

Minimal Representation of Insurance Prices

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Abstract

This paper prices insurance contracts by employing law invariant, coherent risk measures from mathematical finance. We demonstrate that the corresponding premium principle enjoys a minimal representation. Uniqueness—in a sense specified in the paper—of this premium principle is derived from this initial result. The representations are derived from a result by Kusuoka, which is usually given for nonatomic probability spaces. We extend this setting to premium principles for spaces with atoms, as this is of particular importance for insurance.

Further, stochastic order relations are employed to identify the minimal representation. It is shown that the premium principles in the minimal representation are extremal with respect to the order relations. The tools are finally employed to explicitly provide the minimal representation for premium principles, which are important in actuarial practice.

Keywords: Premium principles, Stochastic order relations, Fenchel–Moreau theorem.

1 Introduction

Convex principles for insurance contracts are formulated in Deprez and Gerber [8]. They constitute an entire family of premium principles (cf. Young [28]), which can be employed to price insurance contracts. More than ten years later the axioms contained there became popular as coherent risk measures, which have been introduced in mathematical finance by Artzner et al. [1]) to quantify the risk related to a financial exposure.

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Kusuoka [12] elaborates a comprehensive, explicit representation of these premium principles. The basic, elementary ingredient is well known in insurance as *Conditional Tail Expectation* (CTE, sometimes also *Conditional*, or *Average Value-at-Risk*). The Conditional Tail Expectation is employed in two ways in insurance: to price *individual* insurance contracts and to evaluate the risk related to an entire *portfolio* of contracts (for example by US and Canadian insurance supervisory authorities, cf. Ko et al. [11]). Kusuoka's representation of a premium principle is a supremum of convex combinations of CTEs. In this way the CTE thus can be interpreted as an external point of a Choquet integral.

Choquet integrals of CTEs are also known as distortion risk measures, or Wang premium principle in insurance (cf. Wang [27] or Denneberg [6] for an early discussion of the concept). It is a main characteristic of these premium principles that they overvalue high risks, which are unfavorable for the insurer, while assigning less weight to negligible risks in exchange. Wang premiums constitute a convex premium principle, for which Pichler [20] elaborates the corresponding convex conjugate.

It is known that Kusuoka representations of premiums or risk measures are not unique (cf., e.g., Pflug and Römisch [18]). However, it is demonstrated in Shapiro [23] for the special case of distortion premiums that the corresponding Kusuoka representation allows a unique, minimal representation. This raises the question of in some sense minimality of representations of a general premium principle. It was also posed the question whether such a minimal representation is unique in general? In this paper we give a positive answer to this question and show how such minimal representations can be derived in a constructive way. Further characterizations derived involve stochastic dominance relations of first and second order.

We finally provide minimal representations of two particular premium principles in the concluding section. These premium principles are natural extensions of the Conditional Tail Expectation, but they assign higher weights to the tails. As a special case we provide a new closed form representation of the Dutch premium principle, which is a simple (and for this reason very useful) premium principle allowing a compelling natural interpretation. Its Kusuoka representation turns out to be a mixture of the net premium principle and the Conditional Tail Expectation with a specified weight.

Kusuoka has elaborated his original results on probability spaces without atoms. As an extension we discuss Kusuoka representations for probability spaces which are not nonatomic. This is of particular interest for insurance contracts with a finite number of possible payouts in case of the insurance event, as the underlying samples are typically modeled as atoms with positive probability. Many life insurance contracts are notably of this specific form, where the sum insured is paid in case of death, say, or nothing. The survival of an insured person within one year is naturally given as an atomic insurance event with strictly positive measure, the survival probability. Our characterization of law invariant premium principles provides a necessary condition for existence of a Kusuoka representation on an atomic probability space. It follows from our results that the probabilities of the atoms have to be of a very special form in order to constitute a version independent premium principle, which are beyond the canonical embedding in the standard probability space (the probabilities of the atoms have to be equally weighted). This is not the case in actuarial practice, and it cannot be guaranteed when employing an empirical measure to determine the weights of the atoms. For this we conclude that additional premium principles are essentially not available in the case of a space with atoms, so that one has to use the traditional premium principles.

Kusuoka [12] formulates his original results in $L_\infty(\Omega, \mathcal{F}, P)$ spaces. Jouini et al. demonstrate in [10] that a major assumption on continuity can be dropped in this case (the Fatou property), as it is automatically satisfied. Here we consider the analysis in $L_p(\Omega, \mathcal{F}, P)$ spaces, $p \in [1, \infty)$, although different Banach spaces of random variables can be considered equally well, cf. for example Bellini et al. [3] in this journal. For an extended discussion of a natural choice of an appropriate Banach space we refer to Pichler [19].

Outline of the paper. The following Section provides the mathematical exposition. Section 3 introduces Kusuoka’s representation and the measure preserving transportation map, which is a main tool of our analysis. The following section (Section 4) addresses premium principles on general probability spaces. Section 5 discusses maximality of Kusuoka sets, which can be used to characterize minimal representations by employing stochastic dominance relations of first and second order. Section 6 exposes examples of minimal representations of important premium principles, while we conclude in Section 7.

2 Mathematical Setting

Throughout the paper we work with spaces $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$. That is, $Z \in \mathcal{Z}$ can be viewed as a random variable with finite p -th order moment with respect to the reference probability measure P . This is natural from an applications point of view and theoretically convenient. The space \mathcal{Z} equipped with the norm $\|Z\|_p := (\int |Z|^p dP)^{1/p}$ is a Banach space with the dual space of continuous linear functionals $\mathcal{Z}^* := L_q(\Omega, \mathcal{F}, P)$, where $q \in (1, \infty]$ and $1/p + 1/q = 1$.

It is said that a (real valued) functional $\pi : \mathcal{Z} \rightarrow \mathbb{R}$ is a *coherent risk measure*¹ or a *coherent insurance premium*, if it satisfies the following axioms (Artzner et al. [1]):

(A1) *Monotonicity*: if $Z, Z' \in \mathcal{Z}$ and $Z \leq Z'$, then $\pi(Z) \leq \pi(Z')$.

(A2) *Convexity*:

$$\pi(tZ + (1-t)Z') \leq t\pi(Z) + (1-t)\pi(Z')$$

for all $Z, Z' \in \mathcal{Z}$ and all $t \in [0, 1]$.

(A3) *Translation Equivariance*: if $c \in \mathbb{R}$ and $Z \in \mathcal{Z}$, then $\pi(Z + c) = \pi(Z) + c$.

(A4) *Positive Homogeneity*: if $t \geq 0$ and $Z \in \mathcal{Z}$, then $\pi(tZ) = t\pi(Z)$.

It is said that the risk measure π is *convex* if it satisfies axioms (A1)–(A3). The notation $Z \leq Z'$ expresses that the policy Z' pays more than Z , i.e., $Z(\omega) \leq Z'(\omega)$ for a.e. $\omega \in \Omega$. For an introductory discussion of premium principles (coherent (convex) risk measures) we refer to Young [28].

We say that two policies $Z_1, Z_2 \in \mathcal{Z}$ are *distributionally equivalent* if $F_{Z_1} = F_{Z_2}$, where $F_Z(z) := P(Z \leq z)$ denotes the cumulative distribution function (cdf) of $Z \in \mathcal{Z}$. The risk

¹In a financial context the term *coherent risk measure* is often used for mapping $\mathcal{R}(Z) = \pi(-Z)$, or the concave mapping $\mathcal{R}(Z) = -\pi(-Z)$ instead. The axioms then change accordingly.

measure (premium) $\pi : \mathcal{Z} \rightarrow \mathbb{R}$ is *law invariant* (with respect to the reference probability measure P) if for any distributionally equivalent insurance policies $Z_1, Z_2 \in \mathcal{Z}$ it follows that $\pi(Z_1) = \pi(Z_2)$. An important example of a law invariant coherent risk measure is the *Conditional Tail Expectation* (also called Average Value-at-Risk, or Conditional Value-at-Risk),

$$\text{CTE}_\alpha(Z) := \inf_{t \in \mathbb{R}} \{t + (1 - \alpha)^{-1} \mathbb{E}[Z - t]_+\} = (1 - \alpha)^{-1} \int_\alpha^1 F_Z^{-1}(\tau) d\tau, \quad (2.1)$$

where $\alpha \in [0, 1)$ and $F_Z^{-1}(\tau) := \sup\{t : F_Z(t) \leq \tau\}$ is the right side quantile function. Note that $F_Z^{-1}(\cdot)$ is a monotonically nondecreasing right side continuous function.

- We denote by \mathfrak{P} the set of probability measures on $[0, 1]$ having zero mass at 1. Unless stated otherwise, when talking about topological properties of \mathfrak{P} we use the *weak topology* of probability measures (see, e.g., Billingsley [4] for a discussion of weak convergence of probability measures).

It was elaborated by Kusuoka [12] that when the probability space (Ω, \mathcal{F}, P) is nonatomic, every law invariant, coherent risk measure π has the representation

$$\pi(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \text{CTE}_\alpha(Z) d\mu(\alpha), \quad Z \in \mathcal{Z}, \quad (2.2)$$

where \mathfrak{M} is a set of probability measures on $[0, 1]$.

Definition 2.1 *We say that a set \mathfrak{M} of probability measures on $[0, 1]$ is a Kusuoka set if the representation (2.2) holds.*

Note that the Kusuoka set is associated with a risk measure π and the space \mathcal{Z} . Since it is assumed that $\pi(Z)$ is finite valued for every $Z \in L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$, it follows that every measure $\mu \in \mathfrak{M}$ in representation (2.2) has zero mass at $\alpha = 1$ and hence $\mathfrak{M} \subset \mathfrak{P}$. Note also that if \mathfrak{M} is a Kusuoka set, then its topological closure is a Kusuoka set too (cf. Shapiro [23, Proposition 1]).

Outline of the paper. The paper is organized as follows. In the next section we introduce the notation and discuss minimality and uniqueness of the Kusuoka representations. In Section 4 we consider Kusuoka representations on general, not necessarily nonatomic, probability spaces. In Section 5 we investigate maximality of Kusuoka representations with respect to order, or dominance relations. In Section 6 we discuss some examples, while Section 7 is devoted to conclusions.

3 Uniqueness of Kusuoka sets

It is known that the Kusuoka representation is not unique in general. In this section we elaborate that there is minimal Kusuoka representation in the sense outlined below. To obtain the result we shall relate the Kusuoka representation to distortion premiums (Wang premiums) first and then outline the results.

We can view the integral in the right hand side of (2.2) as the Lebesgue-Stieltjes integral with $\mu : \mathbb{R} \rightarrow [0, 1]$ being a right side continuous monotonically nondecreasing function (distribution function) such that $\mu(t) = 0$ for $t < 0$ and $\mu(t) = 1$ for $t \geq 1$. For example, take $\mu(t) = 0$ for $t < 0$ and $\mu(t) = 1$ for $t \geq 0$. This is a distribution function corresponding to a measure of mass one at $\alpha = 0$. In that case

$$\int_0^1 \text{CTE}_\alpha(Z) d\mu(\alpha) = \text{CTE}_0(Z) = \mathbb{E}[Z],$$

where the integral is understood as taken from 0^- to 1.

Note: When considering integrals of the form $\int_0^\gamma h(\alpha) d\mu(\alpha)$, $\gamma \geq 0$, with respect to a distribution function $\mu(\cdot)$ we always assume that the integral is taken from 0^- , i.e., a neighborhood of 0 is always included in the interval for integration.

Definition 3.1 (Measure-preserving transformation) *We say that $T : \Omega \rightarrow \Omega$ is a measure-preserving transformation if T is one-to-one, onto, measurable and for any $A \in \mathcal{F}$ it follows that $P(A) = P(T^{-1}(A))$. We denote by \mathfrak{F} the group of measure-preserving transformations.*

This definition of measure-preserving transformations is slightly stronger than the standard one in that we also assume that T is one-to-one and onto; hence T is invertible and for any $A \in \mathcal{F}$ it follows that $T(A) \in \mathcal{F}$ and $P(A) = P(T(A))$.

- We refer to the probability space $\Omega = [0, 1]$ equipped with its Borel sigma algebra and uniform probability measure P , as the *standard probability space*. In the nonatomic case without loss of generality it suffices to consider the standard probability space. Unless stated otherwise we assume in the remainder of this section that (Ω, \mathcal{F}, P) is the standard probability space.

Remark 1 Note that two random variables $Z_1, Z_2 : \Omega \rightarrow \mathbb{R}$ have the same probability distribution iff there exists a measure-preserving transformation $T \in \mathfrak{F}$ such that² $Z_2 = Z_1 \circ T$ (e.g., Jouini et al. [10]). It follows that a random variable $Z : \Omega \rightarrow \mathbb{R}$ is distributionally equivalent to F_Z^{-1} , i.e., there exists $T \in \mathfrak{F}$ such that a.e. $F_Z^{-1} = Z \circ T$ (e.g., [23]).

Definition 3.2 (Distortion function, cf. Bellini and Caperdoni [2]) *We say that $\sigma : [0, 1) \rightarrow [0, \infty)$ is a distortion function (or spectral function) if $\sigma(\cdot)$ is right side continuous, monotonically nondecreasing and such that $\int_0^1 \sigma(t) dt = 1$. The set of distortion functions is denoted by \mathfrak{D} .*

Consider the linear mapping $\mathbb{T} : \mathfrak{P} \rightarrow \mathfrak{D}$ defined as

$$(\mathbb{T}\mu)(\tau) := \int_0^\tau (1 - \alpha)^{-1} d\mu(\alpha), \quad \tau \in [0, 1).$$

²The notation $(Z \circ T)(\omega)$ stands for the composition $Z(T(\omega))$.

This mapping is onto, one-to-one with the inverse $\mu = \mathbb{T}^{-1}\sigma$ given by

$$\mu(\alpha) = (\mathbb{T}^{-1}\sigma)(\alpha) = (1 - \alpha)\sigma(\alpha) + \int_0^\alpha \sigma(\tau)d\tau, \quad \alpha \in [0, 1]$$

(cf. [23, Lemma 2]). In particular, let $\mu := \sum_{i=1}^n c_i \delta_{\alpha_i}$, where δ_α denotes the probability measure of mass one at α , c_i are positive numbers such that $\sum_{i=1}^n c_i = 1$, and $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n < 1$. Then

$$\mathbb{T}\mu = \sum_{i=1}^n \frac{c_i}{1 - \alpha_i} \mathbf{1}_{[\alpha_i, 1]},$$

where $\mathbf{1}_A$ denotes the indicator function of the set A .

A (real valued) premium $\pi : \mathcal{Z} \rightarrow \mathbb{R}$ is continuous in the norm topology of $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$ (cf. Ruzsչyński and Shapiro [21]), and by the Fenchel-Moreau theorem has the dual representation

$$\pi(Z) = \sup_{\zeta \in \mathfrak{A}} \langle \zeta, Z \rangle, \quad Z \in \mathcal{Z}, \quad (3.1)$$

where $\mathfrak{A} \subset \mathcal{Z}^*$ is a convex and weakly* compact set of density functions and the scalar product on $\mathcal{Z}^* \times \mathcal{Z}$ is defined as $\langle \zeta, Z \rangle := \int_0^1 \zeta(\tau)Z(\tau)d\tau$. Since π is law invariant, the dual set \mathfrak{A} is invariant under the group \mathfrak{F} of measure preserving transformations, i.e., if $\zeta \in \mathfrak{A}$, then $\zeta \circ T \in \mathfrak{A}$ for any $T \in \mathfrak{F}$ (cf. [23]).

Note that if

$$\pi(Z) = \sup_{\zeta \in \mathfrak{C}} \langle \zeta, Z \rangle, \quad Z \in \mathcal{Z}, \quad (3.2)$$

holds for some set $\mathfrak{C} \subset \mathcal{Z}^*$, then $\mathfrak{C} \subset \mathfrak{A}$.

For a set $\Upsilon \subset \mathcal{Z}^*$ we denote

$$\mathcal{O}(\Upsilon) := \{\zeta \circ T : T \in \mathfrak{F}, \zeta \in \Upsilon\}$$

its orbit with respect to the group \mathfrak{F} .

Consider the following definition (cf. [23, Definition 1]).

Definition 3.3 (Generating sets) *We say that a set $\Upsilon \subset \mathcal{D}$ of distortion functions is a generating set if the representation (3.2) holds for $\mathfrak{C} := \mathcal{O}(\Upsilon)$. That is,*

$$\pi(Z) = \sup_{\sigma \in \Upsilon} \int_0^1 \sigma(\tau)F_Z^{-1}(\tau)d\tau, \quad Z \in \mathcal{Z}. \quad (3.3)$$

It follows that if Υ is a generating set, then $\mathcal{O}(\Upsilon) \subset \mathfrak{A}$.

Proposition 3.1 *A set $\mathfrak{M} \subset \mathfrak{P}$ is a Kusuoka set iff the set $\Upsilon := \mathbb{T}(\mathfrak{M})$ is a generating set.*

Proof. By using the integral representation (2.1) of CTE we can write

$$\int_0^1 \text{CTE}_\alpha(Z)d\mu(\alpha) = \int_0^1 \int_\alpha^1 (1 - \alpha)^{-1} F_Z^{-1}(\tau)d\tau d\mu(\alpha) = \int_0^1 (\mathbb{T}\mu)(\tau)F_Z^{-1}(\tau)d\tau.$$

Therefore representation (2.2) can be stated as

$$\pi(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 (\mathbb{T}\mu)(\tau)F_Z^{-1}(\tau)d\tau.$$

Together with (3.3) this shows that \mathfrak{M} is a Kusuoka set iff $\mathbb{T}(\mathfrak{M})$ is a generating set. ■

Definition 3.4 (Exposed points) *It is said that $\bar{\zeta}$ is a weak* exposed point of $\mathfrak{A} \subset \mathcal{Z}^*$ if there exists $Z \in \mathcal{Z}$ such that $g_Z(\zeta) := \langle \zeta, Z \rangle$ attains its maximum over $\zeta \in \mathfrak{A}$ at the unique point $\bar{\zeta}$. In that case we say that Z exposes \mathfrak{A} at $\bar{\zeta}$. We denote by $\text{Exp}(\mathfrak{A})$ the set of exposed points of \mathfrak{A} .*

A result going back to Mazur [14] says that if \mathcal{X} is a separable Banach space and $K \subset \mathcal{X}^*$ is a nonempty weakly* compact subset of \mathcal{X}^* , then the set of points $x \in \mathcal{X}$ which expose K at some point $x^* \in K$ is a dense (in the norm topology) subset of \mathcal{X} (see, e.g., Larman and Phelps [13] for a discussion of these type results). Since the space $\mathcal{Z} = L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, is *separable* we have the following theorem:³

Theorem 3.1 *Let \mathfrak{A} be the dual set in (3.1). Then the set*

$$\mathcal{D} := \{Z \in \mathcal{Z} : Z \text{ exposes } \mathfrak{A} \text{ at a point } \bar{\zeta}\} \quad (3.4)$$

is a dense (in the norm topology) subset of \mathcal{Z} .

This allows to proceed towards a minimal representation of a risk measure as follows.

Proposition 3.2 *Let $\text{Exp}(\mathfrak{A})$ be the set of exposed points of \mathfrak{A} . Then the representation (3.2) holds with $\mathfrak{C} := \text{Exp}(\mathfrak{A})$. Moreover, if the representation (3.2) holds for some weakly* closed set \mathfrak{C} , then $\text{Exp}(\mathfrak{A}) \subset \mathfrak{C}$.*

Proof. Consider the set \mathcal{D} defined in (3.4). By Theorem 3.1 this set is dense in \mathcal{Z} . So for $Z \in \mathcal{Z}$ fixed, let $\{Z_n\} \subset \mathcal{D}$ be a sequence of points converging (in the norm topology) to Z . Let $\{\zeta_n\} \subset \text{Exp}(\mathfrak{A})$ be a sequence of the corresponding maximizers, i.e., $\pi(Z_n) = \langle \zeta_n, Z_n \rangle$. Since \mathfrak{A} is bounded, we have that $\|\zeta_n\|^*$ is uniformly bounded. Since $\pi : \mathcal{Z} \rightarrow \mathbb{R}$ is real valued it is continuous (cf. [21]), and thus $\pi(Z_n) \rightarrow \pi(Z)$. We also have that

$$|\pi(Z_n) - \langle \zeta_n, Z \rangle| = |\langle \zeta_n, Z_n \rangle - \langle \zeta_n, Z \rangle| \leq \|\zeta_n\|^* \|Z_n - Z\| \rightarrow 0.$$

It follows that

$$\pi(Z) = \sup\{\langle \zeta_n, Z \rangle : n = 1, \dots\},$$

and hence the representation (3.2) holds with $\mathfrak{C} = \text{Exp}(\mathfrak{A})$.

Let \mathfrak{C} be a weakly* closed set such that the representation (3.2) holds. It follows that \mathfrak{C} is a subset of \mathfrak{A} and is weakly* compact. Consider a point $\zeta \in \text{Exp}(\mathfrak{A})$. By the definition of the set $\text{Exp}(\mathfrak{A})$, there is $Z \in \mathcal{D}$ such that $\pi(Z) = \langle \zeta, Z \rangle$. Since \mathfrak{C} is weakly* compact, the maximum in (3.2) is attained and hence $\pi(Z) = \langle \zeta', Z \rangle$ for some $\zeta' \in \mathfrak{C}$. By the uniqueness of the maximizer ζ it follows that $\zeta' = \zeta$. This shows that $\text{Exp}(\mathfrak{A}) \subset \mathfrak{C}$. ■

For a set $A \subset \mathcal{Z}^*$ we denote by \overline{A} the topological closure of A in the weak* topology of the space \mathcal{Z}^* . By the above proposition, $\overline{\text{Exp}(\mathfrak{A})}$ coincides with the intersection of all weakly* closed sets \mathfrak{C} satisfying representation (3.2).

³We need here the dual set \mathfrak{A} to be compact, this is why we assume in this section that π is real valued.

Proposition 3.3 *The sets $\text{Exp}(\mathfrak{A})$ and $\overline{\text{Exp}(\mathfrak{A})}$ are invariant under the group \mathfrak{F} of measure preserving transformations.*

Proof. For $T \in \mathfrak{F}$ we have that

$$\langle \zeta \circ T, Z \rangle = \int_0^1 \zeta(T(\tau))Z(\tau)d\tau = \int_0^1 \zeta(\tau)Z(T^{-1}(\tau))d\tau = \langle \zeta, Z \circ T^{-1} \rangle.$$

Thus $\bar{\zeta} \in \mathfrak{A}$ is a maximizer of $\langle \zeta, Z \rangle$ over $\zeta \in \mathfrak{A}$, iff $\bar{\zeta} \circ T$ is a maximizer of $\langle \zeta, T^{-1} \circ Z \rangle$ over $\zeta \in \mathfrak{A}$. Consequently we have that if $\bar{\zeta} \in \text{Exp}(\mathfrak{A})$, then $\bar{\zeta} \circ T \in \text{Exp}(\mathfrak{A})$, i.e., $\text{Exp}(\mathfrak{A})$ is invariant under \mathfrak{F} .

Now consider a point $\bar{\zeta} \in \overline{\text{Exp}(\mathfrak{A})}$. Then there is a sequence $\zeta_n \in \text{Exp}(\mathfrak{A})$ such that $\langle \zeta_n, Z \rangle$ converges to $\langle \bar{\zeta}, Z \rangle$ for any $Z \in \mathcal{Z}$. For any $T \in \mathfrak{F}$ we have that $\zeta_n \circ T \in \text{Exp}(\mathfrak{A})$, and hence $\langle \zeta_n \circ T, Z \rangle = \langle \zeta_n, Z \circ T^{-1} \rangle$ converges to $\langle \bar{\zeta}, Z \circ T^{-1} \rangle = \langle \bar{\zeta} \circ T, Z \rangle$ for any $Z \in \mathcal{Z}$. It follows that $\bar{\zeta} \circ T \in \overline{\text{Exp}(\mathfrak{A})}$. \blacksquare

By the latter proposition we have that for any exposed point there is a distributional equivalent which is a monotonically nondecreasing function. We employ this observation in the following Corollary 3.1 to reduce the generating set.

Definition 3.5 *Denote by \mathfrak{E} the subset of $\text{Exp}(\mathfrak{A})$ formed by right side continuous monotonically nondecreasing functions, i.e., $\mathfrak{E} := \mathfrak{D} \cap \text{Exp}(\mathfrak{A})$.*

It follows by Proposition 3.2 that the topological closure $\bar{\mathfrak{E}}$ (of the set \mathfrak{E}) is the unique minimal generating set in the following sense.

Corollary 3.1 *The set \mathfrak{E} is a generating set. Moreover, if Υ is any weakly* closed generating set, then the set $\bar{\mathfrak{E}}$ is a subset of Υ , i.e., $\bar{\mathfrak{E}}$ coincides with the intersection of all weakly* closed generating sets.*

As before, let (Ω, \mathcal{F}, P) be the standard probability space and consider the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$, $p \in [1, \infty)$, its dual $\mathcal{Z}^* = L_q(\Omega, \mathcal{F}, P)$, and the set

$$\mathfrak{P}_q := \{\mu \in \mathfrak{P} : \mathbb{T}\mu \in \mathcal{Z}^*\}.$$

Proposition 3.4 *The set \mathfrak{P}_q is closed and the mapping \mathbb{T} is continuous on the set \mathfrak{P}_q with respect to the weak topology of measures and the weak* topology of \mathcal{Z}^* .*

Proof. Let $\{\mu_k\} \subset \mathfrak{P}_q$ be a sequence of measures converging in the weak topology to $\mu \in \mathfrak{P}$, and define $\zeta_k := \mathbb{T}\mu_k$ and $\zeta := \mathbb{T}\mu$. For $Z \in \mathcal{Z}$ we have that

$$\int_0^1 Z(t)\zeta_k(t)dt = \int_0^1 \int_0^t Z(t)(1-\alpha)^{-1}d\mu_k(\alpha)dt = \int_0^1 (1-\alpha)^{-1}h_Z(\alpha)d\mu_k(\alpha),$$

where $h_Z(\alpha) := \int_\alpha^1 Z(t)dt$. Note that $|h_Z(\alpha)| \leq \|Z\|_q$ for $\alpha \in [0, 1]$, and the function $\alpha \mapsto h_Z(\alpha)$ is continuous. Let us choose a dense set $\mathcal{V} \subset \mathcal{Z}$ such that for any $Z \in \mathcal{V}$ the corresponding function $(1-\alpha)^{-1}h_Z(\alpha)$ is bounded on $[0, 1]$. For example, we can take

functions $Z \in \mathcal{Z}$ such that $Z(t) = 0$ for $t \in [\gamma, 1]$ with $\gamma \in (0, 1)$. Then for any $Z \in \mathcal{V}$ we have that

$$\lim_{k \rightarrow \infty} \int_0^1 Z(t) \zeta_k(t) dt = \lim_{k \rightarrow \infty} \int_0^1 (1 - \alpha)^{-1} h_Z(\alpha) d\mu_k(\alpha) = \int_0^1 (1 - \alpha)^{-1} h_Z(\alpha) d\mu(\alpha),$$

and hence

$$\lim_{k \rightarrow \infty} \int_0^1 Z(t) \zeta_k(t) dt = \int_0^1 Z(t) \zeta(t) dt, \quad (3.5)$$

with the integral in the right hand side of (3.5) being finite. This shows that \mathbb{T} is continuous on μ and $\zeta \in \mathcal{Z}^*$, and hence $\mu \in \mathfrak{P}_q$. \blacksquare

It follows that if $\mathfrak{C} \subset \mathfrak{A}$ is a weakly* closed set of right side continuous monotonically nondecreasing functions, then $\mathbb{T}^{-1}(\mathfrak{C})$ is a closed subset of \mathfrak{P} .

Definition 3.6 (Minimality) *We say that a Kusuoka set $\mathfrak{M} \subset \mathfrak{P}$ is minimal if \mathfrak{M} is closed and for any closed Kusuoka set \mathfrak{M}' it holds that $\mathfrak{M} \subset \mathfrak{M}'$. We say that a generating set Υ is minimal if Υ is weakly* closed and for any weakly* closed generating set Υ' it holds that $\Upsilon \subset \Upsilon'$.*

Note that it follows from the above definition that if a minimal Kusuoka set exists, then it is unique.

By Propositions 3.1 and 3.4 we have that a Kusuoka set \mathfrak{M} is minimal iff the set $\mathbb{T}(\mathfrak{M})$ is a minimal generating set. Together with Theorem 3.1 and Proposition 3.2 this implies the following result. Recall the set \mathfrak{E} described in Definition 3.5.

Theorem 3.2 *The set $\mathfrak{M} := \mathbb{T}^{-1}(\mathfrak{E})$ is a Kusuoka set and its closure $\overline{\mathfrak{M}} = \overline{\mathbb{T}^{-1}(\mathfrak{E})} = \mathbb{T}^{-1}(\overline{\mathfrak{E}})$ is the minimal Kusuoka set.*

This shows that the minimal Kusuoka set indeed exists and, as it was mentioned before, by the definition is unique. In particular we obtain the following result (cf. [23]).

Corollary 3.2 *Let $\pi_\sigma : \mathcal{Z} \rightarrow \mathbb{R}$ be a distortion premium (Wang premium) with distortion function $\sigma \in \mathcal{Z}^* \cap \mathfrak{D}$, i.e.,*

$$\pi_\sigma(Z) = \int_0^1 \sigma(t) F_Z^{-1}(t) dt.$$

Then its minimal Kusuoka set is given by the singleton $\{\mathbb{T}^{-1}\sigma\}$.

4 Premium principles on general spaces

In the previous section we considered cases when the reference probability space is nonatomic. Of course this rules out, for example, discrete probability spaces. So what can be said about premiums defined on probability spaces with atoms? Let us consider the following construction. Suppose that the reference probability space can be embedded into the standard probability space. That is, let (Ω, \mathcal{F}, P) be the standard probability space and \mathcal{G} be a sigma

subalgebra of \mathcal{F} such that the considered reference probability space is equivalent (isomorphic) to (Ω, \mathcal{G}, P) . So we can view (Ω, \mathcal{G}, P) as the reference probability space. For example, let the reference probability space be discrete with a countable (finite) number of elementary events and respective probabilities p_1, p_2, \dots . Let us partition $\Omega = [0, 1]$ into intervals A_1, A_2, \dots of respective lengths p_1, p_2, \dots , and consider the sigma algebra $\mathcal{G} \subset \mathcal{F}$ generated by these intervals. This will give the required embedding of the discrete probability space into the standard probability space.

Consider the space $\hat{\mathcal{Z}} := L_p(\Omega, \mathcal{G}, P)$, $p \in [1, \infty)$, and a law invariant risk measure (premium) $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$. Note that ϱ is supposed to be law invariant with respect to the *reference* probability space. That is, if $Z_1, Z_2 \in \hat{\mathcal{Z}}$ are two \mathcal{G} -measurable distributionally equivalent random variables, then $\varrho(Z_1) = \varrho(Z_2)$. Note that \mathcal{G} -measurability of $Z \in \hat{\mathcal{Z}}$ does not necessarily imply \mathcal{F} -measurability of F_Z^{-1} . Note also that the space $\hat{\mathcal{Z}}$ is a subspace of the space $\mathcal{Z} := L_p(\Omega, \mathcal{F}, P)$. Therefore a relevant question is whether it is possible to extend ϱ to a premium on the space \mathcal{Z} . For a premium $\pi : \mathcal{Z} \rightarrow \overline{\mathbb{R}}$ we denote by $\pi|_{\hat{\mathcal{Z}}}$ the restriction of π to the space $\hat{\mathcal{Z}}$, i.e., $\hat{\pi} = \pi|_{\hat{\mathcal{Z}}}$ is defined on the space $\hat{\mathcal{Z}}$ and $\hat{\pi}(Z) = \pi(Z)$ for $Z \in \hat{\mathcal{Z}}$. Note that in this section we consider premiums which may take the value $+\infty$.

Definition 4.1 (Regularity) *We say that a proper lower semicontinuous law invariant coherent (convex) premium $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is regular if there exists a proper lower semicontinuous law invariant coherent (convex) premium $\pi : \mathcal{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\pi|_{\hat{\mathcal{Z}}} = \varrho$.*

A similar definition was given in Noyan and Rudolf [16, Definition 6.1].

For example, the Conditional Tail Expectation and the Dutch premium principle (cf. van Heerwaarden and Kaas [26], also called mean-semideviation measures) are regular premiums irrespective of the reference probability space. We denote by \mathfrak{G} the set of measure-preserving transformations of the *reference* space (Ω, \mathcal{G}, P) . Note that \mathfrak{G} is a subgroup of the group \mathfrak{F} of measure-preserving transformations of the standard probability space (Ω, \mathcal{F}, P) .

For a regular law invariant coherent premium $\varrho = \hat{\pi}$ we can write for $Z \in \hat{\mathcal{Z}}$ the dual representation

$$\varrho(Z) = \sup_{\sigma \in \Upsilon} \int_0^1 \sigma(t) F_Z^{-1}(t) dt, \quad (4.1)$$

where Υ is a generating set of the corresponding premium principle $\pi : \mathcal{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$, and the respective Kusuoka representation

$$\varrho(Z) = \sup_{\mu \in \mathfrak{M}} \int_0^1 \text{CTE}_\alpha(Z) d\mu(\alpha), \quad (4.2)$$

where \mathfrak{M} is the set of probability measures corresponding to the premium $\pi : \mathcal{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$. Conversely, if the representation (4.1) or the Kusuoka representation (4.2) holds and the right hand side of (4.1) (or (4.2)) is well defined and finite for every $Z \in \mathcal{Z}$, then ϱ is a regular law invariant coherent premium. Hence we have the following.

Proposition 4.1 *A proper lower semicontinuous law invariant premium $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is regular iff there exists a set $\Upsilon \subset L_q(\Omega, \mathcal{F}, P)$ of distortion functions such that the representation (4.1) holds, or equivalently iff the Kusuoka representation (4.2) holds.*

Let us give now sufficient conditions for the regularity of a coherent premium principle defined on a reference probability space. Consider the following condition.

- (B) For any \mathcal{G} -measurable random variable $Z : [0, 1] \rightarrow \mathbb{R}$ there exists measure preserving transformation $T \in \mathfrak{G}$, of the reference space (Ω, \mathcal{G}, P) , such that $Z \circ T$ is monotonically nondecreasing.

Proposition 4.2 *Suppose that condition (B) holds. Then every proper lower semicontinuous law invariant coherent premium $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is regular.*

Proof. Let $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous law invariant coherent premium. It has the dual representation

$$\varrho(Z) = \sup_{\zeta \in \hat{\mathfrak{A}}} \int_0^1 \zeta(t) Z(t) dt, \quad Z \in \hat{\mathcal{Z}}, \quad (4.3)$$

where $\hat{\mathfrak{A}} \subset \hat{\mathcal{Z}}^*$ is the corresponding dual set of density functions. Since ϱ is law invariant, the set $\hat{\mathfrak{A}}$ is invariant with respect to measure-preserving transformations $T \in \mathfrak{G}$ (e.g., Shapiro [22]).

Suppose that condition (B) holds. Consider an element $Y \in \hat{\mathcal{Z}}$. By condition (B) there exists $T \in \mathfrak{G}$ such that $Z = Y \circ T$ is monotonically nondecreasing. It follows that $Z \in \hat{\mathcal{Z}}$ and $Z \stackrel{\mathcal{D}}{\sim} Y$, and hence $\varrho(Z) = \varrho(Y)$, and that $Z = F_Y^{-1}$. So let $Z \in \hat{\mathcal{Z}}$ be monotonically nondecreasing and consider an element $\zeta \in \hat{\mathfrak{A}}$. By condition (B) there exists $T \in \mathfrak{G}$ such that $\eta = \zeta \circ T$ is a monotonically nondecreasing function. Since $\hat{\mathfrak{A}}$ is invariant with respect to transformations of the group \mathfrak{G} , it follows that $\eta \in \hat{\mathfrak{A}}$. Also by monotonicity of Z we have that $\int_0^1 \zeta(t) Z(t) dt \leq \int_0^1 \eta(t) Z(t) dt$, and hence it suffices to take the maximum in (4.3) with respect to $\zeta \in \hat{\mathfrak{A}}$, where $\hat{\mathfrak{Y}}$ is the subset of $\hat{\mathfrak{A}}$ formed by monotonically nondecreasing $\eta \in \hat{\mathfrak{A}}$. Define now

$$\pi(Z) := \sup_{\eta \in \hat{\mathfrak{Y}}} \int_0^1 \eta(t) F_Z^{-1}(t) dt, \quad Z \in \hat{\mathcal{Z}}.$$

We have that $\pi : \hat{\mathcal{Z}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous law invariant coherent premium principle and $\pi|_{\hat{\mathcal{Z}}} = \varrho$. This shows that ϱ is regular. \blacksquare

Suppose now that the reference probability space is finite, $\Omega = \{\omega_1, \dots, \omega_n\}$. It can be embedded into the standard probability space. The corresponding space $\hat{\mathcal{Z}}$ consists of all functions $Z : \{\omega_1, \dots, \omega_n\} \rightarrow \mathbb{R}$. If all respective probabilities p_1, \dots, p_n are equal to each other, then the associated group \mathfrak{G} of measure preserving transformations is given by the set of permutations of $\{\omega_1, \dots, \omega_n\}$. It follows that condition (B) of Proposition 4.2 holds and hence we have the following result (cf., [17]).

Corollary 4.1 *Let the reference probability space $\{\omega_1, \dots, \omega_n\}$ be finite, equipped with equal probabilities $p_i = 1/n$, $i = 1, \dots, n$. Then every law invariant coherent risk measure $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R}$ is regular, and hence has a Kusuoka representation.*

In Pflug and Römisch [18, p.63] is given an example of a law invariant coherent risk measure defined on space of random variables $Z : \hat{\Omega} \rightarrow \mathbb{R}$, with $\hat{\Omega} = \{\omega_1, \omega_2\}$ having two elements with unequal probabilities $p_1 \neq p_2$, for which the Kusuoka representation does not hold. We give now an example of a class of law invariant coherent risk measures which are not regular, and hence do not have a Kusuoka representation.

Example 1 Consider the above framework where the reference probability space is embedded into the standard probability space, and $\hat{\mathcal{Z}} = L_p(\Omega, \mathcal{G}, P)$. Suppose that the reference probability space (Ω, \mathcal{G}, P) has atoms. Thus there exists $r > 0$ such that the set $A^r := \{\omega \in \Omega : P(\{\omega\}) = r\}$, of all atoms having probability r , is nonempty. Clearly the set A^r is finite, let $N := |A^r|$ be the cardinality of A^r (the set A^r could be a singleton, i.e., it could be that $N = 1$). The set $A^r = \cup_{i=1}^N A_i$, where $A_i \subset \Omega$, $i = 1, \dots, N$, are intervals of length r . Note that since $Z \in \hat{\mathcal{Z}}$ is \mathcal{G} -measurable, $Z(\cdot)$ is constant on each A_i ; we denote by $Z(\omega)$ this constant when $\omega \in A_i$.

Assume that the reference probability space is not a finite set equipped with equal probabilities, so that $P(\Omega \setminus A^r) > 0$. Define

$$\varrho(Z) := \frac{1}{N} \sum_{\omega \in A^r} Z(\omega), \quad Z \in \hat{\mathcal{Z}}. \quad (4.4)$$

Clearly $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R}$ is a coherent risk measure. Suppose further that the following condition holds.

- (C) If $Z, Z' \in \hat{\mathcal{Z}}$ are distributionally equivalent, then there exists a permutation π of the set A^r such that $Z'(\omega) = Z(\pi(\omega))$ for any $\omega \in A^r$.

Under this condition, the risk measure ϱ is law invariant as well. Let us show that it is not regular.

Let us argue by a contradiction. Indeed, suppose that $\varrho = \hat{\pi}$ for some proper lower semicontinuous law invariant coherent risk measure $\pi : \mathcal{Z} \rightarrow \mathbb{R} \cup \{+\infty\}$ and set $\hat{\pi} := \pi|_{\hat{\mathcal{Z}}}$. Note that $\varrho(Z)$ depends only on values of $Z(\cdot)$ on the set A^r , i.e., for any $Z, Z' \in \hat{\mathcal{Z}}$ we have that $\pi(Z) = \pi(Z')$ if $Z(\omega) = Z'(\omega)$ for all $\omega \in A^r$. In particular, if $Z(\omega) = 0$ for all $\omega \in A^r$, then $\pi(Z) = 0$. It follows that $\pi(\mathbf{1}_B) = 0$ for any \mathcal{G} -measurable set $B \subset \Omega \setminus A^r$. Note that the set $\Omega \setminus A^r$ has a nonempty interior since the set A^r is a union of a finite number of intervals and $P(\Omega \setminus A^r) > 0$, and hence we can take $B \subset \Omega \setminus A^r$ to be an open interval. Since π is law invariant it follows that $\pi(\mathbf{1}_{T(B)}) = 0$ for any measure-preserving transformation $T : \Omega \rightarrow \Omega$ of the standard uniform probability space. Hence there is a family of sets $B_i \in \mathcal{F}$, $i = 1, \dots, m$, such that $\pi(\mathbf{1}_{B_i}) = 0$, $i = 1, \dots, m$, and $[0, 1] = \cup_{i=1}^m B_i$. It follows that for any bounded $Z \in \mathcal{Z}$ there are $c_i > 0$, $i = 1, \dots, m$, such that $Z \leq \sum_{i=1}^m c_i \mathbf{1}_{B_i}$. Consequently

$$\pi(Z) \leq \pi\left(\sum_{i=1}^m c_i \mathbf{1}_{B_i}\right) \leq \sum_{i=1}^m c_i \pi(\mathbf{1}_{B_i}) = 0,$$

this clearly is a contradiction.

It follows that the risk measure ϱ , defined in (4.4), does not have a Kusuoka representation. Note that it was only essential in the above construction that the restriction of the risk measure $\varrho : \hat{\mathcal{Z}} \rightarrow \mathbb{R}$ to the set A^r is a law invariant coherent risk measure. So, for example, under the above assumptions the risk measure $\varrho(Z) := \max\{Z(\omega) : \omega \in A^r\}$ is also not regular.

As far as condition (C) is concerned, suppose for example that the reference space is finite, equipped with respective probabilities p_1, \dots, p_n . For some $r \in \{p_1, \dots, p_n\}$ let $\mathcal{I}_r := \{i : p_i = r, i = 1, \dots, n\}$. Suppose that

$$\sum_{i \in I} p_i \neq \sum_{j \in J} p_j, \quad \forall I \subset \mathcal{I}_r, \quad \forall J \subset \{1, \dots, n\} \setminus \mathcal{I}_r. \quad (4.5)$$

Then condition (C) holds for the set $A^r := \{\omega_i : i \in \mathcal{I}_r\}$. That is, condition (4.5) ensures existence of a nonregular premium ϱ .

5 Maximality of Kusuoka sets

In this section we discuss Kusuoka representations with respect to stochastic dominance relations and investigate the maximality of these sets with respect to those orders. As before we consider probability measures μ supported on the interval $[0, 1]$ and continue identifying the measure μ with its cumulative distribution function $\mu(\alpha) = \mu\{(-\infty, \alpha]\}$, $\alpha \in \mathbb{R}$. Recall that since μ is supported on $[0, 1]$, it follows that $\mu(\alpha) = 0$ for $\alpha < 0$ and $\mu(\alpha) = 1$ for $\alpha \geq 1$.

Definition 5.1 *It is said that μ_1 is dominated in first stochastic order by μ_2 , denoted $\mu_1 \preceq \mu_2$, if $\mu_1(\alpha) \geq \mu_2(\alpha)$ for all $\alpha \in \mathbb{R}$. If moreover, $\mu_1 \neq \mu_2$ we write $\mu_1 \prec \mu_2$.*

A set of measures (\mathfrak{M}, \preceq) , equipped with the first order stochastic dominance relation, is a partially ordered set.

In the space of functions $\sigma \in \mathcal{Z}^*$ we consider the following dominance relation.

Definition 5.2 *For $\sigma_1, \sigma_2 \in \mathcal{Z}^*$ it is said that σ_1 is majorized by σ_2 , denoted $\sigma_1 \preceq \sigma_2$, if*

$$\int_{\gamma}^1 \sigma_1(t) dt \leq \int_{\gamma}^1 \sigma_2(t) dt \quad \text{for all } \gamma \in [0, 1], \quad \text{and} \\ \int_0^1 \sigma_1(t) dt = \int_0^1 \sigma_2(t) dt.$$

Remark 2 For monotonically nondecreasing functions $\sigma_1, \sigma_2 : [0, 1] \rightarrow \mathbb{R}$ the above concept of majorization is closely related to the concept of dominance in the convex order. That is, if σ_1 and σ_2 are monotonically nondecreasing right side continuous functions, then they can be viewed as right side quantile functions $\sigma = F_{Z_1}^{-1}$ and $\sigma_2 = F_{Z_2}^{-1}$ of some respective random variables Z_1 and Z_2 . It is said that Z_1 dominates Z_2 in the convex order if $\mathbb{E}[u(Z_1)] \geq \mathbb{E}[u(Z_2)]$ for all convex functions $u : \mathbb{R} \rightarrow \mathbb{R}$ such that the expectations exist. Equivalently this can be written as (see, e.g., Müller and Stoyan [15])

$$\int_{\gamma}^1 F_{Z_1}^{-1}(t) dt \geq \int_{\gamma}^1 F_{Z_2}^{-1}(t) dt \quad \text{for all } \gamma \in [0, 1], \quad \text{and } \mathbb{E}[Z_1] = \mathbb{E}[Z_2].$$

The dominance in the convex (concave) order was used in studying risk measures in Föllmer and Schied [9] and Dana [5], for example.

Note that if $\sigma_1, \sigma_2 \in \mathcal{Z}^* \cap \mathfrak{D}$ and $\sigma_1 \preceq \sigma_2$, then

$$\int_0^1 \sigma_1(t) F_Z^{-1}(t) dt \leq \int_0^1 \sigma_2(t) F_Z^{-1}(t) dt, \quad Z \in \mathcal{Z}. \quad (5.1)$$

Example 2 Let π be given by maximum of a finite number of distortion premiums (Wang premiums), i.e.,

$$\pi(Z) := \max_{1 \leq i \leq n} \int_0^1 \sigma_i(t) F_Z^{-1}(t) dt, \quad Z \in \mathcal{Z},$$

for some $\sigma_i \in \mathcal{Z}^* \cap \mathfrak{D}$, $i = 1, \dots, n$. Then the set $\Upsilon := \{\sigma_1, \dots, \sigma_n\}$ is a generating set and the convex hull of $\mathcal{O}(\Upsilon)$ is the dual set of π . An element σ_i of $\{\sigma_1, \dots, \sigma_n\}$ is an exposed point of the dual set iff σ_i can be strongly separated from the other σ_j , i.e., iff there exists $Z \in \mathcal{Z}$ such that $\langle \sigma_i, Z \rangle \geq \langle \sigma_j, Z \rangle + \varepsilon$, for some $\varepsilon > 0$ and all $j \neq i$. So the generating set Υ is minimal iff every σ_i can be strongly separated from the other σ_j .

If there is σ_i which is majorized by another σ_j , then it follows by (5.1) that removing σ_i from the set Υ will not change the corresponding maximum, and hence $\Upsilon \setminus \{\sigma_i\}$ is still a generating set. Therefore the condition: “every σ_i is not majorized by any other σ_j ” is necessary for the set Υ to be minimal. For $n = 2$ this condition is also sufficient. However, it is not sufficient already for $n = 3$. For example let σ_1 and σ_2 be such that σ_1 is not majorized by σ_2 , and σ_2 is not majorized by σ_1 , and let $\sigma_3 := (\sigma_1 + \sigma_2)/2$. Then clearly σ_3 can be removed from Υ , while σ_3 is not majorized by σ_1 or σ_2 . Indeed, if σ_3 is majorized say by σ_1 , then

$$\int_{\gamma}^1 \sigma_3(t) dt = \frac{1}{2} \left(\int_{\gamma}^1 \sigma_1(t) dt + \int_{\gamma}^1 \sigma_2(t) dt \right) \leq \int_{\gamma}^1 \sigma_1(t) dt, \quad \gamma \in [0, 1],$$

and hence

$$\int_{\gamma}^1 \sigma_2(t) dt \leq \int_{\gamma}^1 \sigma_1(t) dt, \quad \gamma \in [0, 1],$$

a contradiction with the condition that σ_2 is not majorized by σ_1 .

We show now that first order dominance of measures is transformed by the operator \mathbb{T} into majorization order in the sense of Definition 5.2. The converse implication,

$$\{\sigma_1 \preceq \sigma_2\} \Rightarrow \{\mathbb{T}^{-1}\sigma_1 \preceq \mathbb{T}^{-1}\sigma_2\}$$

does not hold in general. A simple counterexample is provided by the measures $\mu_1 := 0.1 \delta_{0.2} + 0.9 \delta_{0.6}$ and $\mu_2 := 0.2 \delta_{0.5} + 0.8 \delta_{0.9}$. This shows that the first order stochastic dominance indeed is a stronger concept.

Proposition 5.1 For $\mu_1, \mu_2 \in \mathfrak{P}_q$ it holds that if $\mu_1 \preceq \mu_2$, then $\mathbb{T}\mu_1 \preceq \mathbb{T}\mu_2$.

Proof. By definition of \mathbb{T} and reversing the order of integration we can write for $\gamma \in (0, 1)$,

$$\int_0^{\gamma} (\mathbb{T}\mu_1)(t) dt = \int_0^{\gamma} \int_0^t \frac{1}{1-\alpha} d\mu_1(\alpha) dt = \int_0^{\gamma} \int_{\alpha}^{\gamma} \frac{1}{1-\alpha} dt d\mu_1(\alpha) = \int_0^{\gamma} \frac{\gamma-\alpha}{1-\alpha} d\mu_1(\alpha).$$

Now by Riemann-Stieltjes integration by parts

$$\int_0^{\gamma} (\mathbb{T}\mu_1)(t) dt = \frac{\gamma-\alpha}{1-\alpha} \mu_1(\alpha) \Big|_{\alpha=0}^{\gamma} - \int_0^{\gamma} \mu_1(\alpha) d \left(\frac{\gamma-\alpha}{1-\alpha} \right) = \int_0^{\gamma} \mu_1(\alpha) \frac{1-\gamma}{(1-\alpha)^2} d\alpha,$$

where we used that $\mu_1(0^-) = \mu_2(0^-) = 0$. Since $\mu_1 \preceq \mu_2$ we have that $\mu_1(\cdot) \geq \mu_2(\cdot)$ and hence

$$\int_0^\gamma (\mathbb{T}\mu_1)(t)dt = \int_0^\gamma \mu_1(\alpha) \frac{1-\gamma}{(1-\alpha)^2} d\alpha \geq \int_0^\gamma \mu_2(\alpha) \frac{1-\gamma}{(1-\alpha)^2} d\alpha = \int_0^\gamma (\mathbb{T}\mu_2)(t)dt.$$

Because of $\int_0^1 (\mathbb{T}\mu)(t)dt = 1$, this implies that

$$\int_\gamma^1 (\mathbb{T}\mu_1)(t)dt \leq \int_\gamma^1 (\mathbb{T}\mu_2)(t)dt,$$

and hence $\mathbb{T}\mu_1 \preceq \mathbb{T}\mu_2$. ■

Let \mathfrak{M} be a Kusuoka set and $\mu_1, \mu_2 \in \mathfrak{M}$. As it was pointed out above, it follows from (5.1) that if $\mathbb{T}\mu_1 \preceq \mathbb{T}\mu_2$, then the measure μ_1 can be removed from \mathfrak{M} . Hence it follows by Proposition 5.1 that if $\mu_1 \preceq \mu_2$, then the measure μ_1 can be removed from \mathfrak{M} .

Definition 5.3 *A measure $\mu \in \mathfrak{M} \subset \mathfrak{P}$ is called a maximal element, a maximal measure or non-dominated measure of (\mathfrak{M}, \preceq) , if there is no $\nu \in \mathfrak{M}$ satisfying $\mu \prec \nu$.*

The measures δ_0 and δ_1 are extremal in the sense that $\delta_0 \preceq \mu \preceq \delta_1$ for all $\mu \in \mathfrak{P}$, that is δ_1 (δ_0 , resp.) is always a maximal (minimal, resp.) measure.

The next theorem elaborates that it is sufficient to consider the non-dominated measures within the closure of any Kusuoka set.

Theorem 5.1 *Let \mathfrak{M} be a Kusuoka set of the law invariant coherent premium $\pi : \mathcal{Z} \rightarrow \mathbb{R}$, and $\overline{\mathfrak{M}}$ be its topological closure. Then the augmented set $\{\mu' \in \mathfrak{P} : \mu' \preceq \mu \text{ for some } \mu \in \mathfrak{M}\}$ is a Kusuoka set as well. Furthermore, the set of extremal measures of $\overline{\mathfrak{M}}$, i.e., $\mathfrak{M}' := \{\mu \in \overline{\mathfrak{M}} : \nu \not\preceq \mu \text{ for all } \nu \in \mathfrak{M}\}$ is a Kusuoka set.*

Proof. Let $\mu' \preceq \mu \in \mathfrak{M}$. As $\alpha \mapsto \text{CTE}_\alpha(Y)$ is a nondecreasing function and as $\mu(\cdot) \leq \mu'(\cdot)$ it follows by Riemann-Stieltjes integration by parts that

$$\begin{aligned} \int_0^1 \text{CTE}_\alpha(Y) d\mu(\alpha) &= \mu(\alpha) \text{CTE}_\alpha(Y) \Big|_{\alpha=0_-}^1 - \int_0^1 \mu(\alpha) d\text{CTE}_\alpha(Y) \\ &\geq \mu'(\alpha) \text{CTE}_\alpha(Y) \Big|_{\alpha=0_-}^1 - \int_0^1 \mu'(\alpha) d\text{CTE}_\alpha(Y) \\ &= \int_0^1 \text{CTE}_\alpha(Y) \mu'(d\alpha), \end{aligned}$$

which is the first assertion.

For the second assertion recall that (\mathfrak{M}, \preceq) is a partially ordered set and so is $(\overline{\mathfrak{M}}, \preceq)$. Consider a chain $\mathfrak{C} \subset \overline{\mathfrak{M}}$, that is for every μ, ν it holds that $\mu \preceq \nu$ or $\nu \preceq \mu$ (totality). Then the chain \mathfrak{C} has an upper bound in $\overline{\mathfrak{C}} \subset \overline{\mathfrak{M}}$: to accept this (cf. the proof of Helly's Lemma in van der Vaart [25]) define

$$\mu_{\mathfrak{C}}(x) := \inf_{\mu \in \mathfrak{C}} \mu(x)$$

(the upper bound), which is a positive, non-decreasing function satisfying $\mu_{\mathfrak{C}}(1) = 1$. As any $\mu \in \mathfrak{C}$ is right side continuous it is upper semi-continuous, thus $\mu_{\mathfrak{C}}$, as an infimum, is upper semi-continuous as well, hence $\mu_{\mathfrak{C}}$ is right side continuous and $\mu_{\mathfrak{C}}$ thus represents a measure, $\mu_{\mathfrak{C}} \in \mathfrak{P}$. To show that $\mu_{\mathfrak{C}} \in \overline{\mathfrak{C}}$ let x_i be a dense sequence in $[0, 1]$ and choose $\mu_{i,n} \in \mathfrak{C}$ such that $\mu_{i,n}(x_i) < \mu_{\mathfrak{C}}(x_i) + 2^{-n}$. As \mathfrak{C} is a chain one may define $\mu_n := \min\{\mu_{i,n} : i = 1, 2, \dots, n\}$. It holds that $\mu_n(x_i) < \mu_{\mathfrak{C}}(x_i) + 2^{-n}$ for all $i = 1, 2, \dots, n$. As x_i is dense, and as μ_n , as well as $\mu_{\mathfrak{C}}$ are right side continuous it follows that $\mu_n \rightarrow \mu_{\mathfrak{C}}$ uniformly, hence $\mu_{\mathfrak{C}} \in \overline{\mathfrak{C}}$.

By Zorn's Lemma there is at least one maximal element μ^* in $\overline{\mathfrak{M}}$, that is there is no element $\nu \in \overline{\mathfrak{M}}$ such that $\nu \succ \mu^*$. Hence

$$\begin{aligned} \mu^* &\in \{\mu \in \overline{\mathfrak{M}} : \neg \exists \nu \in \mathfrak{M} : \nu \succ \mu\} \\ &= \{\mu \in \overline{\mathfrak{M}} : \forall \nu \in \mathfrak{M} : \nu \not\succeq \mu\} = \mathfrak{M}' \end{aligned}$$

and \mathfrak{M}' thus is a non-empty Kusuoka set. Recall that $\mathfrak{M}' \subset \overline{\mathfrak{M}}$, hence

$$\begin{aligned} \pi(Y) &= \sup_{\mu \in \overline{\mathfrak{M}}} \int_0^1 \text{CTE}_{\alpha}(Y) \mu(d\alpha) \\ &= \sup_{\mu \in \overline{\mathfrak{M}}} \int_0^1 \text{CTE}_{\alpha}(Y) \mu(d\alpha) \geq \sup_{\mu \in \mathfrak{M}'} \int_0^1 \text{CTE}_{\alpha}(Y) \mu(d\alpha). \end{aligned}$$

To establish equality assume that $\pi(\cdot) \neq \sup_{\mu \in \mathfrak{M}'} \int_0^1 \text{CTE}_{\alpha}(\cdot) \mu(d\alpha)$. Then there is a random variable Y satisfying

$$\pi(Y) > \sup_{\mu' \in \mathfrak{M}'} \int_0^1 \text{CTE}_{\alpha}(Y) \mu'(d\alpha). \quad (5.2)$$

For this Y and for some $\varepsilon > 0$ choose $\mu \in \overline{\mathfrak{M}}$ such that $\pi(Y) < \varepsilon + \int_0^1 \text{CTE}_{\alpha}(Y) \mu(d\alpha)$.

Consider the cone $\mathfrak{M}_{\mu} := \{\nu \in \overline{\mathfrak{M}} : \mu \preceq \nu\}$. Notice that $(\mathfrak{M}_{\mu}, \preceq)$ again is a partially ordered set, for which Zorn's Lemma implies that there is a maximal element $\bar{\mu} \in \mathfrak{M}_{\mu}$ with respect to \preceq , and it holds that $\mu \preceq \bar{\mu}$ by construction.

We claim that $\bar{\mu} \in \mathfrak{M}'$. Indeed, if it were not, then there is $\nu' \in \mathfrak{M}$ with $\nu' \succ \bar{\mu}$. But this means $\mu \preceq \bar{\mu} \prec \nu'$, so $\nu' \in \mathfrak{M}_{\mu}$ and hence $\bar{\mu} \prec \nu'$, contradicting the fact that $\bar{\mu}$ is a maximal measure in \mathfrak{M}_{μ} , and hence $\bar{\mu} \in \mathfrak{M}'$.

Now by Riemann-Stieltjes integration by parts

$$\begin{aligned} \pi(Y) - \varepsilon &< \int_0^1 \text{CTE}_{\alpha}(Y) d\mu(\alpha) \\ &= \mu(\alpha) \text{CTE}_{\alpha}(Y) \Big|_{\alpha=0}^1 - \int_0^1 \mu(\alpha) d\text{CTE}_{\alpha}(Y) \\ &\leq \bar{\mu}(\alpha) \text{CTE}_{\alpha}(Y) \Big|_{\alpha=0}^1 - \int_0^1 \bar{\mu}(\alpha) d\text{CTE}_{\alpha}(Y) \\ &= \int_0^1 \text{CTE}_{\alpha}(Y) \bar{\mu}(d\alpha), \end{aligned}$$

as $\alpha \mapsto \text{CTE}_{\alpha}(Y)$ is a non-decreasing function and as $\mu \preceq \bar{\mu}$, that is $\mu(\cdot) \geq \bar{\mu}(\cdot)$.

But as $\bar{\mu} \in \mathfrak{M}'$ this is a contradiction to (5.2), such that the assertion holds indeed. \blacksquare

6 Examples

In order to demonstrate the minimal Kusuoka representation we have chosen two examples. The first is taken from Dentcheva et al. [7] and generalizes the Conditional Tail Expectation. The second example elaborates the Kusuoka representation of the higher order semideviations. This example exposes the minimal Kusuoka representation of the Dutch premium principle.

6.1 Higher order premiums

Consider the premium

$$\pi(Z) := \inf_{t \in \mathbb{R}} \{t + c \|[Z - t]_+\|_p\}, \quad Z \in L_p(\Omega, \mathcal{F}, P), \quad (6.1)$$

where $c \geq 1$ and $p \in [1, \infty)$. For $p = 1$ we have that $\pi = \text{CTE}_\alpha$ with $\alpha := 1 - c^{-1}$.

Proposition 6.1 *The minimal generating set of the premium π , defined in (6.1), is given by*

$$\mathfrak{H} := \{\sigma \in \mathfrak{D} : \|\sigma\|_q = c\}, \quad (6.2)$$

where $1/p + 1/q = 1$.

Proof. For $p = 1$ (and hence $q = \infty$) and $c = (1 - \alpha)^{-1}$, the set defined in the right hand side of (6.2) consists of the single distortion function $\bar{\sigma} = (1 - \alpha)^{-1} \mathbf{1}_{[\alpha, 1]}$. In that case $\pi = \text{CTE}_\alpha$ and its minimal generating set is the singleton $\{\bar{\sigma}\}$.

Let us suppose now that $p > 1$. It is known that $\Upsilon = \{\sigma \in \mathfrak{D} : \|\sigma\|_q \leq c\}$ is a generating set and $\mathfrak{A} = \mathcal{O}(\Upsilon)$ is the dual set of π (cf., [7]). We show first that it is enough to consider distortion functions σ satisfying $\|\sigma\|_q = c$, i.e., $\mathfrak{H} = \{\sigma \in \mathfrak{D} : \|\sigma\|_q = c\}$ is also a generating set of π . Indeed, for σ satisfying $\|\sigma\|_q < c$ define

$$\sigma_\gamma(u) := \mathbf{1}_{[\gamma, 1]}(u) \cdot \left(\sigma(u) + \frac{1}{1 - \gamma} \int_0^\gamma \sigma(t) dt \right), \quad \gamma \in [0, 1).$$

It is evident that $\gamma \mapsto \|\sigma_\gamma\|_q$ is a continuous and unbounded function, hence there is a γ_0 such that $\|\sigma_{\gamma_0}\|_q = c$. Moreover $\sigma \preceq \sigma_{\gamma_0}$ by construction. According to (5.1) this completes the arguments.

Consider now some $\bar{\zeta} \in \mathfrak{H}$, and define $Z := \bar{\zeta}^{q-1}$. Note that $q - 1 = q/p$, and hence $\|Z\|_p = \|\bar{\zeta}\|_q^{q/p} = c^{q/p}$ and $Z \in L_p(\Omega, \mathcal{F}, P)$. Also $\langle \bar{\zeta}, Z \rangle = \int \bar{\zeta}^q dP = \|\bar{\zeta}\|_q^q = c^q$, and for any $\zeta \in \mathfrak{A}$ we have that

$$\langle \zeta, Z \rangle \leq \|\zeta\|_q \|Z\|_p \leq c \cdot c^{q/p} = c^q.$$

Moreover, for $p > 1$ the above inequality is strict for $\zeta \neq \bar{\zeta}$. This shows that $\bar{\zeta}$ is the unique maximizer of $\langle \cdot, Z \rangle$ over the dual set, and hence $\bar{\zeta}$ is an exposed point of the dual set. It remains to note that the set \mathfrak{H} is closed in the weak topology (for $p \in (1, \infty)$ the space $L_p(\Omega, \mathcal{F}, P)$ is reflexive and hence the weak and weak* topologies do coincide here). ■

It follows by Theorem 3.2 that the set $\mathbb{T}^{-1}(\mathfrak{H})$ is the minimal Kusuoka set of π . We also can write (see (3.3))

$$\pi(Z) = \sup_{\sigma \in \mathfrak{H}} \pi_\sigma(Z), \quad Z \in L_p(\Omega, \mathcal{F}, P),$$

where $\pi_\sigma(Z) = \int_0^1 \sigma(\tau) F_Z^{-1}(\tau) d\tau$ is the corresponding distortion premium (Wang premium).

Note also that for any $\sigma \in \mathfrak{D}$ we have that $\int_0^1 \sigma(\tau) d\tau = 1$, i.e., $\|\sigma\|_1 = 1$. Also $\|\sigma\|_1 \leq \|\sigma\|_q$ for any $q \geq 1$, which is in compliance with $c \geq 1$. For $c = 1$ the set \mathfrak{H} is the singleton $\{\mathbf{1}_{[0,1]}\}$ and $\pi(Z) = \mathbb{E}[Z]$. For $p > 1$ and $c > 1$ the set \mathfrak{H} consists of more than one point and hence the corresponding premium π is not a distortion premium (Wang premium).

6.2 Higher order semideviation

Our second example addresses the p -semideviation premium

$$\pi(Z) := \mathbb{E}[Z] + \lambda \left\| [Z - \mathbb{E}Z]_+ \right\|_p, \quad Z \in L_p(\Omega, \mathcal{F}, P),$$

where $p \in [1, \infty)$ and $\lambda \in [0, 1]$. Again we denote by π_σ the distortion premium principle (Wang premium) associated with $\sigma \in \mathfrak{D}$ and show that the following representation holds

$$\pi(Z) = \sup_{\sigma \in \mathfrak{D}} \left(1 - \frac{\lambda}{\|\sigma\|_q} \right) \mathbb{E}[Z] + \frac{\lambda}{\|\sigma\|_q} \pi_\sigma(Z), \quad Z \in L_p(\Omega, \mathcal{F}, P). \quad (6.3)$$

Moreover, we show that for $p > 1$ the above representation is minimal in the sense that the corresponding Kusuoka representation

$$\pi(Z) = \sup_{\mu \in \mathfrak{P}_q} \left\{ \left(1 - \frac{\lambda}{\|\mathbb{T}\mu\|_q} \right) \mathbb{E}[Z] + \frac{\lambda}{\|\mathbb{T}\mu\|_q} \int_0^1 \text{CTE}_\alpha(Z) d\mu(\alpha) \right\}, \quad (6.4)$$

is minimal, i.e.,

$$\mathfrak{M} := \left\{ \left(1 - \frac{\lambda}{\|\mathbb{T}\mu\|_q} \right) \delta_0 + \frac{\lambda}{\|\mathbb{T}\mu\|_q} \mu : \mu \in \mathfrak{P}_q \right\}$$

is the minimal Kusuoka set of π . Notice that $\|\sigma\|_q \geq \|\sigma\|_1 = 1$, and hence $1 - \frac{\lambda}{\|\sigma\|_q} \geq 0$ whenever $0 \leq \lambda \leq 1$.

Proposition 6.2 *The p -semideviation premium π has the representation (6.4), and this Kusuoka representation is minimal whenever $p > 1$.*

Proof. By Hölder's inequality

$$\mathbb{E}[Z\zeta] \leq (\mathbb{E}|Z|^p)^{\frac{1}{p}} (\mathbb{E}|\zeta|^q)^{\frac{1}{q}}$$

and

$$(\mathbb{E}|Z|^p)^{\frac{1}{p}} = \sup \left\{ \mathbb{E}[Z\zeta] : \|\zeta\|_q \leq 1 \right\}.$$

Clearly

$$(\mathbb{E}[Z_+^p])^{\frac{1}{p}} = \sup \left\{ \mathbb{E}[Z\zeta] : \zeta \succeq 0, \|\zeta\|_q \leq 1 \right\},$$

the supremum being attained at $\zeta = \frac{Z_+^{p-1}}{\|Z_+^{p-1}\|_q}$. It follows that

$$\left\| (Z - \mathbb{E}[Z])_+ \right\|_p = \sup \left\{ \mathbb{E}[(Z - \mathbb{E}[Z])\zeta] : \zeta \succeq 0, \|\zeta\|_q \leq 1 \right\}. \quad (6.5)$$

Now

$$\begin{aligned}
\pi(Z) &= \mathbb{E}[Z] + \lambda \|(Z - \mathbb{E}[Z])_+\|_p \\
&= \sup \left\{ \mathbb{E}[Z] + \lambda \mathbb{E}[(Z - \mathbb{E}[Z])\zeta] : \zeta \succeq 0, \|\zeta\|_q \leq 1 \right\} \\
&= \sup \left\{ (1 - \lambda \mathbb{E}[\zeta]) \mathbb{E}[Z] + \lambda \mathbb{E}[Z\zeta] : \zeta \succeq 0, \|\zeta\|_q \leq 1 \right\} \tag{6.6}
\end{aligned}$$

$$\begin{aligned}
&= \sup \left\{ \left(1 - \lambda \frac{\mathbb{E}[\zeta]}{\|\zeta\|_q}\right) \mathbb{E}[Z] + \lambda \mathbb{E} \left[Z \frac{\zeta}{\|\zeta\|_q} \right] : \zeta \succeq 0 \right\} \\
&= \sup \left\{ \left(1 - \frac{\lambda}{\|\zeta\|_q}\right) \mathbb{E}[Z] + \frac{\lambda}{\|\zeta\|_q} \mathbb{E}[Z\zeta] : \zeta \succeq 0, \mathbb{E}[\zeta] = 1 \right\}. \tag{6.7}
\end{aligned}$$

For ζ feasible in (6.7) define the function $\sigma_\zeta(u) := \mathbf{V@R}_u(\zeta)$ and observe that

$$\|\sigma_\zeta\|_q^q = \int_0^1 \sigma_\zeta^q(\alpha) d\alpha = \mathbb{E}[\zeta^q] = \|\zeta\|_q^q.$$

Now notice that every random variable ζ in (6.7) is given by $\zeta = \sigma(U)$ for a uniform U where σ is non-negative, non-decreasing and $\int_0^1 \sigma(u) du = 1$ – that is, σ is a distortion function. It follows that

$$\begin{aligned}
\pi(Z) &= \sup_{\sigma \in \mathfrak{D}} \sup \left\{ \left(1 - \frac{\lambda}{\|\zeta\|_q}\right) \mathbb{E}[Z] + \frac{\lambda}{\|\zeta\|_q} \mathbb{E}[Z\zeta] : \zeta \succeq 0, \mathbb{E}[\zeta] = 1, \mathbf{V@R}(\zeta) = \sigma \right\} \\
&= \sup_{\sigma \in \mathfrak{D}} \sup \left\{ \left(1 - \frac{\lambda}{\|\sigma\|_q}\right) \mathbb{E}[Z] + \frac{\lambda}{\|\sigma\|_q} \mathbb{E}[Z\sigma] : \zeta \succeq 0, \mathbb{E}[\zeta] = 1, \mathbf{V@R}(\zeta) = \sigma \right\} \\
&= \sup_{\sigma \in \mathfrak{D}} \left(1 - \frac{\lambda}{\|\sigma\|_q}\right) \mathbb{E}[Z] + \frac{\lambda}{\|\sigma\|_q} \pi_\sigma(Z).
\end{aligned}$$

This shows that the representation (6.3) holds.

In order to verify the minimality of the representation (6.3) consider the set

$$\mathfrak{C} := \left\{ \zeta = (1 - \lambda \mathbb{E}[\zeta']) \mathbf{1} + \lambda \zeta' : \|\zeta'\|_q = 1, \zeta' \succeq 0 \right\}.$$

In view of (6.6) we have that \mathfrak{C} is a subset of the dual set \mathfrak{A} and the representation (3.2) holds (cf. [24, Example 6.20]). For $Z \in L_p(\Omega, \mathcal{F}, P)$ consider the function $g_Z(\zeta) := \langle \zeta, Z \rangle$ as in Definition 3.4. It holds that

$$\sup_{\zeta \in \mathfrak{A}} g_Z(\zeta) = \sup_{\zeta' \succeq 0, \|\zeta'\|_q \leq 1} (1 - \lambda \mathbb{E}[\zeta']) \mathbb{E}[Z] + \lambda \mathbb{E}[\zeta' Z] = \mathbb{E}[Z] + \lambda \sup_{\zeta' \succeq 0, \|\zeta'\|_q \leq 1} \mathbb{E}[\zeta'(Z - \mathbb{E}[Z])].$$

For $1 < p < \infty$ the latter supremum is *uniquely* attained at

$$\bar{\zeta}' = \frac{(Z - \mathbb{E}[Z])_+^{p-1}}{\|(Z - \mathbb{E}[Z])_+^{p-1}\|_q},$$

such that g_Z attains its *unique* maximum at

$$\bar{\zeta} = (1 - \lambda \mathbb{E}[\bar{\zeta}']) \mathbf{1} + \lambda \bar{\zeta}'.$$

It follows that \mathfrak{C} consists of exposed points only, that is $\mathfrak{C} = \text{Exp}(\mathfrak{A})$. This proves the minimality of the representation (6.3). \blacksquare

Corollary 6.1 (The Dutch premium principle, cf. [23]) For $p = 1$ the minimal Kusuoka representation of the absolute semideviation $\pi(Z) := \mathbb{E}[Z] + \lambda \mathbb{E}[(Z - \mathbb{E}[Z])_+]$, with $\lambda \in [0, 1]$, is

$$\pi(Z) = \sup_{\kappa \in [0,1]} \{(1 - \lambda\kappa) \text{CTE}_0 + \lambda\kappa \text{CTE}_{1-\kappa}(Z)\}.$$

Proof. Note that the supremum in (6.5) is attained at $\zeta = \frac{(Z - \mathbb{E}[Z])_+^0}{\|(Z - \mathbb{E}[Z])_+^0\|_\infty}$, and in (6.7) thus at $\zeta = \frac{(Z - \mathbb{E}[Z])_+^0}{\|(Z - \mathbb{E}[Z])_+^0\|_1}$, which is a function of type $\zeta = \frac{1}{P(B)} \mathbf{1}_B$ (choose $B = \{Z > \mathbb{E}[Z]\}$ to accept the correspondence). Next observe that $\sigma_\zeta(u) = \mathbf{V}@\mathbf{R}_u(\zeta) = \frac{1}{1-\alpha} \mathbf{1}_{[\alpha,1]}(u)$ (for $\alpha = P(B^c)$), which is the distortion function of the Conditional Tail Expectation of level α . It follows that

$$\begin{aligned} \pi(Z) &= \sup_{\sigma = \frac{1}{1-\alpha} \mathbf{1}_{[\alpha,1]}} \left\{ \left(1 - \frac{\lambda}{\|\sigma\|_\infty}\right) \mathbb{E}[Z] + \frac{\lambda}{\|\sigma\|_\infty} \pi_\sigma(Z) \right\}, \\ &= \sup_{\alpha} \{(1 - \lambda(1 - \alpha)) \mathbb{E}[Z] + \lambda(1 - \alpha) \text{CTE}_\alpha(Y)\}, \\ &= \sup_{\kappa \in (0,1)} \{(1 - \lambda\kappa) \mathbb{E}[Z] + \lambda\kappa \text{CTE}_{1-\kappa}(Z)\}. \end{aligned}$$

That is, the set $\mathfrak{M} := \cup_{\kappa \in (0,1)} \{(1 - \lambda\kappa)\delta_0 + \lambda\kappa\delta_{1-\kappa}\}$ is a Kusuoka set and its closure is the minimal Kusuoka set. ■

7 Conclusion

This paper describes insurance premiums by Kusuoka representations, which have been introduced for law invariant coherent risk measures in Kusuoka [12]. In general many representations may describe the same premium, but we demonstrate that there is a minimal representation available and this minimal representation is moreover unique.

The minimal representation can be extracted by identifying the set of exposed points of the dual set and then applying the corresponding one-to-one transformation. Moreover, it turns out that the minimal representation only consists of measures which are nondominated in first order stochastic dominance. This relation, as well as the convex order stochastic dominance relation can be employed to identify the necessary measures. The general results are elaborated for two dedicated, important examples to demonstrate this fact. In particular we provide an explicit minimum representation for the Dutch premium principle.

We also consider Kusuoka representations on a class of general probability spaces, which potentially contain atoms. We discuss law invariant risk measures on spaces, which contain—or do not contain—atoms and characterize premium principles on these spaces, which allow a Kusuoka representation.

We want to note that many of the results can be extended to convex risk measures or premiums, which are not necessarily positively homogeneous.

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