The Natural Banach Space for Version Independent Risk Measures

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Abstract

Risk measures, or coherent measures of risk are often considered on the space L^∞, and important theorems on risk measures build on that space. Other risk measures, among them the most important risk measure—the Average Value-at-Risk—are well defined on the larger space L^1 and this seems to be the natural domain space for this risk measure. Spectral risk measures constitute a further class of risk measures of central importance, and they are often considered on some L^p space. But in many situations this is possibly unnatural, because any L^p with $p > p_0$, say, is suitable to define the spectral risk measure as well. In addition to that risk measures have also been considered on Orlicz and Zygmund spaces. So it remains for discussion and clarification, what the natural domain to consider a risk measure is?

This paper introduces a norm, which is built from the risk measure, and a new Banach space, which carries the risk measure in a natural way. It is often strictly larger than its original domain, and obeys the key property that the risk measure is finite valued and continuous on that space in an elementary and natural way.

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Classification: 90C15, 60B05, 62P05

1 Introduction

This paper addresses coherent measures of risk (risk measures, for short) and the natural domain (the natural space), where they can be considered. Coherent measures of risk have been introduced in the seminal paper [4] in an axiomatic way and have been investigated in a series of subsequent papers in mathematical finance since then. In the actuarial literature, however, risk measures and axiomatic treatments have been considered already earlier, for example in Denneberg [10] and in this journal by Wang et al. [27].

We shall assume throughout the paper that the risk measure ρ is a function mapping $\mathbb{R}$-valued random variables into the real numbers $\mathbb{R}$ or to $+\infty$, which satisfies the following axioms (cf. [5]):

(M) Monotonicity: $\rho(Y_1) \le \rho(Y_2)$ whenever $Y_1 \le Y_2$ almost surely;

(H) Positive homogeneity: $\rho(\lambda Y) = \lambda\rho(Y)$ whenever $\lambda > 0$;

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(C) Convexity: $\rho\left((1-\lambda)Y_0+\lambda Y_1\right)\leq(1-\lambda)\,\rho\left(Y_0\right)+\lambda\rho\left(Y_1\right)$ for $0\leq\lambda\leq 1$;

(T) Translation equivariance[1]: $\rho\left(Y+c\right)=\rho\left(Y\right)+c$ if $c\in\mathbb{R}$.

Here, the initial axioms have been adapted to follow the interpretation of loss instead of profit—the common modification in insurance—in the usual and appropriate way.

The main observation in this paper starts with the fact that the risk measure ρ can be associated in a natural way with a seminorm, which is a norm in important cases. It is an elementary property that the risk measure is continuous with respect to the norm introduced.

We investigate this new norm for specific risk measures, starting with spectral risk measures. It turns out that the domain, where the spectral risk measure can be defined in a meaningful way, is always strictly larger than L^∞. As an initial example consider the simple expectation, $\rho\left(\cdot\right)=\mathbb{E}\left(\cdot\right)$, which satisfies the axioms (M)–(T): ρ is naturally defined on the much larger space of integrable random variables, $L^1\supsetneqq L^\infty$.

In general, the respective domain is a Banach space. We study its topology, which can be compared with spaces of measurable random variables having a finite p^{th} moment,

$$L^p=\left\{Y\in L^0:\,\|Y\|_p<\infty\right\},$$

where $\|Y\|_p=\left(\mathbb{E}\left|Y\right|^p\right)^{1/p}$ is the norm. However, the topology of the new Banach space always differs from the topology of an L^p space.

A risk measure ρ—being a convex function—has a convex conjugate function, and the Fenchel–Moreau theorem allows recovering the initial function, the initial risk measure ρ in our situation. The convex conjugate function involves the dual of the initial space, for this reason it is essential to understand the dual of the Banach space associated with the risk measure. The norm on the dual space measures the growth of the random variable by involving second order stochastic dominance relations.

It is elaborated moreover in this paper that a risk measure cannot be defined in a meaningful way on a space larger than L^1 (cf. also [13]).

The domain and the co-domain of spectral risk measures

The axioms characterizing risk measures have been stated above without giving the domain and the co-domain precisely. Indeed, important results are well known when considering ρ as a function on L^∞, $\rho:\,L^\infty\to\mathbb{R}$: the results include Kusuoka's representation (cf. [18] and (3) below) and results on continuity. We state the following example.

Proposition 1. *Every $\mathbb{R}$-valued risk measure ρ on L^∞ is Lipschitz-continuous with constant 1, it satisfies $\left|\rho\left(Y_2\right)-\rho\left(Y_1\right)\right|\leq\|Y_2-Y_1\|_\infty$.*

Proof. See, e.g., [14, Lemma 4.3] for a proof. □

In many situations, for example when considering the trivial risk measure $\rho\left(\cdot\right):=\mathbb{E}\left(\cdot\right)$ or the Average Value-at-Risk, the domain L^∞ is not satisfactory large enough, the domain L^1 is perhaps more natural and convenient to consider in this situation.

[1]In an economic or monetary environment this is often called Cash invariance instead.

Depending on the domain chosen for a risk measure, the co-domain is often specified to be $\mathbb{R}$, or the extended reals $\mathbb{R} \cup \{\infty\}$, in some publications even $\mathbb{R} \cup \{\infty, -\infty\}$. In this context it should be emphasized that there is an intimate relationship between the properties *continuity* of a risk measure and its *range*, the following important result clarifies the connections:

Proposition 2. *Consider a $\mathbb{R} \cup \{\infty\}$-valued risk measure ρ defined on L^p, $1 \leq p < \infty$, satisfying (M), (C) and (T). Suppose further that $\{\rho < \infty\}$ has a nonempty interior. Then ρ is finite valued and continuous on the entire L^p.*

The proof is contained in [23] and in [24], Proposition 6.7. The preceding discussion of the latter reference also contains the following reformulation of the statement, which is more striking: A risk measure satisfying (M), (C) and (T) is either finite valued and continuous on the *entire* L^p, or it takes the value $+\infty$ on a dense subset.

Both results suggest to consider $\mathbb{R}$ (i.e., $\mathbb{R}\setminus\{\pm\infty\}$) valued risk measures solely, because these are precisely the finite valued and continuous risk measures.

Outline of the paper: The following Section 2 introduces the associated norm and elaborates its elementary property. The subsequent section, Section 3, addresses an elementary risk measure, the spectral risk measure. This risk measure is elementary, as every version independent risk measure can be built from spectral risk measures.

A space is introduced, called the space of *natural domain*, which is as large as possible to carry a spectral risk measure. It is verified that the associated space is a Banach space. The new norm can be used in a natural way to extend the domain of elementary risk measures, and it is elaborated which L^p spaces the space of natural domain comprises.

This section contains moreover the remarkable result, that there is no finite valued risk measure on a space larger than L^1.

We study further the topological dual of the Banach space introduced (Section 5). It turns out the dual norm can be characterized by use of the Average Value-at-Risk, the simplest risk measure, and by second order stochastic dominance. The investigations are pushed further to more general risk measures, and an even more general Banach space to carry a general risk measure is highlighted in Section 6.

2 The norm associated with a risk measure

The results presented in this paper start along with the observation that a risk measure ρ induces a (semi-)norm in the following elementary way.

Definition 3. Let L be a vector space of $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{F}, P)$ and $\rho : L \to \mathbb{R} \cup \{-\infty, \infty\}$ be a risk measure. Then

$$\|\cdot\|_\rho := \rho\left(|\cdot|\right)$$

is called *associated norm*, associated with the risk measure ρ.

Remark 4. If no confusion may occur we shall simply write $\|\cdot\|$ to refer to $\|\cdot\|_\rho$.

The following proposition verifies that $\|\cdot\|_\rho$ is indeed a seminorm on the appropriate vector space and thus justifies the notation.

Proposition 5 (Finiteness, and the seminorm property)**.** *Let ρ be a risk measure on a vector space of $\mathbb{R}$-valued random variables. Then $\|\cdot\| = \rho(|\cdot|)$ is a seminorm on $L := \{Y : \rho(|Y|) < \infty\}$ and ρ is finite valued on L.*

Proof. We show first that that ρ is $\mathbb{R}$-valued on $L = \{Y : \rho(|Y|) < \infty\}$. For this observe that $Y \le |Y|$, and by monotonicity thus $\rho(Y) \le \rho(|Y|) = \|Y\|$. Moreover it holds that $\rho(0) = 0$ [2] and thus

$$0 = 2 \cdot \rho\left(\frac{1}{2}Y + \frac{1}{2}(-Y)\right) \le 2 \cdot \left(\frac{1}{2}\rho(Y) + \frac{1}{2}\rho(-Y)\right) = \rho(Y) + \rho(-Y),$$

such that $-\rho(Y) \le \rho(-Y)$. Now $-Y \le |Y|$ and, again by monotonicity, $-\rho(Y) \le \rho(-Y) \le \rho(|Y|) = \|Y\|$. Summarizing thus $|\rho(Y)| \le \|Y\|$, such that ρ is finite valued on L.

Note that

$$\|\lambda \cdot Y\| = \rho(|\lambda \cdot Y|) = \rho(|\lambda| \cdot |Y|) = |\lambda| \cdot \rho(|Y|) = |\lambda| \cdot \|Y\|,$$

and $\|\cdot\|$ thus is positively homogeneous.

Next it follows from monotonicity, positive homogeneity and convexity that

$$\begin{aligned}\|Y_1 + Y_2\| =& \rho(|Y_1 + Y_2|) \le \rho(|Y_1| + |Y_2|) = 2 \cdot \rho\left(\frac{1}{2}|Y_1| + \frac{1}{2}|Y_2|\right)\\ &\le 2 \cdot \left(\frac{1}{2}\rho(|Y_1|) + \frac{1}{2}\rho(|Y_2|)\right) = \rho(|Y_1|) + \rho(|Y_2|)\\ &= \|Y_1\| + \|Y_2\|,\end{aligned}$$

and this is the triangle inequality. □

The next proposition elaborates that the risk measure is continuous with respect to its associated norm. This consistency result on continuity generalizes Proposition 1.

Proposition 6 (Continuity)**.** *Let ρ be a risk measure, defined on a vector space of $\mathbb{R}$-valued random variables. Then ρ is Lipschitz continuous with constant 1 with respect to the seminorm $\|\cdot\| = \rho(|\cdot|)$.*

Proof. As for continuity note that

$$\begin{aligned}\rho(Y_2) &= 2 \cdot \rho\left(\frac{1}{2}Y_1 + \frac{1}{2}(Y_2 - Y_1)\right)\\ &\le 2\left(\frac{1}{2}\rho(Y_1) + \frac{1}{2}\rho(Y_2 - Y_1)\right) \le \rho(Y_1) + \rho(|Y_2 - Y_1|)\end{aligned}$$

by convexity and monotonicity. It follows that $\rho(Y_2) - \rho(Y_1) \le \|Y_2 - Y_1\|$. Interchanging the roles of Y_1 and Y_2 reveals that

$$|\rho(Y_2) - \rho(Y_1)| \le \|Y_2 - Y_1\|,$$

the assertion. To accept that the Lipschitz constant 1 cannot be improved consider the particular choices $Y_1 := 0$ and $Y_2 := \mathbb{1}$ in view of translation equivariance (T). □

[2] Otherwise, $\rho(0) = \rho(2 \cdot 0) = 2 \cdot \rho(0)$ would imply $1 = 2$, a contradiction.

3 Spectral risk measures

Among initial attempts to introduce premium principles to price insurance contracts are distorted probabilities: a concept which can be found nowadays under different names as distorted acceptability functionals (cf. [19]), distortion risk measures, under the more suggestive term weighted Value-at-Risk (cf. [8]) or the term we use here, spectral risk measures (cf. [1] and [2]). This risk measure constitutes a basic risk measure in the sense that every risk measure can be built of spectral risk measures, what is the content of Kusuoka's famous theorem (Theorem 7 below). Spectral risk measures are employed in insurance pricing as they incorporate a safety margin by overvaluing high losses and undervaluing small losses in exchange.

The spectral risk measure involves the Value-at-Risk at level p,

$$\mathsf{V@R}_p(Y) := F_Y^{-1}(p) := \inf\{y:\ P(Y \le y) \ge p\},$$

which is the left-continuous, lower semi-continuous (lsc.) *quantile*; the spectral risk measure (or weighted $\mathsf{V@R}$) then is the functional

$$\rho_\sigma(Y) := \int_0^1 \sigma(u)\, \mathsf{V@R}_u(Y)\, \mathrm{d}u, \tag{1}$$

mapping a random variable Y to a real number, if the integral exists.

The function $\sigma: [0,1] \to \mathbb{R}_0^+$, called the *spectrum* or *spectral function*, is a weight function. To build a reasonable premium principle the function σ should obey some properties to be consistent with the axioms imposed on risk measures: first, associating Y with loss, σ should evaluate to nonnegative reals, $\mathbb{R}_0^+$. Higher losses should be weighted higher, thus σ should be nondecreasing. And finally, as σ represents a weight function, it is natural to request $\int_0^1 \sigma(u)\, \mathrm{d}u = 1$.

An important, elementary spectral risk measure satisfying all axioms above is the Average Value-at-Risk, which is specified by the spectral function

$$\sigma_\alpha(u) := \begin{cases} 0 & \text{if } u < \alpha \\ \frac{1}{1-\alpha} & \text{else,} \end{cases}$$

that is

$$\mathsf{AV@R}_\alpha(Y) := \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_u(Y)\,\mathrm{d}u \qquad (\alpha < 1), \tag{2}$$

and for $\alpha = 1$ the Average Value-at-Risk per definition is

$$\mathsf{AV@R}_1(Y) := \lim_{\alpha \nearrow 1} \mathsf{AV@R}_\alpha(Y) = \operatorname{ess\,sup} Y \qquad (\alpha = 1).$$

The domain of spectral risk measures

It is obvious that the Average Value-at-Risk ($\alpha < 1$) may be well defined on L^1, with the result that

$$|\mathsf{AV@R}_\alpha(Y)| \le \frac{1}{1-\alpha}\mathbb{E}\,|Y| = \frac{1}{1-\alpha}\,\|Y\|_1 < \infty \qquad \left(Y \in L^1\right),$$

that means that $\mathsf{AV@R}_\alpha$ is finite valued whenever $Y \in L^1$. This is not the case, however, for $\alpha = 1$: a restriction to the smaller space $L^\infty \subset L^1$ is necessary in order to ensure that $\mathsf{AV@R}_1$ is finite valued,

$$|\mathsf{AV@R}_1(Y)| \le \|Y\|_\infty < \infty \qquad (Y \in L^\infty).$$

Even more peculiarities appear when considering the spectral function $\sigma(u) := \frac{1}{2\sqrt{1-u}}$. Clearly, $\sigma \in L^q$ whenever $q < 2$, but $\sigma \notin L^2$. Hölder's inequality can be employed to insure that ρ_σ is finite valued on L^p ($p > 2$, $\frac{1}{q} + \frac{1}{p} = 1$), because

$$|\rho_\sigma(Y)| \leq \|\sigma\|_q \cdot \left(\int_0^1 F_Y^{-1}(u)^p\right)^{\frac{1}{p}} = \frac{1}{2}\left(\frac{2}{2-q}\right)^{\frac{1}{q}} \cdot \|Y\|_p,$$

and the constant $\frac{1}{2}\left(\frac{2}{2-q}\right)^{\frac{1}{q}}$ again exceeds every finite bound whenever q approaches 2 from below.

So what is a good space to consider ρ_σ? Any L^p ($p > 2$) guarantees that ρ_σ is finite valued and continuous, but L^2 is obviously too large. The naïve choice $\bigcup_{p>2} L^p$ does not have a satisfying norm, or topology neither. (See, for different configurations, [6, 7].)

Further properties and importance of spectral risk measures

A well known and essential representation of risk measures was elaborated by Kusuoka in [18] (see [17] for the statement presented below). Kusuoka's result considers risk measures on L^∞ which are *version independent* (also: *law invariant*), i.e., which satisfy $\rho(Y) = \rho(Y')$ whenever Y and Y' share the same law, that is if $P(Y \leq y) = P(Y' \leq y)$ for every $y \in \mathbb{R}$.

Theorem 7 (Kusuoka's representation)**.** *A version independent risk measure ρ on L^∞ of an atomless probability space $(\Omega, \mathcal{F}, P)$ has the representation*

$$\rho(Y) = \sup_{\mu \in \mathscr{M}} \int_0^1 \mathsf{AV@R}_\alpha(Y)\; \mu(\mathrm{d}\alpha), \tag{3}$$

where $\mathscr{M}$ is a collection of probability measures on $[0, 1]$.

Kusuoka representation *of* a spectral risk measure. The Kusuoka representation of a spectral risk measure ρ_σ is provided by the probability measure $\mu_\sigma((a, b]) := \int_a^b \mathrm{d}\mu_\sigma(\alpha)$ on $[0, 1]$, where μ_σ is the nondecreasing function

$$\mu_\sigma(p) := (1-p)\,\sigma(p) + \int_0^p \sigma(u)\,\mathrm{d}u \quad (0 \leq p \leq 1), \qquad \mu_\sigma(p) := 0 \quad (p < 0),$$

which satisfies $\mu_\sigma(1) = 1$ and $\mathrm{d}\mu_\sigma(p) = (1-p)\,\mathrm{d}\sigma(p)$. It holds that

$$\rho_\sigma(Y) = \int_0^1 \mathsf{AV@R}_\alpha(Y)\; \mu_\sigma(\mathrm{d}\alpha),$$

which exposes the Kusuoka representation of a spectral risk measure (cf. [25]).

Kusuoka representation *by* spectral risk measures. Conversely, any measure μ (provided that $\mu(\{1\}) = 0$) of the representation (3) can be related to the function

$$\sigma_\mu(\alpha) = \int_0^\alpha \frac{1}{1-u}\,\mu(\mathrm{d}u), \tag{4}$$

and it holds that

$$\int_0^1 \mathsf{AV@R}_\alpha(Y)\ \mu(\mathrm{d}\alpha) = \int_0^1 \sigma_\mu(\alpha)\, \mathsf{V@R}_\alpha(Y)\ \mathrm{d}\alpha = \rho_{\sigma_\mu}(Y),$$

which is a spectral risk measure.

But even the requirement $\mu(\{1\}) = 0$ can be dropped: indeed, there is a set $\mathscr{S}$ of continuous (and thus bounded) spectral functions on [0, 1], such that the relation

$$\rho(Y) = \sup_{\mu\in\mathscr{M}} \int_0^1 \mathsf{AV@R}_\alpha(Y)\ \mu(\mathrm{d}\alpha) = \sup_{\sigma\in\mathscr{S}} \int_0^1 \mathsf{V@R}_\alpha(Y)\, \sigma(\alpha)\ \mathrm{d}\alpha = \sup_{\sigma\in\mathscr{S}} \rho_\sigma(Y) \tag{5}$$

holds (cf. [20]). This again exposes the importance of spectral risk measures, as every version independent risk measure ρ can be built from spectral risk measures by (5).

Recall that Kusuoka's representation builds on the space L^∞. But again it is not clear, if, and to which larger space this risk measure can be extended, because every σ might allow a different domain.

4 The space of natural domain, L_σ

Let σ be a nonnegative, nondecreasing, integrable function with $\int_0^1 \sigma(u)\mathrm{d}u = 1$. For Y a random variable we consider the function

$$\rho_\sigma(Y) = \int_0^1 \sigma(u)\ F_Y^{-1}(u)\ \mathrm{d}u$$

already defined in (1). For $\sigma \in L^1$ (which is a minimal requirement to insure that $\int_0^1 \sigma(u)\mathrm{d}u = 1$), ρ_σ is certainly well defined for $Y \in L^\infty$, but for other random variables the integral possibly diverges. And it might diverge to $+\infty$, to $-\infty$, or be even of the indefinite form $\infty - \infty$. The following definition respects the finiteness of the spectral risk measure in view of Proposition 5.

Definition 8. The *natural domain* corresponding to a spectral risk measure ρ_σ induced by a spectral function σ is

$$L_\sigma := \left\{Y \in L^0 : \|Y\|_\sigma < \infty\right\},$$

where

$$\|Y\|_\sigma := \rho_\sigma(|Y|).$$

Note that $|Y| \geq 0$ is positive, such that $F_{|Y|}^{-1}(\cdot) \geq 0$ is positive as well and the condition $\rho_\sigma(|Y|) < \infty$ makes perfect sense for any measurable random variable $Y \in L^0$.

Remark 9. The seminorm $\|\cdot\|_\sigma$ has the representation

$$\|Y\|_\sigma = \int_0^\infty \tau_\sigma\left(F_{|Y|}(y)\right)\mathrm{d}y$$

in terms of the cdf. $F_{|Y|}$ directly, without involving the inverse $F_{|Y|}^{-1}$ ($\tau_\sigma(\alpha) := \int_\alpha^1 \sigma(u)\mathrm{d}u$).

Proposition 10. $\|\cdot\|_\sigma = \rho_\sigma(|\cdot|)$ *is a norm on* L_σ.

Proof. It was already shown in Proposition 5 that $\|\cdot\|_\sigma$ is a seminorm. What remains to be shown is that $\|\cdot\|_\sigma$ separates points. For this recall that σ is positive, nondecreasing, and satisfies $\int_0^1 \sigma(p)\,\mathrm{d}p = 1$, and $F_{|Y|}(\cdot)$ is a nondecreasing and positive function as well. Hence if $\int_0^1 \sigma(p)\, F_{|Y|}^{-1}(p)\,\mathrm{d}p = 0$, then $F_{|Y|}^{-1}(\cdot) \equiv 0$, that is $Y = 0$ almost everywhere. The function $\|\cdot\|_\sigma$ thus separates points in L_σ and $\|\cdot\|_\sigma$ hence is a norm. $\square$

The next theorem already elaborates that the set L_σ is large enough and at least contains L^p, whenever $\sigma \in L^q$ (and the exponents are conjugate, $\frac{1}{p} + \frac{1}{q} = 1$).

Theorem 11 (Comparison with L^p)**.** *Let σ be fixed.*

(i) If $\sigma \in L^q$ for some $q \in [1,\infty]$ with conjugate exponent p, then

$$L^\infty \subset L^p \subset L_\sigma \subset L^1$$

and

$$\|Y\|_1 \le \|Y\|_\sigma \le \|\sigma\|_q \cdot \|Y\|_p \tag{6}$$

whenever $Y \in L^p$.

(ii) For σ bounded (i.e., $\sigma \in L^\infty$) it holds moreover that $L_\sigma = L^1$, the norms are equivalent and satisfy

$$\|Y\|_1 \le \|Y\|_\sigma \le \|\sigma\|_\infty \cdot \|Y\|_1 .$$

It follows in particular from (ii) that $P(A) \le \|\mathbb{1}_A\|_\sigma \le 1$ for measurable sets A, and $\|Y\|_\sigma = \|Y\|_1$ for the function being constantly 1 ($\sigma = \mathbb{1}$).

Proof. Note that $\int_0^1 \sigma(u)\,\mathrm{d}u = 1$ and $\sigma(\cdot)$ is nondecreasing, hence there is a $\tilde{u} \in (0,1)$ such that $\sigma(u) \le 1$ for $u < \tilde{u}$ and $\sigma(u) \ge 1$ for $u > \tilde{u}$. Note as well that $\int_0^{\tilde{u}} 1 - \sigma(u)\,\mathrm{d}u = \int_{\tilde{u}}^1 \sigma(u) - 1\,\mathrm{d}u$. Then it follows that

$$\begin{aligned}\int_0^{\tilde{u}} (1-\sigma(u))\, F_{|Y|}^{-1}(u)\,\mathrm{d}u &\le \int_0^{\tilde{u}} (1-\sigma(u))\, F_{|Y|}^{-1}(\tilde{u})\,\mathrm{d}u \\ &= \int_{\tilde{u}}^1 (\sigma(u)-1)\, F_{|Y|}^{-1}(\tilde{u})\,\mathrm{d}u \le \int_{\tilde{u}}^1 (\sigma(u)-1)\, F_{|Y|}^{-1}(u)\,\mathrm{d}u,\end{aligned}$$

because $F_{|Y|}^{-1}(\cdot)$ is increasing. After rearranging thus

$$\|Y\|_1 = \mathbb{E}\,|Y| = \int_0^1 F_{|Y|}^{-1}(u)\,\mathrm{d}u \le \int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u = \rho_\sigma(|Y|) = \|Y\|_\sigma ,$$

which is the first assertion. The inclusion $L_\sigma \subset L^1$ is immediate as well, as $\|Y\|_\sigma < \infty$ implies that $\|Y\|_1 < \infty$.

The remaining inequality

$$\|Y\|_\sigma = \int_0^1 F_{|Y|}^{-1}(u)\,\sigma(u)\,\mathrm{d}u \le \left(\int_0^1 \sigma(u)^q\right)^{\frac{1}{q}} \cdot \left(\int_0^1 F_{|Y|}^{-1}(u)^p\right)^{\frac{1}{p}} = \|\sigma\|_q \cdot \left(\mathbb{E}\,|Y|^p\right)^{\frac{1}{p}}$$

is Hölder's inequality. $\square$

Remark 12. The inequality $\|Y\|_1 \leq \|Y\|_\sigma$ is also a direct consequence of Chebyshev's sum inequality in its continuous form, which states that $\int_0^1 f(u)\,\mathrm{d}u \cdot \int_0^1 g(u)\,\mathrm{d}u \leq \int_0^1 f(u)\,g(u)\,\mathrm{d}u$ whenever f and g are both nondecreasing (choose $f=\sigma$ and $g=F_{|Y|}^{-1}$; cf. [15]).

Corollary 13. *For $\sigma \in L^q$ the risk measure ρ_σ is continuous with respect to $\|\cdot\|_p$, it holds that*

$$|\rho_\sigma(Y_2) - \rho_\sigma(Y_1)| \leq \|\sigma\|_q \cdot \|Y_2 - Y_1\|_p .$$

Proof. This is an immediate consequence of Proposition 6 and (6). □

Theorem 14 (Comparability of L_σ-spaces)**.** *Suppose that*

$$c := \sup_{0\leq\alpha<1} \frac{\int_\alpha^1 \sigma_2(u)\,\mathrm{d}u}{\int_\alpha^1 \sigma_1(u)\,\mathrm{d}u} \tag{7}$$

is finite ($c<\infty$), then

$$\|Y\|_{\sigma_2} \leq c\cdot \|Y\|_{\sigma_1} \qquad (Y \in L_{\sigma_1}) \tag{8}$$

and $L_{\sigma_1} \subset L_{\sigma_2}$; c is moreover the smallest constant satisfying (8), the identity

$$\mathrm{id} : \left(L_{\sigma_1}, \|\cdot\|_{\sigma_1}\right) \to \left(L_{\sigma_2}, \|\cdot\|_{\sigma_2}\right)$$

thus is continuous with norm $\|\mathrm{id}\| = c$.

Proof. To accept (8) define the functions $S_i(\alpha) := \int_\alpha^1 \sigma_i(u)\,\mathrm{d}u$ ($i=1,2$), then by Riemann–Stieltjes integration by parts and as $u \mapsto F_{|Y|}^{-1}(u)$ is nondecreasing,

$$\begin{aligned}
\|Y\|_{\sigma_2} &= \int_0^1 F_{|Y|}^{-1}(u)\,\sigma_2(u)\,\mathrm{d}u = -\int_0^1 F_{|Y|}^{-1}(u)\,\mathrm{d}S_2(u) \\
&= -\,F_{|Y|}^{-1}(u)\,S_2(u)\Big|_0^1 + \int_0^1 S_2(u)\,\mathrm{d}F_{|Y|}^{-1}(u) = F_{|Y|}^{-1}(0) + \int_0^1 S_2(u)\,\mathrm{d}F_{|Y|}^{-1}(u) \\
&\leq F_{|Y|}^{-1}(0) + c\cdot \int_0^1 S_1(u)\,\mathrm{d}F_{|Y|}^{-1}(u) \\
&= F_{|Y|}^{-1}(0) + c\cdot F_{|Y|}^{-1}(u)\,S_1(u)\Big|_0^1 - c\cdot \int_0^1 F_{|Y|}^{-1}(u)\,\mathrm{d}S_1(u) \\
&= -F_{|Y|}^{-1}(0)\,(c-1) + c\cdot \int_0^1 F_{|Y|}^{-1}(u)\,\sigma_1(u)\,\mathrm{d}u \leq c\cdot \|Y\|_{\sigma_1} ,
\end{aligned}$$

because $F_{|Y|}^{-1}(0) \geq 0$ and $c \geq 1$ (choose $\alpha = 0$ in (7)).

To accept that c is the smallest constant satisfying (8) just consider the random variable $Y = \mathbb{1}_{A^c}$, for which $\|Y\|_\sigma = \rho_\sigma(\mathbb{1}_{A^c}) = \int_{P(A)}^1 \sigma(u)\,\mathrm{d}u$. The assertion follows, as the measurable set A may be chosen arbitrarily. □

It is a particular consequence of (8) that

$$\mathsf{AV@R}_{\alpha_1}(|Y|) \leq \mathsf{AV@R}_{\alpha_2}(|Y|) \leq \frac{1-\alpha_1}{1-\alpha_2}\mathsf{AV@R}_{\alpha_1}(|Y|),$$

which holds whenever $\alpha_1 \leq \alpha_2 < 1$. It should be noted, however, that $\mathsf{AV@R}_{\alpha_1}(Y) \leq \mathsf{AV@R}_{\alpha_2}(Y) \not\leq \frac{1-\alpha_1}{1-\alpha_2}\mathsf{AV@R}_{\alpha_1}(Y)$ for general, not necessarily positive random variables Y.

The following representation result for spectral risk measures is well known for σ in an appropriate space. We extend it to L_σ, the result will be used in the sequel.

Proposition 15 (Representation of the spectral risk measure). *ρ_σ has the equivalent representation* [3]

$$\rho_\sigma(Y) = \sup\left\{\mathbb{E}\, Y \cdot \sigma(U) : U \text{ is uniformly distributed}\right\} \tag{9}$$

on L_σ.

Remark 16. For the Average Value-at-Risk it holds in particular that

$$\mathsf{AV@R}_\alpha(Y) = \sup\left\{\mathbb{E}\, Y \cdot Z : \mathbb{E}\, Z = 1,\, 0 \leq Z \leq \frac{1}{1-\alpha}\right\} \tag{10}$$

in view of the spectral function (2).

Proof. Consider the random variable $Z = \sigma(U)$ for a uniformly distributed random variable U, then $P(Z \leq \sigma(\alpha)) = P(\sigma(U) \leq \sigma(\alpha)) \geq P(U \leq \alpha) = \alpha$, that is $\mathsf{V@R}_\alpha(Z) \geq \sigma(\alpha)$. But as $1 = \int_0^1 \sigma(\alpha)\,\mathrm{d}\alpha \leq \int_0^1 \mathsf{V@R}_\alpha(\sigma(U))\,\mathrm{d}\alpha = \mathbb{E}\,\sigma(U) = \int_0^1 \sigma(p)\,\mathrm{d}p = 1$ it follows that

$$\mathsf{V@R}_\alpha(Z) = \sigma(\alpha). \tag{11}$$

Now $F_Y^{-1}(\cdot)$ is an increasing function, and so is $\sigma(\cdot)$. By the Hardy–Littlewood rearrangement inequality (cf. [16] and [19, Proposition 1.8] for the respective rearrangement inequality, sometimes also referred to as *Hardy–Littlewood–Pólya inequality*, cf. [9]) it follows thus that

$$\mathbb{E}\, Y \cdot \sigma(U) \leq \int_0^1 F_Y^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha.$$

However, if Y and U are coupled in a co-monotone way, then equality is attained, that is $\mathbb{E}\, Y \cdot \sigma(U) = \int_0^1 F_Y^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha$. This proves the statement in view of the definition of the spectral risk measure, (1). $\square$

The next theorem demonstrates that the spaces L_σ really add something to L^p spaces, the space L_σ is *strictly larger* than L^p.

Theorem 17 (L_σ is larger than L^p). *The following hold true:*

(i) Suppose that $\sigma \in L^q$ for some $1 \leq q < \infty$. Then the space of natural domain L_σ is strictly larger than L^p, $L^p \subsetneqq L_\sigma$ ($\frac{1}{p} + \frac{1}{q} = 1$).

(ii) In particular the space of natural domain L_σ is (always) strictly larger than L^∞, $L^\infty \subsetneqq L_\sigma$ ($q = 1$).

Remark 18. It should be noted that the statement of the latter theorem does not hold for $\sigma \in L^\infty$: In this situation ρ_σ is well defined on L^1, and $L_\sigma = L^1$ by the preceding Theorem 11, (i).

[3] A random variable U is uniformly distributed if $P(U \leq u) = u$ whenever $u \in [0,\, 1]$.

Proof. To prove the first assertion assume that $\sigma \in L^q$ for $1 < q < \infty$. Consider the uniquely determined numbers $t_0 := 0 < t_1 < t_2 < \dots < 1$ for which $\int_0^{t_n} \sigma(u)^q \mathrm{d}u = \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{j=1}^n \frac{1}{j^{p+1}}$ and observe that $\int_{t_{n-1}}^{t_n} \sigma(u)^q \mathrm{d}u = \frac{\|\sigma\|_q^q}{\zeta(p+1)} \frac{1}{n^{p+1}}$.[4] Define the function

$$\tau(u) := \begin{cases} n & \text{if } \ t_{n-1} \le u < t_n, \end{cases}$$

let U be uniformly distributed and consider the random variable

$$Y := \sigma(U)^{q-1} \cdot \tau(U). \tag{12}$$

Note, by (9), that

$$\begin{aligned}
\rho_\sigma(Y) &= \mathbb{E}\,\sigma(U)\,Y = \mathbb{E}\,\sigma(U)\,\sigma(U)^{q-1}\,\tau(U) = \mathbb{E}\,\sigma(U)^q\,\tau(U) \\
&= \int_0^1 \sigma(u)^q\,\tau(u)\,\mathrm{d}u = \sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} \sigma(u)^q \cdot n\,\mathrm{d}u \\
&= \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{n}{n^{p+1}} = \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{1}{n^p} = \|\sigma\|_q^q \frac{\zeta(p)}{\zeta(p+1)} < \infty,
\end{aligned}$$

because $p > 1$. Next,

$$\begin{aligned}
\|Y\|_p^p &= \mathbb{E}\,|Y|^p = \int_0^1 \sigma(u)^{(q-1)p}\,\tau(u)^p\,\mathrm{d}u \\
&= \int_0^1 \sigma(u)^q\,\tau(u)^p\,\mathrm{d}u = \sum_{n=1}^\infty \int_{t_{n-1}}^{t_n} \sigma(u)^q \cdot n^p\,\mathrm{d}u \\
&= \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{n^p}{n^{p+1}} = \frac{\|\sigma\|_q^q}{\zeta(p+1)} \sum_{n=1}^\infty \frac{1}{n} = \infty.
\end{aligned}$$

Hence, $Y \in L_\sigma$, but $Y \notin L^p$.

The second statement of the theorem is actually the first statement with $q = 1$, but the above proof needs a modification: To accept it define, as above, an increasing sequence of values by $t_0 := 0 < t_1 < t_2 < \dots < 1$ satisfying $\int_0^{t_n} \sigma(t)\mathrm{d}t \ge 1 - 2^{-n}$. Note, that

$$\int_{t_{n-1}}^{t_n} \sigma(u)\mathrm{d}u \le \int_{t_{n-1}}^1 \sigma(u)\,\mathrm{d}u = 1 - \int_0^{t_{n-1}} \sigma(u)\mathrm{d}u \le 2^{1-n}.$$

Define moreover the increasing function

$$\tau(\cdot) := \sum_{n=0} \mathbb{1}_{[t_n,\,1]}(\cdot)$$

(i.e., $\tau(t) = n$ if $t_{n-1} \le t < t_n$) and observe that $\tau(t) \nearrow \infty$ whenever $t \to 1$.

[4] $\zeta(p) := \sum_{n=1}^\infty \frac{1}{n^p}$ is Riemann's Zeta function, the series converges whenever $p > 1$.

Now let U be a uniformly distributed random variable and set $Y := \tau(U)$. Then

$$\begin{aligned}\rho_\sigma(Y) &= \int_0^1 \sigma(u)\tau(u)\mathrm{d}u = \sum_{n=1} \int_{t_{n-1}}^{t_n} \sigma(u)\tau(u)\mathrm{d}u \\ &= \sum_{n=1} n \cdot \int_{t_{n-1}}^{t_n} \sigma(u)\mathrm{d}u \le \sum_{n=1} n \cdot 2^{1-n} = 4 < \infty,\end{aligned}$$

so $Y \in L_\sigma$. But $Y \notin L^\infty$, because $P(Y \ge n) \ge 1 - t_{n-1} > 0$ by definition of τ. □

Remark 19. Notably the preceding proof applies for the random variable $Y = \sigma(U)^{q-1} \cdot \tau(U)^\alpha$ in (12) equally well whenever $1 \le \alpha < p$, such that L_σ is larger than L^p by an entire infinite dimensional manifold.

It was demonstrated above that the space L_σ is contained in L^1. The above inequality (6), $\|\cdot\|_1 \le \|\cdot\|_\sigma$, allows to prove an even much stronger result: a finite valued risk measure cannot be considered on a space larger than L^1. This is the content of the following theorem, which was communicated to the author by Prof. Alexander Shapiro (Georgia Tech). In brief: it does not make sense to consider risk measures on a space larger than L^1.

Theorem 20. *Let $L \subset L^0$ be a vector space collecting $\mathbb{R}$-valued random variables on $([0,1], \mathscr{B}, \lambda)$ (the standard probability space equipped with its Borel sets) such that $L \supsetneqq L^1$ and $|Y| \in L$, if $Y \in L$. Then there does not exist a version independent, finite valued risk measure on L.*

Proof. Suppose that $\rho: L \to \mathbb{R}$ is a version independent, and finite valued risk measure on L. Restricted to L^∞, Kusuoka's theorem (Theorem 7) applies and ρ takes the form $\rho(\cdot) = \sup_{\sigma \in \mathscr{S}} \rho_\sigma(\cdot)$. Choose $Y \in L \backslash L^1$, that is $\mathbb{E}|Y| = \infty$, or $\int_0^p F_{|Y|}^{-1}(u)\,\mathrm{d}u \to \infty$ whenever $p \to 1$.

Next, pick any $\sigma \in \mathscr{S}$. Define $Y_n := \min\{n, |Y|\}$ and observe that $\rho(Y_n) \le \rho(|Y|)$ by monotonicity. Note that $Y_n \in L^\infty$ and hence, by Kusuoka's representation, (6) and the particular choice of Y,

$$\rho(|Y|) \ge \rho(Y_n) \ge \rho_\sigma(Y_n) = \|Y_n\|_\sigma \ge \|Y_n\|_1 \ge \int_0^{P(|Y| \le n)} F_{|Y|}^{-1}(u)\,\mathrm{d}u \to \infty,$$

as $n \to \infty$. Hence, ρ is not finite valued on L. □

Theorem 21. *$(L_\sigma, \|\cdot\|_\sigma)$ is a Banach space over $\mathbb{R}$.*

Proof. It remains to be shown that $(L_\sigma, \|\cdot\|_\sigma)$ is complete. For this let $(Y_k)_k$ be a Cauchy sequence for $\|\cdot\|_\sigma$. By (6) the sequence $(Y_k)_k$ is a Cauchy sequence for $\|\cdot\|_1$ as well, and from completeness of L^1 it follows that there exists a limit $Y \in L^1$. We shall show that $Y \in L_\sigma$.

It follows from convergence in L^1 that $(Y_k)_k$ converges in distribution, that is $F_{Y_k}(y) \to F_Y(y)$ for every point y where F_Y is continuous and moreover $F_{|Y_k|}^{-1}(\cdot) \to F_{|Y|}^{-1}(\cdot)$ (cf. [26, Chapter 21]). Now

$$\begin{aligned}\|Y\|_\sigma &= \rho_\sigma(|Y|) = \int_0^1 \sigma(t) F_{|Y|}^{-1}(t)\,\mathrm{d}t = \int_0^1 \sigma(t) \lim_{k\to\infty} F_{|Y_k|}^{-1}(t)\,\mathrm{d}t \\ &= \int_0^1 \liminf_{k\to\infty} \sigma(t) F_{|Y_k|}^{-1}(t)\,\mathrm{d}t \le \liminf_{k\to\infty} \int_0^1 \sigma(t) F_{|Y_k|}^{-1}(t)\,\mathrm{d}t = \liminf_{k\to\infty} \|Y_k\|_\sigma\end{aligned}$$

by Fatou's Lemma, which is applicable because $F_{|Y_k|}^{-1}(\cdot) \geq 0$.

As $(Y_k)_k$ is a Cauchy sequence one may pick $k^* \in \mathbb{N}$ such that $\|Y_k - Y_{k^*}\|_\sigma < 1$ for all $k > k^*$, and hence $\|Y_k\|_\sigma \leq \|Y_{k^*}\|_\sigma + \|Y_k - Y_{k^*}\|_\sigma < \|Y_{k^*}\|_\sigma + 1 < \infty$ by the triangle inequality. The sequence $(Y_k)_k$ thus is uniformly bounded in its norm. Hence,

$$\|Y\|_\sigma \leq \liminf_{k\to\infty} \|Y_k\|_\sigma \leq \|Y_{k^*}\|_\sigma + 1 < \infty,$$

that is $Y \in L_\sigma$ and L_σ thus is complete. □

Example 22. Consider the spectrum $\sigma(\alpha) = \frac{1}{2\sqrt{1-\alpha}}$. It should be noted that $L_\sigma \supset \bigcup_{p>2} L^p$, and $\|\cdot\|_\sigma$ provides a reasonable norm on that set.

Restricted to L^p, for some $p > 2$, the open mapping theorem (cf. [22] or [3]) insures that the norms are equivalent, that is there are constants c_1 and c_2 such that

$$c_1 \cdot \|Y\|_p \leq \|Y\|_\sigma \leq c_2 \cdot \|Y\|_p \qquad (Y \in L^p \subset L_\sigma).$$

The latter inequalities hold just for $Y \in L^p$, but not for $Y \in L_\sigma$.

Proposition 23. *Measurable, simple (step) functions are dense in L_σ, and in particular L^∞ is dense in L_σ.*

Proof. Given $Y \in L_\sigma$ and $\varepsilon > 0$, find $t_0 \in (0,1)$ such that $\int_0^{t_0} F_Y^{-1}(u)\sigma(u)\mathrm{d}u < \frac{\varepsilon}{3}$ and set $s(t) := F_Y^{-1}(t_0)$ whenever $t \leq t_0$. Moreover, find $t_1 \in (0,1)$ such that $\int_{t_1}^1 F_Y^{-1}(u)\sigma(u)\mathrm{d}u < \frac{\varepsilon}{3}$ and set $s(t) := F_Y^{-1}(t_1)$ whenever $t \geq t_1$. In between, as $F_Y^{-1}(t)$ is nondecreasing on the compact $[t_0, t_1]$, there is an increasing step function $s(t)$ such that $\left|s(t) - F_Y^{-1}(t)\right| \sigma(t) < \frac{\varepsilon}{3}$. Let U be uniformly distributed and co-monotone with Y. Then it holds that $\|Y - s(U)\|_\sigma < \varepsilon$ by construction of the step function s. □

5 The Dual of the natural domain L_σ

Risk measures are convex and lower semi-continuous (cf. [17]) functions, hence they have a dual representation by involving the Fenchel–Moreau Theorem (also Legendre transformation, see below). This representation involves the dual space in a natural way, and hence it is of interest to understand the dual of the Banach space $(L_\sigma, \|\cdot\|_\sigma)$. We describe the norm of the dual and identify the dual with a subspace of L^1. The respective results are proven in this section, moreover essential properties of the dual are highlighted.

Theorem 24 (Fenchel–Moreau)**.** *Let $\mathscr{Y}$ be a Banach space and $f : \mathscr{Y} \to \mathbb{R} \cup \{\infty\}$ be convex and lower semi-continuous with $f(Y_0) < \infty$ for an $Y_0 \in \mathscr{Y}$. Then*

$$f^{**} = f,$$

where

$$f^*(Z^*) := \sup_{Y\in\mathscr{Y}} Z^*(Y) - f(Y) \quad \text{and} \quad f^{**}(Y) := \sup_{Z^*\in\mathscr{Y}^*} Z^*(Y) - f^*(Z^*).$$

Proof. cf. [21]. □

Note, that a risk measure ρ_σ is not only lower semicontinuous, by Proposition 6 it is continuous with respect to the norm $\|\cdot\|_\sigma$ on the Banach space $\mathscr{Y} = (L_\sigma, \|\cdot\|_\sigma)$. By the Fenchel–Moreau theorem thus $\rho_\sigma^{**} = \rho_\sigma$. To involve it on its natural domain $\mathscr{Y} = (L_\sigma, \|\cdot\|_\sigma)$ its dual $\mathscr{Y}^* = (L_\sigma, \|\cdot\|_\sigma)^*$ has to be available, and this is elaborated in the sequel.

Definition 25. For a spectral function σ and a random variable $Z \in L^1$ define the binary relation

$$Z \preccurlyeq \sigma \ \text{ iff } \quad \mathsf{AV@R}_\alpha\left(|Z|\right) \leq \frac{1}{1-\alpha} \int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \leq \alpha < 1, \tag{13}$$

the gauge function (Minkowski functional)

$$\begin{aligned} \|Z\|_\sigma^* :&= \inf\left\{\eta \geq 0 : \ \mathsf{AV@R}_\alpha\left(|Z|\right) \leq \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \leq \alpha < 1\right\} \\ &= \inf\left\{\eta \geq 0 : \ |Z| \preccurlyeq \eta \cdot \sigma\right\} \end{aligned} \tag{14}$$

and the set $L_\sigma^* := \left\{Z \in L^1 : \ \|Z\|_\sigma^* < \infty\right\}$.

It should be noted that the relation (13), which is a kind of second order stochastic dominance relation (cf. [12, 11]), can be interpreted as a growth condition for $|Z|$, which is a condition on Z's tails: $Z \preccurlyeq \eta \cdot \sigma$ can only hold true if $|Z|$ does not grow (in terms of average quantiles) faster towards ∞ than $\eta \cdot \sigma$.

Notice as well that

$$\|Z\|_\sigma^* \leq \eta \text{ if and only if } \mathsf{AV@R}_\alpha\left(|Z|\right) \leq \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \text{ for all } 0 \leq \alpha < 1, \tag{15}$$

and in particular the closed form expression

$$\|Z\|_\sigma^* = \sup_{\alpha \in (0,1)} \frac{\mathsf{AV@R}_\alpha\left(|Z|\right)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u}. \tag{16}$$

Example 26. For U a uniformly distributed random variable it follows readily from (11) and Definition 25 that

$$\|\sigma\left(U\right)\|_\sigma^* = 1. \tag{17}$$

Example 27. The norm of the indicator function has the explicit form

$$\|\mathbb{1}_A\|_\sigma^* = \frac{1}{\frac{1}{P(A)}\int_{1-P(A)}^1 \sigma\left(u\right)\mathrm{d}u}, \tag{18}$$

which derives from $\mathsf{AV@R}_\alpha\left(\mathbb{1}_A\right) = \min\left\{1, \frac{P(A)}{1-\alpha}\right\}$ and the particular choice $\alpha = 1 - P\left(A\right)$ in (13). This choice is justified by considering the mappings

$$\tau_1\left(\alpha\right) := \int_\alpha^1 \sigma(u)\mathrm{d}u \text{ and } \tau_2\left(\alpha\right) := \frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u$$

with derivatives

$$\tau_1'\left(\alpha\right) = -\sigma(\alpha) \leq 0 \text{ and } \tau_2'\left(\alpha\right) = \frac{1}{\left(1-\alpha\right)^2}\int_\alpha^1 \sigma(u) - \sigma(\alpha)\mathrm{d}u \geq 0.$$

Hence, $\alpha \mapsto \frac{\frac{P(A)}{1-\alpha}}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} = \frac{P(A)}{\tau_1(\alpha)}$ is increasing, and $\alpha \mapsto \frac{1}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} = \frac{1}{\tau_2(\alpha)}$ decreasing, such that the supremum in

$$\|\mathbb{1}_A\|_\sigma^* = \sup_{0\le\alpha\le 1} \frac{\min\left\{\frac{P(A)}{1-\alpha}, 1\right\}}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u}$$

is attained whenever $P(A) = 1-\alpha$.

Immediate consequences of (18) are further the bounds $P(A) \le \|\mathbb{1}_A\|_\sigma^* \le 1$.

Remark 28. Given Kusuoka's representation one may employ the measure μ directly instead of the spectral density σ by involving (4). It holds that

$$\frac{1}{1-\alpha}\int_\alpha^1 \sigma_\mu(u)\mathrm{d}u = \int_0^1 \min\left\{\frac{1}{1-u}, \frac{1}{1-\alpha}\right\}\mathrm{d}\mu(u),$$

the condition $Z \preccurlyeq \sigma_\mu$ thus reads directly

$$Z \preccurlyeq \sigma_\mu \;\text{ iff }\; \mathsf{AV@R}_\alpha(|Z|) \le \int_0^1 \min\left\{\frac{1}{1-u}, \frac{1}{1-\alpha}\right\}\mathrm{d}\mu(u) \;\text{ for all } 0\le\alpha<1.$$

Notice as well that $\int_0^1 \min\left\{\frac{1}{1-u}, \frac{1}{1-\alpha}\right\}\mathrm{d}\mu(u)$ represents an expectation of a (bounded) function with respect to the measure μ.

Monotonicity. It follows from monotonicity of the Average Value-at-Risk that

$$\|Y_1\|_\sigma^* \le \|Y_2\|_\sigma^*, \text{ if } |Y_1| \le |Y_2|. \tag{19}$$

Comparison with L^1. For $Z \in L^\sigma$, $\|Z\|_\sigma^* \le \eta$ implies that $\mathbb{E}|Z| \le \eta$ (by the choice $\alpha = 0$ in (15)), hence

$$\|Z\|_1 \le \|Z\|_\sigma^* \tag{20}$$

and $L_\sigma^* \subset L^1$.

Comparison with L^∞. It follows from Example 27 that $\tau_2(\alpha) \ge \tau_2(0) = 1$ and consequently

$$\|Z\|_\sigma^* = \sup_{\alpha\in(0,1)} \frac{\mathsf{AV@R}_\alpha(|Z|)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} \le \sup_{\alpha\in(0,1)} \frac{\|Z\|_\infty}{\tau_2(\alpha)} \le \|Z\|_\infty. \tag{21}$$

Suppose moreover that σ is bounded and $Z \in L^\infty$. Then $\mathsf{AV@R}_\alpha(|Z|) \to \|Z\|_\infty$ and $\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u \to \|\sigma\|_\infty$, as $\alpha \to 1$, and consequently $\|Z\|_\infty \le \eta \cdot \|\sigma\|_\infty$ has to hold by (15) for η to be feasible. That is,

$$\|Z\|_\infty \le \|Z\|_\sigma^* \cdot \|\sigma\|_\infty. \tag{22}$$

Upper bound. An upper bound for the norm $\|\cdot\|_\sigma^*$ is given by

$$\|Z\|_\sigma^* \le \sup_{0\le u<1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)},$$

where the conventions $\frac{0}{0}=0$ and $\frac{1}{0}=\infty$ have to be employed. Indeed, if $\frac{F_{|Z|}^{-1}(u)}{\sigma(u)} \le \eta$, then integrating gives $(1-\alpha)\,\mathsf{AV@R}_\alpha(|Z|) = \int_\alpha^1 F_{|Z|}^{-1}(u)\mathrm{d}u \le \eta \cdot \int_\alpha^1 \sigma(u)\,\mathrm{d}u$, which in turn means that $\|Z\|_\sigma^* \le \eta$. Notice, however, that $Z \mapsto \sup_{0\le u<1} \frac{F_{|Z|}^{-1}(u)}{\sigma(u)}$ is not a norm, it does not satisfy the triangle inequality.

Simple functions. For $Z=\sum_{j=1}^n a_j \mathbb{1}_{A_j}$ a simple (step) function, $\alpha \mapsto (1-\alpha)\,\mathsf{AV@R}_\alpha(|Z|) = \int_0^1 F_{|Z|}^{-1}(u)\,\mathrm{d}u$ is piecewise linear. As $\alpha \mapsto \int_\alpha^1 \sigma(u)\,\mathrm{d}u$ is concave (this is, because σ is increasing), the defining condition (15) has to be verified on finite many points only, such that simple functions are contained in L_σ^*.

Proposition 29. *The pair $\left(L_\sigma^*, \|\cdot\|_\sigma^*\right)$ is a Banach space.*

Proof. Notice first that $\|Z\|_\sigma^*=0$ implies that $\mathsf{AV@R}_\alpha(|Z|)=0$ for all $\alpha<1$, so

$$0=\lim_{\alpha\nearrow 1} \mathsf{AV@R}_\alpha(|Z|) = \operatorname{ess\,sup}|Z|,$$

that is $Z=0$ almost everywhere, such that $\|\cdot\|_\sigma^*$ separates points in L_σ^*.

Positive homogeneity is immediate and inherited from the Average Value-at-Risk.

As for the triangle inequality let η_1 and η_2, resp. satisfy (14) for Z_1 and Z_2, resp.. Then, by monotonicity and sub-additivity of the Average Value-at-Risk,

$$\mathsf{AV@R}_\alpha(|Z_1+Z_2|) \le \mathsf{AV@R}_\alpha(|Z_1|+|Z_2|) \le \mathsf{AV@R}_\alpha(|Z_1|) + \mathsf{AV@R}_\alpha(|Z_2|)$$

such that

$$\mathsf{AV@R}_\alpha(|Z_1+Z_2|) \le \frac{\eta_1+\eta_2}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u,$$

that is finally $\|Z_1+Z_2\|_\sigma^* \le \|Z_1\|_\sigma^* + \|Z_2\|_\sigma^*$, the triangle inequality.

Finally completeness remains to be shown. For this let Z_k be a Cauchy sequence. Hence there is a natural number k^*, such that $\|Z_k\|_\sigma^* \le \|Z_{k^*}\|_\sigma^* + \|Z_k - Z_{k^*}\|_\sigma^* \le \|Z_{k^*}\|_\sigma^* + 1$, that is there is $\eta \ge 0$ (η satisfies $\eta \le \|Z_{k^*}\|_\sigma^* + 1$) such that

$$\mathsf{AV@R}_\alpha(|Z_k|) \le \frac{\eta}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u$$

for all $k>k^*$ and $\alpha\in(0,1)$. Next, by (20), Z_k is a Cauchy sequence for L^1 as well, hence there is a limit $Z\in L^1$, and Z_k converges in distribution and in quantiles. By Fatou's inequality,

$$\begin{aligned}
\mathsf{AV@R}_\alpha(|Z|) &= \frac{1}{1-\alpha}\int_\alpha^1 F_{|Z|}^{-1}(u)\mathrm{d}u = \frac{1}{1-\alpha}\int_\alpha^1 \liminf_{k\to\infty} F_{|Z_k|}^{-1}(u)\mathrm{d}u \\
&\le \frac{1}{1-\alpha}\liminf_{k\to\infty}\int_\alpha^1 F_{|Z_k|}^{-1}(u)\mathrm{d}u = \liminf_{k\to\infty}\mathsf{AV@R}_\alpha(|Z_k|) \\
&\le \frac{\eta}{1-\alpha}\cdot\int_\alpha^1 \sigma(u)\mathrm{d}u.
\end{aligned}$$

The limit $Z \in L^1$ thus satisfies the defining conditions to qualify for L_σ^* and $\|Z\|_\sigma^* \leq \eta$. It follows that $Z \in L_\sigma^*$ and $\left(L_\sigma^*, \|\cdot\|_\sigma^*\right)$ thus is a Banach space. $\square$

Theorem 30. *The space $\left(L_\sigma^*, \|\cdot\|_\sigma^*\right)$ is the dual of $\left(L_\sigma, \|\cdot\|_\sigma\right)$.*

Proof. Let $Y \in L_\sigma$ and $Z \in L_\sigma^*$ with $\|Z\|_\sigma^* =: \eta$ be chosen. Then note that

$$|\mathbb{E}\, Y Z| \leq \mathbb{E}\,|Y| \cdot |Z| \leq \int_0^1 F_{|Y|}^{-1}(u)\, F_{|Z|}^{-1}(u)\, \mathrm{d}u$$

by the Hardy–Littlewood–Pólya inequality. To abbreviate the notation we introduce the functions $S(u) := \int_u^1 \sigma(p)\mathrm{d}p$ and $G(u) := \int_u^1 F_{|Z|}^{-1}(p)\,\mathrm{d}p$ (the functions are well defined, because $\sigma \in L^1$ and $Z \in L^1$). Then, by Riemann–Stieltjes integration by parts,

$$\begin{aligned}
\int_0^1 F_{|Y|}^{-1}(u)\, F_{|Z|}^{-1}(u)\, \mathrm{d}u &= -\int_0^1 F_{|Y|}^{-1}(u)\, \mathrm{d}G(u) \\
&= -\left.F_{|Y|}^{-1}(u)\, G(u)\right|_{u=0}^1 + \int_0^1 G(u)\, \mathrm{d}F_{|Y|}^{-1}(u) \\
&= F_{|Y|}^{-1}(0) \cdot \mathbb{E}\,|Z| + \int_0^1 G(u)\, \mathrm{d}F_{|Y|}^{-1}(u).
\end{aligned}$$

Now note that $F_{|Y|}^{-1}(\cdot)$ is an increasing function, and $G(u) = \int_u^1 F_{|Z|}^{-1}(p)\,\mathrm{d}p \leq \eta \cdot \int_u^1 \sigma(p)\,\mathrm{d}p = \eta \cdot S(u)$ because $\|Z\|_\sigma^* \leq \eta$. Thus, and employing again Riemann–Stieltjes integration by parts,

$$\begin{aligned}
|\mathbb{E} Y Z| &\leq F_{|Y|}^{-1}(0) \cdot \|Z\|_1 + \eta \cdot \int_0^1 S(u)\, \mathrm{d}F_{|Y|}^{-1}(u) \\
&= F_{|Y|}^{-1}(0) \cdot \|Z\|_1 + \eta \cdot \left. S(u)\, F_{|Y|}^{-1}(u)\right|_{u=0}^1 - \eta \cdot \int_0^1 F_{|Y|}^{-1}(u)\, \mathrm{d}S(u) \\
&= F_{|Y|}^{-1}(0) \cdot \|Z\|_1 - \eta \cdot F_{|Y|}^{-1}(0) + \eta \cdot \int_0^1 F_{|Y|}^{-1}(u)\, \sigma(u)\, \mathrm{d}u \\
&= F_{|Y|}^{-1}(0) \cdot \left(\|Z\|_1 - \eta\right) + \eta \cdot \int_0^1 F_{|Y|}^{-1}(u)\, \sigma(u)\, \mathrm{d}u \\
&= F_{|Y|}^{-1}(0) \cdot \left(\|Z\|_1 - \|Z\|_\sigma^*\right) + \|Z\|_\sigma^* \cdot \int_0^1 F_{|Y|}^{-1}(u)\, \sigma(u)\, \mathrm{d}u.
\end{aligned}$$

Finally observe that $F_{|Y|}^{-1}(0) = \operatorname{ess\,inf} |Y| \geq 0$ and $\|Z\|_1 - \|Z\|_\sigma^* \leq 0$ by (20), hence

$$|\mathbb{E}\, Y Z| \quad \leq \quad \|Z\|_\sigma^* \cdot \int_0^1 F_{|Y|}^{-1}(u)\, \sigma(u)\, \mathrm{d}u = \rho_\sigma(|Y|) \cdot \|Z\|_\sigma^* = \|Y\|_\sigma \cdot \|Z\|_\sigma^*.$$

This proves that for every $Z \in L_\sigma^*$ the linear mapping $Y \mapsto \mathbb{E}\, Y Z$ is continuous with respect to the norm $\|\cdot\|_\sigma$.

It remains to be shown that every linear, continuous mapping ζ in the dual of L_σ ($\zeta \in \left(L_\sigma, \|\cdot\|_\sigma\right)^*$) takes the form $\zeta(Y) = \mathbb{E} Y Z$ for some $Z \in L_\sigma^*$. For this consider the (signed) measure $\mu(A) := \zeta(\mathbb{1}_A)$. If $A = \bigcup_{i=1}^\infty A_i$ is a disjoint union of countably measurable sets, then

$\mathbb{1}_A=\sum_{i=1}^\infty \mathbb{1}_{A_i}$. Clearly,

$$\left\| \mathbb{1}_A-\sum_{i=1}^n \mathbb{1}_{A_i}\right\|_\sigma=\int_{1-\sum_{i=n+1}^\infty P(A_i)}^1 \sigma(u)\mathrm{d}u \xrightarrow[n\to\infty]{} 0,$$

as P is sigma-finite and $\sigma\in L^1$. It follows by continuity of ζ with respect to $\|\cdot\|_\sigma$ that

$$\mu\left(A\right)=\zeta\left(\mathbb{1}_A\right)=\zeta\left(\sum_{i=1}^\infty \mathbb{1}_{A_i}\right)=\sum_{i=1}^\infty \zeta\left(\mathbb{1}_{A_i}\right)=\sum_{i=1}^\infty \mu\left(A_i\right),$$

hence μ is a sigma-finite measure. If $P\left(A\right)=0$, then

$$\left|\mu\left(A\right)\right|=\left|\zeta\left(\mathbb{1}_A\right)\right|\le\|\zeta\|\cdot\left\|\mathbb{1}_A\right\|_\sigma=\|\zeta\|\cdot\int_0^1\sigma\left(u\right)F_{\mathbb{1}_A}^{-1}\left(u\right)\mathrm{d}u=0,$$

because $F_{\mathbb{1}_A}^{-1}\left(u\right)=0$ for every $u<1$. It follows that $\mu\left(A\right)=0$, such that μ is moreover absolutely continuous with respect to P.

Let $Z\in L^0$ be the Radon–Nikodým derivative, $\mathrm{d}\mu=Z\mathrm{d}P$. Then $\zeta\left(\mathbb{1}_A\right)=\mu\left(A\right)=\int_A Z\mathrm{d}P=\int Z\mathbb{1}_A\mathrm{d}P=\mathbb{E}\,Z\mathbb{1}_A$ and hence $\zeta\left(\phi\right)=\mathbb{E}\,Z\phi$ for all simple functions ϕ by linearity and $\left|\mathbb{E}\,Z\phi\right|=\left|\zeta\left(\phi\right)\right|\le\|\zeta\|\cdot\|\phi\|_\sigma$ by continuity of ζ.

Choose the function $\phi:=\operatorname{sign}Z$ (a simple function) to see that $\mathbb{E}\left|Z\right|\le\|\zeta\|$, that is $Z\in L^1$.

Note as well that $\mathbb{E}\left|Z\right|\phi=\mathbb{E}\,Z\cdot\operatorname{sign}\left(Z\right)\phi\le\|\zeta\|\cdot\left\|\operatorname{sign}\left(Z\right)\phi\right\|_\sigma\le\|\zeta\|\cdot\|\phi\|_\sigma$, because ρ_σ is monotone and $\left|\operatorname{sign}\left(Z\right)\cdot\phi\right|\le\left|\phi\right|$. For any measurable set A (with complement denoted A^c) thus

$$\mathbb{E}\left|Z\right|\mathbb{1}_{A^c}\le\|\zeta\|\cdot\left\|\mathbb{1}_{A^c}\right\|_\sigma=\|\zeta\|\cdot\rho_\sigma\left(\mathbb{1}_{A^c}\right)=\|\zeta\|\cdot\int_{P(A)}^1\sigma(u)\mathrm{d}u,$$

and hence $\mathbb{E}\left|Z\right|\frac{\mathbb{1}_{A^c}}{P(A^c)}\le\|\zeta\|\cdot\frac{1}{1-P(A)}\int_{P(A)}^1\sigma(u)\mathrm{d}u$. Taking the supremum over all sets A with $P\left(A\right)\le\alpha$ gives

$$\begin{aligned}\mathsf{AV@R}_\alpha\left(\left|Z\right|\right)&=\sup_{P(A^c)\ge1-\alpha}\mathbb{E}\left|Z\right|\frac{\mathbb{1}_{A^c}}{P\left(A^c\right)}\le\|\zeta\|\cdot\sup_{P(A)\le\alpha}\frac{1}{1-P\left(A\right)}\int_{P(A)}^1\sigma(u)\mathrm{d}u\\&=\frac{\|\zeta\|}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u\end{aligned}$$

by (10) and because σ is increasing. It follows that $\|Z\|_\sigma^*\le\|\zeta\|$ and thus $Z\in L_\sigma^*$. This completes the proof. □

The Hahn–Banach functional in L_σ. Let $Y\in L_\sigma$ be fixed, and let U be coupled in a co-monotone way with $\left|Y\right|$. Define $Z_Y:=\sigma\left(U\right)\cdot\operatorname{sign}Y$ and observe that $F_{\sigma(U)}^{-1}\left(\alpha\right)=\sigma\left(\alpha\right)$ by (11). Hence $\mathsf{AV@R}_\alpha\left(\sigma\left(U\right)\right)=\frac{1}{1-\alpha}\int_\alpha^1\sigma\left(u\right)\mathrm{d}u$, and it follows that $\left\|Z_Y\right\|_\sigma^*=1$. On the other side $\mathbb{E}\,Y\cdot Z_Y=\mathbb{E}\left|Y\right|\cdot\sigma\left(U\right)=\int_0^1F_{|Y|}^{-1}\left(u\right)\sigma\left(u\right)\mathrm{d}u=\|Y\|_\sigma$. Z_Y thus is a maximizer of the problem

$$\|Y\|_\sigma=\max\left\{\mathbb{E}\,Y\cdot Z:\ \|Z\|_\sigma^*\le1\right\}.$$

Reflexivity. In order to investigate reflexivity of the space $(L_\sigma,\, \|\cdot\|_\sigma)$ the following theorem is perhaps an unexpected surprise.

Theorem 31. *If σ is unbounded ($\sigma \notin L^\infty$), then simple functions and L^∞ are not dense in $\left(L_\sigma^*,\, \|\cdot\|_\sigma^*\right)$.*

Proof. Let U be uniformly distributed and consider the random variable $Z := \sigma(U)$. Moreover, let $Z_n = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$ be any simple function with disjoint, measurable sets A_i, and we assume—without loss of generality—that $|\lambda_1| \le |\lambda_2| \le \dots |\lambda_n|$.

By the triangle inequality it holds that $|Z| \le |Z - Z_n| + |Z_n|$, and by monotonicity and subadditivity hence $\mathsf{AV@R}_\alpha\left(|Z|\right) \le \mathsf{AV@R}_\alpha\left(|Z - Z_n|\right) + \mathsf{AV@R}_\alpha\left(|Z_n|\right) \le \mathsf{AV@R}_\alpha\left(|Z - Z_n|\right) + |\lambda_n|$. It follows that

$$\|Z - Z_n\|_\sigma^* \ge \lim_{\alpha\to 1} \frac{\mathsf{AV@R}_\alpha\left(|Z|\right) - |\lambda_n|}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} = \lim_{\alpha\to 1} \frac{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} - \lim_{\alpha\to 1} \frac{|\lambda_n|}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u}.$$

As σ is unbounded it holds that $0 \le \frac{|\lambda_n|}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} \le \frac{|\lambda_n|}{\sigma(1-\alpha)} \xrightarrow[\alpha\to 1]{} 0$, and hence $\|Z - Z_n\|_\sigma^* \ge 1$. As the simple function Z_n was chosen arbitrarily it follows that Z cannot be approximated arbitrarily close by a simple function.

Now choose $\tilde Z \in L^\infty$ and a simple function Z_n such that $\left\|\tilde Z - Z_n\right\|_\infty < \varepsilon$ for some $\varepsilon > 0$. Then, by the reverse triangle inequality and (21),

$$\left\|\tilde Z - Z\right\|_\sigma^* \ge \|Z - Z_n\|_\sigma^* - \left\|Z_n - \tilde Z\right\|_\sigma^* \ge \|Z - Z_n\|_\sigma^* - \left\|Z_n - \tilde Z\right\|_\infty \ge 1 - \varepsilon.$$

As $\tilde Z \in L^\infty$ and $\varepsilon > 0$ were chosen arbitrarily it follows that

$$d\left(\sigma(U),\, L^\infty\right) = \inf_{\tilde Z \in L^\infty} \left\|\tilde Z - \sigma(U)\right\|_\sigma^* = 1 > 0,$$

and L^∞ thus certainly is not dense in $\left(L_\sigma^*,\, \|\cdot\|_\sigma^*\right)$. □

Theorem 32. *The Banach space $(L_\sigma,\, \|\cdot\|_\sigma)$ is not reflexive.*

Proof. If σ is bounded, then $(L_\sigma,\, \|\cdot\|_\sigma) = \left(L^1,\, \|\cdot\|_1\right)$, which is not a reflexive space. One thus may assume that σ is not bounded.

Consider the particular random variable $Z := U \cdot \sigma(U)$. Note, that $Z \le \sigma(U)$, and thus $\|Z\|_\sigma^* \le \|\sigma(U)\|_\sigma^* = 1$ by monotonicity of the norm (cf. (19)), such that $Z \in L_\sigma^*$.

Define the level sets $S_\alpha = \{U \ge \alpha\}$. If $(L_\sigma,\, \|\cdot\|_\sigma)$ were reflexive, then it is the dual of $\left(L_\sigma^*,\, \|\cdot\|_\sigma^*\right)$, and by the Banach–Alaoglu theorem (cf., e.g., [28]) there is a weak* accumulation point Y of the sequence $Y_n := \frac{\mathbb{1}_{S_{1-\frac{1}{n}}}}{\rho_\sigma\left(\mathbb{1}_{S_{1-\frac{1}{n}}}\right)}$ satisfying $\rho_\sigma(Y) \le 1$. In view of

$$1 \ge \|Z\|_\sigma^* > \frac{\frac{1}{1-\alpha}\int_\alpha^1 u \cdot \sigma(u)\mathrm{d}u}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\mathrm{d}u} \xrightarrow[\alpha\to 1]{} 1$$

it holds that

$$\begin{aligned}
\|Z\|_\sigma^* &= \lim_{\alpha\nearrow 1} \frac{\frac{1}{1-\alpha}\mathbb{E}\,\mathbb{1}_{S_\alpha} U \cdot \sigma(U)}{\frac{1}{1-\alpha}\mathbb{E}\,\mathbb{1}_{S_\alpha}\sigma(U)} = \lim_{\alpha\nearrow 1} \frac{\mathbb{E}\,\mathbb{1}_{S_\alpha} U \cdot \sigma(U)}{\rho_\sigma\left(\mathbb{1}_{S_\alpha}\right)} \\
&= \lim_{\alpha\nearrow 1} \mathbb{E}\frac{\mathbb{1}_{S_\alpha}}{\rho_\sigma\left(\mathbb{1}_{S_\alpha}\right)} Z = \lim_{n\to\infty} \mathbb{E}\, Y_n Z = \mathbb{E}\, Y Z.
\end{aligned}$$

However, $\{Y>0\}\subset\{Y_n>0\}$ and $P(Y_n>0)\le\frac{1}{n}$ by construction, such that $Y=0$ almost everywhere. This is hence a contradiction, and the norm of the specific random variable $Z=U\cdot\sigma(U)$ cannot be given in the form $\|Z\|_\sigma^*=\mathbb{E}\,YZ$ for some $Y\in L_\sigma$ satisfying $\rho_\sigma(Y)\le 1$. The space $(L_\sigma,\|\cdot\|_\sigma)$ thus is not reflexive. □

The Hahn–Banach functional in L_σ^*. It follows from the previous theorem that there does not always exist a random variable $Y\in L_\sigma$ such that $\|Z\|_\sigma^*=\frac{\mathbb{E}YZ}{\rho_\sigma(Y)}$. However, suppose that the maximum in (16) is attained, that is $\|Z\|_\sigma^*=\frac{\mathsf{AV@R}_\alpha(|Z|)}{\frac{1}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u}$ for some $\alpha\in[0,1)$. Then it holds for

$$Y_Z:=\frac{1}{1-\alpha}\mathbb{1}_{\left\{Z\ge F_{|Z|}^{-1}(\alpha)\right\}}\cdot\operatorname{sign}Z \tag{23}$$

that

$$\|Z\|_\sigma^*=\frac{\mathsf{AV@R}_\alpha(|Z|)}{\frac{1}{1-\alpha}\int_\alpha^1\sigma(u)\mathrm{d}u}=\frac{\mathbb{E}\,Y_Z\cdot Z}{\mathbb{E}\,|Y_Z|\,\sigma(U)}=\frac{\mathbb{E}\,Y_Z\cdot Z}{\rho_\sigma(|Y_Z|)}=\frac{\mathbb{E}\,Y_Z\cdot Z}{\|Y_Z\|_\sigma},$$

where U is coupled in a co-monotone way with $|Y_Z|$. Hence, $Y_Z\in L_\sigma$ (Eq. (23)) represents a Hahn–Banach functional corresponding to Z.

Note as well that the random variable Z in the proof of Theorem 32 was chosen such that the maximum in (16) is *not* attained.

The following statement compares L_σ^* spaces with spaces L^q, and it generalizes the relations (20) and (22) for general L^q spaces. It is the dual statement to Theorem 11.

Theorem 33 (Comparison with L^q)**.** *For $\sigma\in L^q$ $(1\le q\le\infty)$ it holds that*

$$\|Z\|_q\le\|Z\|_\sigma^*\cdot\|\sigma\|_q$$

whenever $Z\in L_\sigma^*$, and thus $L_\sigma^*\subset L^q$.

Moreover,

$$\frac{\|Z\|_\infty}{\|\sigma\|_\infty}\le\|Z\|_\sigma^*\le\|Z\|_\infty$$

such that the norms $\|\cdot\|_\infty$ and $\|\cdot\|_\sigma^*$ are equivalent whenever $\sigma\in L^\infty$, and in this case $L_\sigma^*=L^\infty$.

Proof. Employing L^p-L^q duality and $L_\sigma-L_\sigma^*$ duality it holds that

$$\|Z\|_q=\sup_{Y\ne 0}\frac{\mathbb{E}YZ}{\|Y\|_p}\le\sup_{Y\ne 0}\frac{\|Y\|_\sigma\|Z\|_\sigma^*}{\|Y\|_p}\le\sup_{Y\ne 0}\frac{\|\sigma\|_q\|Y\|_p\|Z\|_\sigma^*}{\|Y\|_p}=\|\sigma\|_q\cdot\|Z\|_\sigma^*$$

by (6).

The missing inequality (21) is given by

$$\|Z\|_\sigma^*=\sup_{Y\ne 0}\frac{\mathbb{E}YZ}{\|Y\|_\sigma}\le\sup_{Y\ne 0}\frac{\|Y\|_1\|Z\|_\infty}{\|Y\|_\sigma}\le\sup_{Y\ne 0}\frac{\|Y\|_\sigma\|Z\|_\infty}{\|Y\|_\sigma}=\|Z\|_\infty,$$

again by (6). □

6 The general natural domain space $L_{\mathscr{S}}$

Kusuoka's theorem (Theorem 7) and (5) suggest to consider risk measures of the form

$$\rho_{\mathscr{S}}(\cdot) := \sup_{\sigma\in\mathscr{S}} \rho_\sigma(\cdot).$$

To investigate this general type of risk measure we define the according norm and space first.

Definition 34. The *natural domain* of $\rho_{\mathscr{S}}$, where $\mathscr{S}$ is a collection of spectral functions, is

$$L_{\mathscr{S}} := \left\{Y \in L^1 : \|Y\|_{\mathscr{S}} < \infty\right\},$$

where

$$\|\cdot\|_{\mathscr{S}} := \rho_{\mathscr{S}}(|\cdot|) = \sup_{\sigma\in\mathscr{S}} \rho_\sigma(|\cdot|) = \sup_{\sigma\in\mathscr{S}} \|\cdot\|_\sigma.$$

Obviously, $L_{\mathscr{S}} \subset \bigcap_{\sigma\in\mathscr{S}} L_\sigma$. In view of Theorem 11 (ii) it is obvious as well that

$$L^\infty \subset L_{\mathscr{S}} \subset L^1,$$

even more, it holds that $\|Y\|_{\mathscr{S}} \leq \|Y\|_\infty$ whenever $Y \in L^\infty$, and $\|Y\|_1 \leq \|Y\|_{\mathscr{S}}$, whenever $Y \in L_{\mathscr{S}}$. Further, if $\sup_{\sigma\in\mathscr{S}} \|\sigma\|_q < \infty$ is finite as well, then

$$\|Y\|_{\mathscr{S}} \leq \|Y\|_p \cdot \sup_{\sigma\in\mathscr{S}} \|\sigma\|_q$$

by Theorem 11, (i).

Theorem 35. *The pair $(L_{\mathscr{S}}, \|\cdot\|_{\mathscr{S}})$ is a Banach space.*

Proof. First of all it is clear that $\|\cdot\|_{\mathscr{S}}$ is a norm on $L_{\mathscr{S}}$, as it separates points, is positively homogeneous and satisfies the triangle inequality: these properties are inherited from the spaces $(L_\sigma, \|\cdot\|_\sigma)_{\sigma\in\mathscr{S}}$.

It remains to be shown that $(L_{\mathscr{S}}, \|\cdot\|_{\mathscr{S}})$ is complete. So if $(Y_k)_k$ is a Cauchy sequence in $L_{\mathscr{S}}$, then because of $\|\cdot\|_\sigma \leq \|\cdot\|_{\mathscr{S}}$ it is a Cauchy sequence in any of the spaces $(L_\sigma, \|\cdot\|_\sigma)$ and it has a limit Y there. The limit is the same for all L_σ, so $Y \in \bigcap_{\sigma\in\mathscr{S}} L_\sigma$. Following (13) it holds that

$$\|Y\|_{\mathscr{S}} = \sup_{\sigma\in\mathscr{S}} \|Y\|_\sigma \leq \sup_{\sigma\in\mathscr{S}} \liminf_{k\to\infty} \|Y_k\|_\sigma \leq \liminf_{k\to\infty} \sup_{\sigma\in\mathscr{S}} \|Y_k\|_\sigma = \liminf_{k\to\infty} \|Y_k\|_{\mathscr{S}}$$

by the max-min inequality. Now choose $k^* \in \mathbb{N}$ such that $\|Y_k - Y_{k^*}\|_{\mathscr{S}} < 1$ for all $k > k^*$, which is possible because the sequence is Cauchy. It follows that

$$\|Y\|_{\mathscr{S}} \leq \liminf_{k\to\infty} \|Y_k\|_{\mathscr{S}} \leq \|Y_{k^*}\|_{\mathscr{S}} + 1 < \infty,$$

and hence $Y \in L_{\mathscr{S}}$, that is $L_{\mathscr{S}}$ is complete. □

Theorem 36. *The risk measure $\rho_{\mathscr{S}}$ is finite valued on $L_{\mathscr{S}}$, it is moreover continuous with respect to the norm $\|\cdot\|_{\mathscr{S}}$ with Lipschitz constant 1.*

Proof. The assertion follows from the more general Proposition 6. □

Comparison of different $L_{\mathscr{S}}$ spaces. The norm of the identity

$$\mathrm{id}:\left(L_{\mathscr{S}_1},\|\cdot\|_{\mathscr{S}_1}\right)\to\left(L_{\mathscr{S}_2},\|\cdot\|_{\mathscr{S}_2}\right)$$

is

$$\|\mathrm{id}\|=\sup_{\sigma_2\in\mathscr{S}_2}\inf_{\sigma_1\in\mathscr{S}_1}\sup_{0\leq\alpha<1}\frac{\int_\alpha^1\sigma_2(u)\,\mathrm{d}u}{\int_\alpha^1\sigma_1(u)\,\mathrm{d}u},$$

and $L_{\mathscr{S}_1}\subset L_{\mathscr{S}_2}$ iff $\|\mathrm{id}\|<\infty$. This is immediate from (7), (8) and

$$\|\mathrm{id}\|=\inf\left\{c>0:\ \forall\sigma_2\in\mathscr{S}_2\ \exists\sigma_1\in\mathscr{S}_1:\ \int_\alpha^1\sigma_2(u)\,\mathrm{d}u\leq c\cdot\int_\alpha^1\sigma_1(u)\,\mathrm{d}u\text{ for all }\alpha\in(0,1)\right\}.$$

Examples

We give finally two examples for which the norm $\|\cdot\|_{\mathscr{S}}$ induced by a set of spectral functions $\mathscr{S}$ coincides with the norm $\|\cdot\|_p$ on L^p. Note, that this is contrast to the space L_σ, as Theorem 17 insures that L_σ is strictly larger than L^p.

Example 37 (Higher order semideviation)**.** The $p-$semideviation risk measure for $0<\lambda\leq1$ is

$$\rho(Y):=\mathbb{E}Y+\lambda\cdot\left\|(Y-\mathbb{E}Y)_+\right\|_p.$$

Then $L_{\mathscr{S}}=L^p$, where $\mathscr{S}$ is an appropriate spectrum to generate $\rho=\rho_{\mathscr{S}}$, and the norms $\|\cdot\|_{\mathscr{S}}$ and $\|\cdot\|_p$ are equivalent.

Proof. The generating set $\mathscr{S}$ is provided in [24] and in [25], the higher order semideviation risk measure takes the alternative form

$$\rho(Y)=\rho_{\mathscr{S}}(Y)=\sup_{\sigma\in L^q}\left(1-\frac{\lambda}{\|\sigma\|_q}\right)\mathbb{E}Y+\frac{\lambda}{\|\sigma\|_q}\rho_\sigma(Y).$$

It is evident that $\rho_{\mathscr{S}}(|Y|)\leq\left(1-\frac{\lambda}{\|\sigma\|_q}\right)\|Y\|_1+\lambda\|Y\|_p\leq(1+\lambda)\|Y\|_p$, such that $\rho_{\mathscr{S}}$ is finite valued for $Y\in L^p$. We claim that the natural domain is $L_{\mathscr{S}}=L^p$. For this suppose that $Y\in L_{\mathscr{S}}\backslash L^p$, i.e., $\|Y\|_1<\infty$, but $\|Y\|_p=\infty$. So it holds that

$$\rho_{\mathscr{S}}(Y)\geq\lambda\cdot\sup_{\sigma\in L^q}\frac{\rho_\sigma(Y)}{\|\sigma\|_q}=\lambda\cdot\sup_{Z\in L^q}\mathbb{E}Y\frac{Z}{\|Z\|_q}=\lambda\cdot\|Y\|_p=\infty$$

by L^p-L^q duality, hence $Y\notin L_{\mathscr{S}}$ and thus $L_{\mathscr{S}}=L^p$.

It follows by the open mapping theorem that the norms are equivalent. □

Example 38. Theorem 17 states that $L_\sigma\supsetneqq L^\infty$, that is to say L_σ is strictly larger than L^∞. This is not the case any more for the space $L_{\mathscr{S}}$: for this consider just the risk measure

$$\rho(Y):=\sup_{\alpha<1}\mathsf{AV@R}_\alpha(Y)\qquad(=\operatorname{ess\,sup}Y).$$

Then $\rho(Y)<\infty$ if and only if $\operatorname{ess\,sup}Y<\infty$, that is $L_{\mathscr{S}}=L^\infty$.

7 Summary

In this paper we associate a norm with a risk measure in a natural way. The risk measure is continuous with respect to the associated norm. This point of view allows considering spectral risk measures on its natural domain, which is a new Banach space and as large as possible. The space of natural domain is considerably larger than an accordant L^p space for spectral risk measures.

As important representation theorems, as the Fenchel–Moreau theorem, involve the dual space, we study the dual space as well. Its norm can be described by a gauge functional, and the underlying set is characterized by second order stochastic dominance constraints, which measure the pace of growth of the random variable considered. An important consequence of the results of this paper is given by the fact that finite valued risk measures cannot be defined on a space lager than L^1 in a meaningful way.

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