\sigma_1}, A_{\sigma_2}, b), \quad \sigma_j = \infty: \ \text{span}\{V^{(1)}\} = K_q (A, b),
$$

$$
\text{span}\{V^{(k)}\} = K_q \left( A_{\sigma_k}, A_{\sigma_k} NV^{(k-1)} \right), k \leq r. \quad \text{span}\{V^{(k)}\} = K_q (A, NV^{(k-1)}), k \leq r.
$$

Then $m(l_1, \ldots, l_k) = \hat{m}(l_1, \ldots, l_k)$ for $k = 1, \ldots, r$ and $l_1, \ldots, l_k = 1, \ldots, q$.

Note that the orders of the Krylov spaces as well as the number of the starting vectors may be varied in order to match only a selection of multimoments. The previous result also generalizes the common approach $\sigma_j = 0$ proposed in the existing literature, see [3,4]. Furthermore, we can improve the approximation quality by using two-sided projection methods and a set of interpolation points which will lead to a generalized interpolation of the transfer functions over a broader frequency range.

### 3 Numerical results

We have tested different reduction approaches on a large-scale discrete-time bilinear system resulting from a spatial (finite differences) and time (semi-implicit Euler) discretization of the heat equation subject to mixed Dirichlet and Robin boundary conditions. For a more detailed derivation of the model, we refer to [5]. The reduced models were constructed by two-sided projection methods that lead to an approximation of the first four transfer functions of the original system. While we were not able to generate reasonable approximations with $\sigma_j = 0$ at all, we obtained nice results with $\sigma_j = \infty$, which were again outperformed by a generalized rational interpolation at the optimal linear points obtained by IRKA (see [6]).

### References