Premiums And Reserves, Adjusted By Distortions

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Abstract

The net-premium principle is considered to be the most genuine and fair premium principle in actuarial applications. However, actuarial due diligence requires additional caution in pricing of insurance contracts to avoid, for example, at least bankruptcy of the insurer.

This paper addresses the distorted premium principle from various angles. Distorted premiums are typically computed by under-weighting or ignoring low, but over-weighting high losses. Dual characterizations, which are elaborated in a first part of the paper, support this interpretation.

The main contribution consists in an opposite point of view—an alternative characterization—which leaves the probability measure unchanged, but modifies (increases) the outcomes instead in a consistent way. It turns out that this new point of view is natural in actuarial practice, as it can be used for premium calculations, but equally well to determine the reserve process in subsequent years in a time consistent way.

Keywords: Premium Principles, Dual Representation, Fenchel–Young inequality, Stochastic dominance

Classification: 90C15, 60B05, 62P05

1 Introduction

Axiomatic characterizations of insurance premiums have been outlined by Wang et al. in [34, 32] and summarized by Young in [35]. These axiomatic treatments, initiated in an actuarial context first (early attempts by Denneberg appeared already 1989 in [7]), have been developed further in financial mathematics, for example in the celebrated seminal paper [2] by Artzner et alii. The connection between actuarial and financial mathematics is striking here, as premium principles in an actuarial context correspond to risk measures in financial mathematics, so that risk measures constitute a premium principle and vice versa. What perhaps is surprising is that the name—risk measure—is a term that should be expected in actuarial science rather than in financial mathematics.

Risk adjusted insurance prices by employing *distorted probability measures* have been considered in this journal by Wang [33] and, among others, recently in the paper [12] by Heras, Balbás and Vilar. It is the essential idea of distorted insurance prices to over-value outstanding, potential losses, and to under-weight small claims in exchange. This approach provides a risk-adjusted premium, which

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always exceeds the net premium (cf. also the paper by Furman et al. [10]) and thus incorporates a natural safety margin.

In this paper we provide a different perspective in a way, which leaves the probabilities unchanged (the measure is not changed), but the claims are adjusted in an appropriate way. Comparing just the premiums, both approaches provide the same result. However, the new perspective allows computing the reserves as well in a concise and time-consistent way, and this is an essential, additional property.

The distorted probability relates directly to a special class of risk measures—the spectral measures—introduced by Acerbi et al. in [1]. An important study of spectral risk measures, although under the different name *distortion functional*, was provided by Pflug in [19]. The concepts of (i) premium principles by distorting probability measures, (ii) distortion functionals, and (iii) spectral risk measures are essentially the same, they differ only in sign conventions, resulting in a convex or a concave mathematical environment.

Distorted premium principles constitute an elementary and important class of premium principles, as every premium functional can be described by premium functionals involving distortions. They are moreover defined in an explicit way, hence there is an explicit evaluation scheme available, which is of course important for an applied actuary.

The most important distorted premium functional, which made its way to the top, is the *conditional tail expectation*, CTE (in a financial context the alternative terms Conditional Value-at-Risk or Average Value-at-Risk are more common). The conditional tail expectation is usually associated and employed for loss distributions of entire portfolios (for example by the US and Canadian insurance supervisory authorities: cf. Ko et al. [15], and also Chi et al. [5]). In what follows we shall exploit that the CTE constitutes an elementary pricing principle for single policies as well (cf. Heras et al. [12]). It is the essential advantage of the conditional tail expectation that different representations are known, which makes this premium principle eligible in varying situations: by conjugate duality there is an expression in the form of a supremum, but in applications and for quick computations a different formulation as an infimum is extremely convenient: developed in the paper *Optimization of Conditional Value-at-Risk* by Rockafellar and Uryasev [23] (cf. also [24]), the general formula is given in *Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk* by Pflug in [18]. The main results of this article extend both formulations to distorted premium functionals. Further, both representations can be associated with different views on distortions, providing different interpretations in an actuarial context.

A description of the distorted premium principle as a *supremum* is a first result of this article. The description builds on dual representations and on second order stochastic dominance. Stochastic dominance relations have been considered in the literature, but typically for the risk measure itself (the primal functions) with the result that law invariant, coherent risk measures are consistent with second order stochastic dominance, cf. Theorem 6.29 in [27]. A concise formulation, in addition, is available by imposing stochastic dominance constraints on the convex conjugate function (the dual function) instead of considering stochastic orders on the primal, and this is elaborated here.

Besides that—and this is of particular importance for applications and a further result in this paper—a formula for the distorted premium is elaborated by involving an *infimum*. The infimum description builds on the Fenchel–Young inequality. This alternative representation of distorted premium functionals is the converse of the initial description, as it does not change the measure, but the outcomes instead.

The article is organized as follows. The premium principle is introduced in the following Section 2. Its description as a supremum by means of stochastic order relations is contained in Section 3. The infimum representation is elaborated in Section 4. Further implications for actuarial sciences are outlined and explained in Section 5; this section contains illustrating examples as well.

2 The distorted distribution

In this paper—as usual in an actuarial context—we shall associate a $\mathbb{R}$-valued random variable with a loss and therefore write L to denote a random variable. We shall assume throughout the paper that L is at least integrable ($L \in \mathbb{L}^1$, that is $\mathbb{E}(|L|) < \infty$), and often in addition that L is bounded ($L \in \mathbb{L}^\infty$).

$F_L(x) := P(L \leq x)$ is the *cumulative distribution function* (cdf), and

$$F_L^{-1}(u) := \inf\{x : F_L(x) \geq u\} \tag{1}$$

denotes the *generalized inverse* or *quantile*. The random variable L can be given by employing the probability integral transform (or inverse sampling) as

$$L = F_L^{-1}(U) \quad \text{a.s.,} \tag{2}$$

where U is a uniformly distributed random variable[1] on the same probability space as L and coupled in a comonotone way with L (for example $U := F_L(L)$, if F_L is invertible; we may refer to van der Vaart [29] for basic, but essential properties of the random variables and related functions).

We shall call a nonnegative, nondecreasing function

$$\sigma : [0,1] \to [0,\infty)$$

satisfying $\int_0^1 \sigma(u)\,\mathrm{d}u = 1$ a *distortion*, and define the antiderivative $\tau_\sigma(p) := \int_0^p \sigma(u)\mathrm{d}u$. By the conditions imposed on σ the function τ_σ is nonnegative, convex, continuous on $[0,1]$ and satisfies $\tau_\sigma(1) = 1$. Moreover it has a generalized inverse, τ_σ^{-1}, defined in accordance with (1).

The *distorted loss* L_σ (distorted by the distortion σ) then is

$$L_\sigma := F_L^{-1}\left(\tau_\sigma^{-1}(U)\right), \tag{3}$$

where U is chosen as in (2). L and L_σ notably have the same outcomes, but their probabilities differ. It holds that $\tau_\sigma(u) \leq u$ (for all $u \in [0,1]$) by convexity of τ_σ, so that

$$L_\sigma \geq L \text{ and } F_{L_\sigma}(\cdot) \leq F_L(\cdot)$$

(it is said that L_σ stochastically dominates L in first order, cf. Stoyan and Müller [17]). Applying the simple net premium principle to L_σ and L reveals that

$$\mathbb{E}(L) \leq \mathbb{E}(L_\sigma) = \int_0^1 F_L^{-1}\left(\tau_\sigma^{-1}(u)\right)\mathrm{d}u = \int_0^1 F_L^{-1}(u)\,\mathrm{d}\tau_\sigma(u) = \int_0^1 F_L^{-1}(u)\,\sigma(u)\,\mathrm{d}u$$

by monotonicity of the expectation, ensuring thus that $\mathbb{E}(L_\sigma)$ is a plausible price for the insurance contract, the price $\mathbb{E}(L_\sigma)$ at least exceeding the net-premium $\mathbb{E}(L)$.

[1] U is *uniformly distributed* if $P(U \leq u) = u$ for all $u \in [0,1]$.

The premium $\mathbb{E}\left(L_\sigma\right)$ is moreover easily accessible to the actuary, because

$$F_{L_\sigma}\left(y\right)=P\left(L_\sigma\leq y\right)=P\left(U\leq\tau_\sigma\left(F_L\left(y\right)\right)\right)=\tau_\sigma\circ F_L\left(y\right)\quad\text{a.e.,}\tag{4}$$

the actuary just has to replace the cdf F_L by $F_{L_\sigma}=\tau_\sigma\circ F_L$ in his or her computations for the premium or reserves, or consider the density

$$f_{L_\sigma}\left(y\right)=f_L\left(y\right)\cdot\sigma\left(F_L\left(y\right)\right),$$

if available; cf. Valdez et al. [28]. So the premium $\mathbb{E}\left(L_\sigma\right)$ is an expectation again—as the net premium principle—just with probabilities modified (distorted) according (4).

These considerations give rise for the following definition.

Definition 1. Let $\sigma\in\mathbb{L}^q$ ($q\in[1,\infty]$) be a distortion and $L\in\mathbb{L}^p$ be a random variable, where $\frac{1}{p}+\frac{1}{q}=1$ (the conjugate exponent), then

$$\pi_\sigma\left(L\right):=\int_0^1 F_L^{-1}\left(u\right)\sigma\left(u\right)\mathrm{d}u\tag{5}$$

is called σ-distorted premium, or simply distorted premium for the loss L. π_σ is called *distorted premium functional*.

Remark 2. The premium $\pi_\sigma\left(L\right)$ is well defined and finite valued, it satisfies $\mathbb{E}\left(L\right)\leq\pi_\sigma\left(L\right)\leq\left\|\sigma\right\|_q\cdot\left\|L\right\|_p$ by Hölder's inequality.

The distorted premium functional π_σ satisfies the following axioms, which have been proposed and formulated in a different context—for risk measures in mathematical finance—by Artzner et al. in [3]. The axioms here have been adapted to account for insurance instead of financial risk (cf. also Wang and Dhaene [31], and for reinsurance cf. Balbás [4]).

Definition 3. A function $\pi:\mathbb{L}^p\to\mathbb{R}$ is called *premium functional* (or *premium principle*) if the following axioms are satisfied:

(M) Monotonicity: $\pi\left(L_1\right)\leq\pi\left(L_2\right)$ whenever $L_1\leq L_2$ almost surely;

(C) Convexity: $\pi\left(\left(1-\lambda\right)L_0+\lambda L_1\right)\leq\left(1-\lambda\right)\pi\left(L_0\right)+\lambda\,\pi\left(L_1\right)$ for $0\leq\lambda\leq1$;

(T) Translation equivariance:[2] $\pi\left(L+c\right)=\pi\left(L\right)+c$ if $c\in\mathbb{R}$;

(H) Positive homogeneity: $\pi\left(\lambda L\right)=\lambda\cdot\pi\left(L\right)$ whenever $\lambda>0$.

Remark 4. In a banking or investment environment the interpretation of a reward is more natural, in this context the mapping $\rho\left(L\right)=\pi\left(-L\right)$ is often considered and called *coherent risk measure* instead (note that essentially the monotonicity condition (M) and translation property (T) reverse for ρ).

The term *acceptability functional* was introduced in energy or decision theory to quantify and classify acceptable strategies. In this context the concave mapping $\mathcal{A}\left(L\right)=-\pi\left(-L\right)$, the acceptability functional, is employed instead (here, (C) modifies to concavity).

[2] In an economic or monetary environment this is often called Cash invariance instead.

The conditional tail expectation is the most important premium principle.

Definition 5 (Conditional tail expectation)**.** The premium principle with distortion

$$\sigma_\alpha(\cdot) := \frac{1}{1-\alpha}\mathbb{1}_{(\alpha,1]}(\cdot) \tag{6}$$

is the *conditional tail expectation* at level α $(0 \le \alpha < 1)$,

$$\mathsf{CTE}_\alpha(L) := \pi_{\sigma_\alpha}(L) = \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(p)\,\mathrm{d}p \qquad (L \in \mathbb{L}^1).$$

The conditional tail expectation at level $\alpha = 1$ is

$$\mathsf{CTE}_1(L) := \lim_{\alpha \nearrow 1} \mathsf{CTE}_\alpha(L) = \operatorname{ess\,sup}(L) \qquad (L \in \mathbb{L}^\infty).$$

Due to the defining equation (5) of the distorted premium the same real number is assigned to all random variables L sharing the same law, irrespective of the underlying probability space. This gives rise to the following notion of version independence.

Definition 6 (version independence)**.** A premium principle π is *version independent*,[3] if $\pi(L_1) = \pi(L_2)$ whenever L_1 and L_2 share the same law, that is if $P(L_1 \le y) = P(L_2 \le y)$ for all $y \in \mathbb{R}$.

The following representation underlines the central role of the conditional tail expectation for version independent premium principles. It is moreover the basis and justification for investigating distorted premium principles in much more detail.

Theorem 7 (Kusuoka's representation)**.** *For any version independent premium principle π satisfying (M), (C), (T) and (H) on $\mathbb{L}^\infty$ of an atomless probability space there exists a set $\mathscr{M}$ of probability measures on $[0,1]$ such that π obeys the representation*

$$\pi(L) = \sup_{\mu \in \mathscr{M}} \int_0^1 \mathsf{CTE}_\alpha(L)\,\mu(\mathrm{d}\alpha)\,. \tag{7}$$

Proof. The proof of the statement, involving an additional assumption called *Fatou property*, is given by Kusuoka in [16], and contained as well in [20] or [26]. Schachermayer et al. [14] proved that the Fatou property is automatically satisfied and thus can be dropped from the assumptions. □

In the present context of distorted premiums it is essential to observe that any distorted premium has an immediate representation as in (7). To accept this define the measure μ_σ corresponding to the density σ as

$$\mu_\sigma(A) := \sigma(0)\,\delta_0(A) + \int_A (1-\alpha)\,\mathrm{d}\sigma(\alpha) \qquad (A \subset [0,1],\ \text{measurable}), \tag{8}$$

where δ_0 is the Dirac measure at 0 and the integral is a Riemann–Stieltjes integral with integrator σ. The cumulative distribution function of the measure μ_σ (which we may denote again by μ_σ) then is

$$\begin{aligned}\mu_\sigma(p) :&= \mu_\sigma([0,p]) = \sigma(0) + \int_0^p 1-\alpha\,\mathrm{d}\sigma(\alpha)\\ &= (1-p)\,\sigma(p) + \int_0^p \sigma(\alpha)\,\mathrm{d}\alpha \quad (0 \le p \le 1) \qquad (\text{and } \mu_\sigma(p) = 0 \text{ if } p < 0),\end{aligned}$$

[3] sometimes also *law invariant* or *distribution based.*

where equality follows by applying Riemann–Stieltjes integration by parts. μ_σ is a positive measure by (8) since σ is nondecreasing, and it is a probability measure on $[0,1]$ as $\mu_\sigma([0,1]) = \mu_\sigma(1) = 1$. Kusuoka's representation is immediate for the set $\mathscr{M} = \{\mu_\sigma\}$ now, as

$$\begin{aligned}\int_0^1 \mathsf{CTE}_\alpha(L)\,\mu_\sigma(\mathrm{d}\alpha) &= \sigma(0)\,\mathsf{CTE}_0(L) + \int_0^1 \frac{1}{1-\alpha}\int_\alpha^1 F_L^{-1}(p)\,\mathrm{d}p\cdot(1-\alpha)\,\mathrm{d}\sigma(\alpha)\\ &= \sigma(0)\,\mathsf{CTE}_0(L) + \int_0^1\int_\alpha^1 F_L^{-1}(p)\,\mathrm{d}p\,\mathrm{d}\sigma(\alpha)\\ &= -\int_0^1 \sigma(\alpha)\,\mathrm{d}\int_\alpha^1 F_L^{-1}(p)\,\mathrm{d}p\,\mathrm{d}\alpha = \int_0^1 F_L^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha = \pi_\sigma(L),\end{aligned} \tag{9}$$

again by Riemann–Stieltjes integration by parts.

Conversely, the premium functional $\int_0^1 \mathsf{CTE}_\alpha(L)\,\mu(\mathrm{d}\alpha)$ in Kusuoka's representation (7) can often be expressed as a distorted premium functional with distortion σ_μ. This is accomplished by the function

$$\sigma_\mu(\alpha) := \int_0^\alpha \frac{1}{1-p}\mu(\mathrm{d}p). \tag{10}$$

Provided that σ_μ is well defined (notice that possibly $\mu(\{1\}) > 0$ has to be excluded when computing $\sigma_\mu(1)$), it is positive and a density, as $\int_0^1 \sigma_\mu(\alpha)\mathrm{d}\alpha = \int_0^1 \frac{1}{1-p}\int_p^1 \mathrm{d}\alpha\mu(\mathrm{d}p) = 1$.

Kusuoka representation by means of distorted premium principles. By the preceding discussion there is a one-to-one relationship $\sigma \mapsto \mu_\sigma$ given by (8) (with inverse $\mu \mapsto \sigma_\mu$ given by (10)) such that Kusuoka's representation (Theorem 7) can be formulated with distorted premium functionals equally well,

$$\pi(L) = \sup_{\sigma\in\mathcal{S}} \pi_\sigma(L). \tag{11}$$

$\mathcal{S}$ is a set of distortions. $\mathcal{S}$ can be restricted to consist of continuous and strictly increasing (thus invertible) density functions. A rigorous discussion is rather straightforward (cf. [21]), although beyond the scope of this article. Here it is just important to observe that any premium principle is built of distorted premium functionals by (11).

3 Supremum-representation of distorted premium functionals

The supremum representation of distorted premium functionals is derived from the convex conjugate relation for convex functionals. To formulate the result in a concise way we employ the notion of (second order) stochastic dominance.

Definition 8 (Convex ordering)**.** Let τ, $\sigma : [0,1] \to \mathbb{R}$ be integrable functions.

(i) σ majorizes τ (denoted $\sigma \succcurlyeq \tau$ or $\tau \preccurlyeq \sigma$) iff

$$\int_\alpha^1 \tau(p)\,\mathrm{d}p \le \int_\alpha^1 \sigma(p)\,\mathrm{d}p \quad \text{for all } \alpha\in[0,1) \text{ and } \int_0^1 \tau(p)\,\mathrm{d}p = \int_0^1 \sigma(p)\,\mathrm{d}p.$$

(ii) The distortion σ majorizes the random variable $Z \in \mathbb{L}^1$ ($Z \preccurlyeq \sigma$) iff

$$(1-\alpha)\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(p)\,\mathrm{d}p \quad \text{for all } \alpha \in [0,1) \text{ and } \mathbb{E}(Z) = \int_0^1 \sigma(p)\,\mathrm{d}p.$$

Remark 9. Recall that for the conditional tail expectation it holds that

$$(1-\alpha)\,\mathsf{CTE}_\alpha(Z) = \int_\alpha^1 F_Z^{-1}(p)\,\mathrm{d}p.$$

It thus is evident that

$$Z \preccurlyeq \sigma \text{ if and only if } F_Z^{-1} \preccurlyeq \sigma.$$

Moreover $Z \preccurlyeq \sigma$ is related to *convex order* or *stochastic dominance* conditions, which are studied for example in Stoyan and Müller [17] or Shaked and Shanthikumar [25]. The dominance in convex (concave) order was used in studying risk measures for example in Föllmer and Schied [9], and in Dana [6], as well as in Shapiro [26].

The following Theorem 10 characterizes distorted premium functionals by employing the convex conjugate relationship for the dual. It is formulated for $\mathbb{L}^p$ spaces, although it can be proved for an even larger space, the space of natural domain $\mathbb{L}_\sigma$ ($\mathbb{L}_\sigma \supset \mathbb{L}^p$, cf. [22]). The proof below is elaborated for $L \in \mathbb{L}^p$, although $L \in \mathbb{L}^\infty$ is sufficient in practical actuarial applications.

Theorem 10 (Representation of distorted premium functionals as a supremum by stochastic order constraints.)**.** *Let $\sigma \in \mathbb{L}^q$ and π_σ be a distorted premium functional on $\mathbb{L}^p$. Then the representation*

$$\begin{aligned}\pi_\sigma(L) &= \sup\left\{\mathbb{E}(LZ):\ Z \in \mathbb{L}^q,\ Z \preccurlyeq \sigma\right\} \qquad (12)\\ &= \sup\left\{\mathbb{E}(LZ):\ Z \in \mathbb{L}^q,\ \mathbb{E}(Z) = 1,\ (1-\alpha)\,\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(u)\,\mathrm{d}u,\ 0 \leq \alpha < 1\right\}\end{aligned}$$

holds true, where p and q are conjugate exponents ($\frac{1}{p} + \frac{1}{q} = 1$).

Remark 11. First note that the conditions $(1-\alpha)\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(u)\,\mathrm{d}u$ and $\mathbb{E}(Z) = 1$ together imply that $Z \geq 0$ almost everywhere. Indeed, suppose that $P(Z < 0) =: \alpha > 0$. Then $1 = \mathbb{E}(Z) = \int_{\{Z<0\}} Z\mathrm{d}P + \int_{\{Z\geq 0\}} Z\mathrm{d}P = \int_{\{Z<0\}} Z\mathrm{d}P + (1-\alpha)\,\mathsf{CTE}_\alpha(Z)$. As $\int_{\{Z<0\}} Z\mathrm{d}P < 0$ it follows that $(1-\alpha)\ \mathsf{CTE}_\alpha(Z) > 1$. But this contradicts the fact that $(1-\alpha)\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(u)\,\mathrm{d}u \leq 1$, hence Z is nonnegative, $Z \geq 0$ almost surely.

Proof of Theorem 10. Recall first that π_σ is continuous by [27], Proposition 6.5, such that π_σ is given by the Legendre–Fenchel transformation for convex functions as

$$\begin{aligned}\pi_\sigma(L) &= \sup_{Z \in \mathbb{L}^q} \mathbb{E}(L \cdot Z) - \pi_\sigma^*(Z), \text{ where}\\ \pi_\sigma^*(Z) &= \sup_{L \in \mathbb{L}^p} \mathbb{E}(L \cdot Z) - \pi_\sigma(L) \qquad (Z \in \mathbb{L}^q). \qquad (13)\end{aligned}$$

The conjugate function π_σ^* is further

$$\pi_\sigma^*(Z) = \sup_{L \in \mathbb{L}^\infty} \mathbb{E}(L \cdot Z) - \pi_\sigma(L),$$

because π_σ is continuous and as L^∞ is dense in L^p.

As π_σ is version independent the random variable L minimizing (13) is coupled in a comonotone way with Z (cf. [13] and [20, Proposition 1.8] for the respective rearrangement inequality, sometimes referred to as *Hardy and Littlewood's inequality* or *Hardy–Littlewood–Pólya inequality*, cf. [6]). It follows that

$$\begin{aligned}\pi_\sigma^*(Z) &= \sup_{L\in\mathbb{L}^\infty} \mathbb{E}(LZ) - \pi_\sigma(L)\\ &= \sup_{F_L,\, L\in\mathbb{L}^\infty} \int_0^1 F_L^{-1}(\alpha)\, F_Z^{-1}(\alpha)\,\mathrm{d}\alpha - \int_0^1 F_L^{-1}(\alpha)\,\sigma(\alpha)\,\mathrm{d}\alpha,\end{aligned}$$

the supremum being among all cumulative distribution functions $F_L(y) = P(L \le y)$ of a bounded random variable L. Define $G(\alpha) := \int_\alpha^1 F_Z^{-1}(u)\,\mathrm{d}u$ and $S(\alpha) := \int_\alpha^1 \sigma(u)\,\mathrm{d}u$, then

$$\begin{aligned}\pi_\sigma^*(Z) &= \sup_{F_L} \int_0^1 F_L^{-1}(\alpha)\,\mathrm{d}\left(S(\alpha) - G(\alpha)\right)\\ &= \sup_{F_L} F_L^{-1}(\alpha)\left(S(\alpha) - G(\alpha)\right)\Big|_{\alpha=0}^1 - \int_0^1 S(\alpha) - G(\alpha)\,\mathrm{d}F_L^{-1}(\alpha)\\ &= \sup_{F_L} F_L^{-1}(0)\left(G(0) - S(0)\right) + \int_0^1 G(\alpha) - S(\alpha)\,\mathrm{d}F_L^{-1}(\alpha) \qquad (14)\end{aligned}$$

by integration by parts of the Riemann–Stieltjes integral and as $L \in \mathbb{L}^\infty$.

Consider the constant random variables $L \equiv c$ ($c \in \mathbb{R}$), then $F_L^{-1} \equiv c$ and, by (14),

$$\pi_\sigma^*(Z) \ge \sup_{c\in\mathbb{R}} c\left(G(0) - S(0)\right).$$

Note now that $S(0) = \int_0^1 \sigma(u)\,\mathrm{d}u = 1$, hence

$$\pi_\sigma^*(Z) \ge \sup_{c\in\mathbb{R}} c\left(G(0) - 1\right) = \begin{cases} 0 & \text{if } G(0) = 1\\ \infty & \text{else}\end{cases} \quad = \begin{cases} 0 & \text{if } \mathbb{E}(Z) = 1\\ \infty & \text{else,}\end{cases}$$

because

$$G(0) = \int_0^1 F_Z^{-1}(u)\,\mathrm{d}u = \mathbb{E}(Z). \qquad (15)$$

Assuming $\mathbb{E}(Z) = 1$ it follows from (14) that

$$\pi_\sigma^*(Z) = \sup_{F_L} \int_0^1 G(\alpha) - S(\alpha)\,\mathrm{d}F_L^{-1}(\alpha).$$

Then choose an arbitrary measurable set B and consider the random variable $L_c := c \cdot \mathbb{1}_{B^\complement}$ for some $c > 0$. Note that $F_{L_c}^{-1} = c \cdot \mathbb{1}_{[\alpha_0,1]}$, where $\alpha_0 = P(B)$. With this choice

$$\begin{aligned}\pi_\sigma^*(Z) &\ge \sup_{F_{L_c}} \int_0^1 G(\alpha) - S(\alpha)\,\mathrm{d}F_{L_c}^{-1}(\alpha) = \sup_{c\ge 0} c\left(G(\alpha_0) - S(\alpha_0)\right) =\\ &= \begin{cases} 0 & \text{if } G(\alpha_0) \le S(\alpha_0)\\ \infty & \text{else.}\end{cases}\end{aligned}$$

As B was chosen arbitrarily it follows that $G(\alpha) \leq S(\alpha)$ has to hold for any $0 \leq \alpha \leq 1$ for Z to be feasible, which means in turn that $(1-\alpha)\,\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(u)\,\mathrm{d}u$.

Conversely, if (15) and $G(\alpha) \leq S(\alpha)$ for all $0 \leq \alpha \leq 1$, then

$$\sup_{F_L} \int_0^1 G(\alpha) - S(\alpha)\,\mathrm{d}F_L^{-1}(\alpha) \leq 0,$$

because $F_L^{-1}(\cdot)$ is a nondecreasing function. Note now that

$$\int_\alpha^1 \sigma(u)\,\mathrm{d}u = S(\alpha) \geq G(\alpha) = \int_\alpha^1 F_Z^{-1}(u)\,\mathrm{d}u = (1-\alpha)\,\mathsf{CTE}_\alpha(Z),$$

from which finally follows that

$$\pi_\sigma^*(Z) = \begin{cases} 0 & \text{if } \mathbb{E}(Z) = 1 \text{ and } (1-\alpha)\,\mathsf{CTE}_\alpha(Z) \leq \int_\alpha^1 \sigma(u)\,\mathrm{d}u \;\; (0 \leq \alpha \leq 1) \\ \infty & \text{else,} \end{cases}$$

which is the assertion. □

The following statement, which will be essential in what follows, derives naturally as a corollary of Theorem 10 (cf. also [20]).

Corollary 12. *Let π_σ be a distortion risk functional on $\mathbb{L}^p$ ($p \in [1,\infty]$, $\sigma \in \mathbb{L}^q$), then*

$$\pi_\sigma(L) = \sup\left\{\mathbb{E}(L \cdot \sigma(U)) : U \text{ is uniformly distributed}\right\}, \tag{16}$$

where the supremum is attained if L and U are coupled in a comonotone way.

Remark 13. The statement of the corollary implicitly and tacitly assumes that the probability space is rich enough to carry a uniform random variable. This is certainly the case if the probability space does not contain atoms. But even if the probability space has atoms, then this is not a restriction either, as any probability space with atoms can be extended as follows to allow a uniformly distributed random variable.

Indeed, for a probability space (Ω, Σ, P) with a single atom[4] $A \in \Sigma$ the extension is the space $\left(\Omega \times [0,1], \tilde{\Sigma}, \tilde{P}\right)$, where $\tilde{\Sigma}$ is the product sigma algebra and $\tilde{P}$ is the measure defined by $\tilde{P}(S \times T) := P(S \backslash A) + P(S \cap A) \cdot P_\lambda(T)$, where P_λ is the Lebesgue measure on the standard space $[0,1]$; the extension of the random variable L is provided by the projection $\tilde{L}(s,t) := L(s)$. For a space with more than one atom the construction is obtained analogously (cf. Ferguson [8], or Propisition 1.3 in [20] for a similar construction).

Proof. Consider $Z := \sigma(U)$ for a uniformly distributed random variable U, then $P(Z \leq \sigma(u)) = P(\sigma(U) \leq \sigma(u)) \geq P(U \leq u) = u$, that is $F_Z^{-1}(u) \geq \sigma(u)$. But as $1 = \int_0^1 \sigma(u)\,\mathrm{d}u \leq \int_0^1 F_{\sigma(U)}^{-1}(u)\,\mathrm{d}u$ $= \mathbb{E}(\sigma(U)) = 1$ it follows that

$$F_{\sigma(U)}^{-1}(\cdot) = \sigma(\cdot)$$

[4] $A \subset \Omega$ is an atom if $P(A) > 0$ and for $B \subset A$ either $P(B) = 0$, or $P(B) = P(A)$ holds.

almost everywhere. Observe now that any Z with $F_Z^{-1}(u) \le \sigma(u)$ is feasible for (12), because

$$\int_\alpha^1 \sigma(u)\,\mathrm{d}u \ge \int_\alpha^1 F_Z^{-1}(u)\,\mathrm{d}u = (1-\alpha)\mathsf{CTE}_\alpha(Z)$$

and $\mathbb{E}(Z) = \mathbb{E}(\sigma(U)) = \int_0^1 \sigma(u)\,\mathrm{d}u = 1$. Now let U be coupled in an comonotone way with L, then $\mathbb{E}(LZ) = \int_0^1 F_L^{-1}(u)\,F_Z^{-1}(u)\,\mathrm{d}u = \int_0^1 F_L^{-1}(u)\,F_{\sigma(U)}^{-1}(u)\,\mathrm{d}u = \int_0^1 F_L^{-1}(u)\,\sigma(u)\,\mathrm{d}u$ so that

$$\pi_\sigma(L) = \sup\{\mathbb{E}(L\cdot\sigma(U)) : U \text{ uniformly distributed}\},$$

which is finally the second assertion. □

4 Infimum representation of distortion premium functionals

The latter Theorem 10 exposes the distorted risk premium as a supremum and characterizes the convex conjugate function by stochastic dominance constraints. The following theorem, the second main result of this article, provides a description in opposite terms, as an infimum. The representation extends the well known formula for the conditional tail expectation (Average Value-at-Risk) provided in [23], finally stated in the present form in [18].

This alternative description allows an alternative view on distortions and alternative simulations, as is the content of the following section.

Theorem 14 (Representation as an infimum)**.** *For any $L \in \mathbb{L}^p$ the distorted premium functional with distortion $\sigma \in \mathbb{L}^q$ has the representation*

$$\pi_\sigma(L) = \inf_h \mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u, \tag{17}$$

where the infimum is among all measurable functions $h\colon \mathbb{R}\to\mathbb{R}$ with $\mathbb{E}(h(L)) > -\infty$, and h^* is h's convex conjugate function.[5]

Remark 15. Having a look at representation (17) it is not immediate that the axioms of Definition 3 are satisfied. The transformations listed in Lemma 20 in the Appendix can be used in a straightforward manner to deduce the properties directly from (17).

The statement of the inf-representation, Theorem 14, can be formulated equivalently in the following ways.

Corollary 16. *For any $L \in \mathbb{L}^p$ the distorted risk premium with distortion σ allows the representations*

$$\pi_\sigma(L) = \inf_{h \text{ convex}} \mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u \tag{18}$$

and

$$\pi_\sigma(L) = \inf\left\{\mathbb{E}(h(L)) : \int_0^1 h^*(\sigma(u))\,\mathrm{d}u \le 0\right\}, \tag{19}$$

where the infimum in (18) is among convex functions $h\colon\mathbb{R}\to\mathbb{R}$, and the infimum in (19) is among arbitrary measurable functions with $\mathbb{E}(h(L)) > -\infty$.

[5]The convex conjugate function of h is $h^*(y) := \sup_x x\cdot y - h(x)$. The convex conjugate may become $+\infty$.

Proof of Corollary 16. If h is convex there is $c \in \mathbb{R}$ (the subderivative) such that $h(x) \geq h(x_0) + c(x - x_0)$. It follows that $\mathbb{E}(h(L)) \geq h(x_0) + c(\mathbb{E}(L) - x_0) > -\infty$, as $L \in \mathbb{L}^p \subset \mathbb{L}^1$, and the condition $\mathbb{E}(h(L)) > -\infty$ thus is automatically satisfied for convex functions.

It is well known that the bi-conjugate function $h^{**} := (h^*)^*$ is a convex and lower semicontinuous function satisfying $h^{**} \leq h$ and $h^{***} = h^*$ (cf. the analogous Fenchel–Moreau Theorem and equation (13)). Assuming Theorem 14 the infimum in (17) hence—without any loss of generality—can be restricted to *convex* functions, that is (18), i.e.,

$$\pi_\sigma(L) = \inf_{h \text{ convex}} \mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u.$$

As for the second assertion, (19), notice first that clearly

$$\begin{aligned}\pi_\sigma(L) &\leq \inf\left\{\mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u : \int_0^1 h^*(\sigma(u))\,\mathrm{d}u \leq 0\right\}\\ &\leq \inf\left\{\mathbb{E}(h(L)) : \int_0^1 h^*(\sigma(u))\,\mathrm{d}u \leq 0\right\}.\end{aligned}$$

Consider $h_\alpha(x) := h(x) - \alpha$ (where α a constant and h arbitrary). It holds that $h_\alpha^*(y) = h^*(y) + \alpha$, as follows from the auxiliary Lemma 20 in the Appendix. Hence $\int_0^1 h_\alpha^*(\sigma(u))\,\mathrm{d}u = \int_0^1 h^*(\sigma(u))\,\mathrm{d}u + \alpha$ and

$$\mathbb{E}(h_\alpha(L)) + \int_0^1 h_\alpha^*(\sigma(u))\,\mathrm{d}u = \mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u. \tag{20}$$

Choose $\alpha := \int_0^1 h^*(\sigma(u))\,\mathrm{d}u$ such that $\int_0^1 h_\alpha^*(\sigma(u))\,\mathrm{d}u = 0$. h_α hence is feasible for (19) with the same objective as h by (20), from which the assertion follows. □

Proof of Theorem 14. From the definition of the convex conjugate h^* it is immediate that

$$h^*(\sigma) \geq y \cdot \sigma - h(y)$$

for all numbers y and σ (this is often called *Fenchel–Young inequality*), hence

$$h(L) + h^*(\sigma(U)) \geq L \cdot \sigma(U),$$

where U is any uniformly distributed random variable, i.e., U satisfies $P(U \leq u) = u$. Taking expectations it follows that

$$\mathbb{E}(h(L)) + \mathbb{E}(h^*(\sigma(U))) \geq \mathbb{E}(L \cdot \sigma(U)).$$

As U is uniformly distributed it holds that

$$\mathbb{E}(h^*(\sigma(U))) = \int_0^1 h^*(\sigma(u))\,\mathrm{d}u,$$

such that

$$\mathbb{E}(h(L)) + \int_0^1 h^*(\sigma(u))\,\mathrm{d}u \geq \mathbb{E}(L \cdot \sigma(U)),$$

irrespective of the uniform random variable U. Hence, by (16) in Corollary 10,

$$\mathbb{E}\left(h\left(L\right)\right)+\int_0^1 h^*\left(\sigma\left(u\right)\right)\mathrm{d}u \geq \sup_{U \text{ uniform}} \mathbb{E}\left(L\cdot\sigma\left(U\right)\right)=\pi_\sigma\left(L\right),$$

establishing the inequality

$$\pi_\sigma\left(L\right)\leq \mathbb{E}\left(h\left(L\right)\right)+\int_0^1 h^*\left(\sigma\left(u\right)\right)\mathrm{d}u.$$

As for the converse inequality consider the function

$$h_\sigma\left(y\right):=\int_0^1 F_L^{-1}\left(\alpha\right)+\frac{1}{1-\alpha}\left(y-F_L^{-1}\left(\alpha\right)\right)_+\mu_\sigma\left(\mathrm{d}\alpha\right). \tag{21}$$

$h_\sigma\left(y\right)$ is well defined for all y because $L\in\mathbb{L}^1$; $h_\sigma\left(y\right)$ is moreover increasing and convex, because $y\mapsto\left(y-q\right)_+$ is increasing and convex, and because μ_σ is a positive measure.

Recall the formula

$$\mathsf{CTE}_\alpha\left(L\right)=\inf_{q\in\mathbb{R}} q+\frac{1}{1-\alpha}\mathbb{E}\left(L-q\right)_+$$

and the fact that the infimum is attained at $q=F_L^{-1}\left(\alpha\right)$ (cf. [18] or [11, Section 4.1] for the general formula), thus providing the explicit form

$$\mathsf{CTE}_\alpha\left(L\right)=F_L^{-1}\left(\alpha\right)+\frac{1}{1-\alpha}\mathbb{E}\left(L-F_L^{-1}\left(\alpha\right)\right)_+.$$

Note now that, by Fubini's Theorem and (9),

$$\begin{aligned}\pi_\sigma\left(L\right)&=\int_0^1 \mathsf{CTE}_\alpha\left(L\right)\mu_\sigma\left(\mathrm{d}\alpha\right)\\&=\int_0^1 F_L^{-1}\left(\alpha\right)+\frac{1}{1-\alpha}\mathbb{E}\left(L-F_L^{-1}\left(\alpha\right)\right)_+\mu_\sigma\left(\mathrm{d}\alpha\right)\\&=\mathbb{E}\left(\int_0^1 F_L^{-1}\left(\alpha\right)+\frac{1}{1-\alpha}\left(L-F_L^{-1}\left(\alpha\right)\right)_+\mu_\sigma\left(\mathrm{d}\alpha\right)\right)\\&=\mathbb{E}\left(h_\sigma\left(L\right)\right).\end{aligned} \tag{22}$$

To establish assertion (17) it remains to be shown that $\int_0^1 h_\sigma^*\left(\sigma\left(u\right)\right)\mathrm{d}u\leq 0$. For this observe first that h_σ is almost everywhere differentiable (because it is convex), with derivative

$$\begin{aligned}h_\sigma'\left(y\right)&=\int_{\left\{\alpha:\ F_L^{-1}\left(\alpha\right)\leq y\right\}}\frac{1}{1-\alpha}\mu_\sigma\left(\mathrm{d}\alpha\right)\\&=\int_0^{F_L(y)}\frac{1}{1-\alpha}\mu_\sigma\left(\mathrm{d}\alpha\right)=\sigma\left(F_L(y)\right)\end{aligned} \tag{23}$$

(almost everywhere) by relation (10). Moreover $h_\sigma^*\left(\sigma(u)\right)=\sup_y \sigma(u)\cdot y-h_\sigma\left(y\right)$, the supremum being attained at every y satisfying $\sigma(u)=h_\sigma'\left(y\right)=\sigma\left(F_L(y)\right)$, hence at $y=F_L^{-1}\left(u\right)$. It follows that

$$h_\sigma^*\left(\sigma\left(u\right)\right)=\sigma\left(u\right)\cdot F_L^{-1}\left(u\right)-h_\sigma\left(F_L^{-1}\left(u\right)\right).$$

Now

$$\begin{aligned}\int_0^1 h_\sigma^*(\sigma(u))\,\mathrm{d}u &= \int_0^1 \sigma(u)\cdot F_L^{-1}(u)\,\mathrm{d}u - \int_0^1 h_\sigma\left(F_L^{-1}(u)\right)\mathrm{d}u \\ &= \pi_\sigma(L) - \mathbb{E}\left(h_\sigma(L)\right).\end{aligned}$$

But it was established already in (22) that $\pi_\sigma(L) = \mathbb{E}(h_\sigma(L))$, so that

$$\int_0^1 h_\sigma^*(\sigma(u))\,\mathrm{d}u = 0.$$

This finally proves the second inequality. □

Remark 17. Notice that σ has its range in the interval $\{\sigma(u) : u \in [0,1]\} \subset [\sigma(0), \sigma(1)]$, and from the convexity of h^* it follows that the level set $\{h^* < \infty\}$ is convex. Hence $h^*(y) < \infty$ necessarily has to hold for all $y \in (\sigma(0), \sigma(1))$ to ensure that $\int_0^1 h^*(\sigma(u))\,\mathrm{d}u < \infty$. For h convex this means in turn that

$$\lim_{y\to-\infty} h'(y) \le \sigma(0) \text{ and } \lim_{y\to\infty} h'(y) \ge \sigma(1),$$

thus limiting the class of interesting functions in Corollary 16 to convex functions satisfying $h'(\mathbb{R}) \supset (\sigma(0), \sigma(1))$.

CTE as a special case. The conditional tail expectation is a special case of the infimum in (17). Indeed, it follows from (21) in the proof that the infimum is attained at a function of the form $h_q(y) = q + \frac{1}{1-\alpha}(y-q)_+$ with conjugate

$$h_q^*(\sigma) = \begin{cases} -q + q\,\sigma & \text{if } 0 \le \sigma \le \frac{1}{1-\alpha} \\ \infty & \text{else.}\end{cases}$$

It holds that

$$\begin{aligned}\int_0^1 h_\sigma^*(\sigma_\alpha(u))\,\mathrm{d}u &= \int_0^\alpha h_\sigma^*(0)\,\mathrm{d}u + \int_\alpha^1 h_\sigma^*\left(\frac{1}{1-\alpha}\right)\mathrm{d}u \\ &= -\alpha q + \left(-q + \frac{q}{1-\alpha}\right)(1-\alpha) = 0,\end{aligned}$$

so that

$$\mathsf{CTE}_\alpha(L) = \inf_{q\in\mathbb{R}} \mathbb{E}\left(h_q(L)\right) = \inf_q\, q + \frac{1}{1-\alpha}\mathbb{E}(L-q)_+, \tag{24}$$

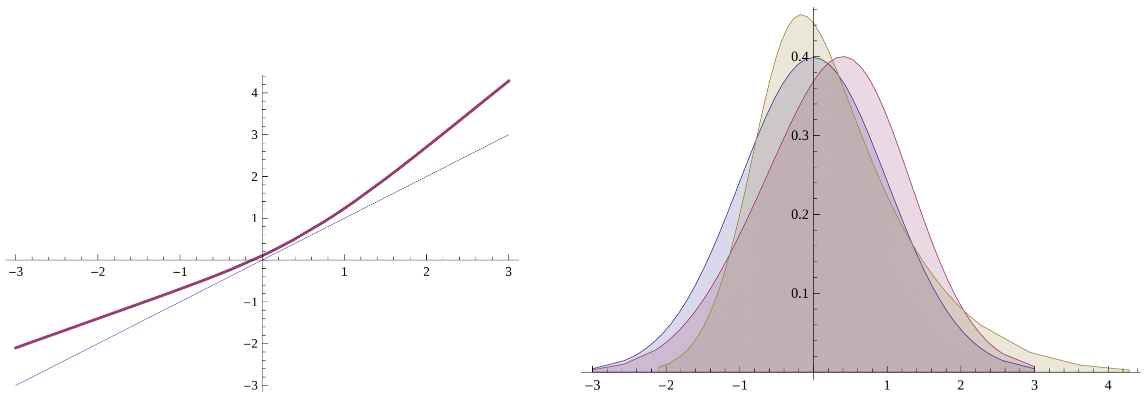
the classical result. Clearly, the infimum in (24) is over $\mathbb{R}$, a much smaller space than convex functions from $\mathbb{R}$ to $\mathbb{R}$ as required in (17).

5 Implications for actuarial science and claim sampling

5.1 Comparison of L_σ and L'_σ

In the introductory discussion it was outlined that claims can be sampled (based on (2)) by use of

$$L_\sigma = F_L^{-1}\left(\tau_\sigma^{-1}(U)\right).$$



(a) The function h_σ of the normal distribution for distortion function $\sigma(u) = 0.7 + 0.9u^2$.

(b) The density of the normal distribution, together with the density of L_σ (right mode) and density of L'_σ (left mode).

Figure 1: Distortion of the standard normal distribution.

It is obvious by this formula that the distorted claims L_σ have the same outcomes as L, but their probability is disturbed by involvement of the function τ_σ.

The infimum representation developed in Section 4 suggests to consider the random variable

$$L'_\sigma := h_\sigma(L),$$

where h_σ is the function defined in (21), and which is the optimal function for problem (19). For this function it holds that

$$\mathbb{E}(L'_\sigma) = \pi_\sigma(L) = \mathbb{E}(L_\sigma),$$

because $\int_0^1 h^*_\sigma(\sigma(u))\,\mathrm{d}u = 0$. We have moreover that

$$h_\sigma(y) = \int_0^1 F_L^{-1}(\alpha) + \frac{1}{1-\alpha}\left(y - F_L^{-1}(\alpha)\right)_+ \mu_\sigma(\mathrm{d}\alpha) \geq \int_0^1 y\,\mu_\sigma(\mathrm{d}\alpha) = y,$$

from which follows that

$$L'_\sigma \geq L,$$

that is, L'_σ stochastically dominates L in first order. The cumulative distribution function of L'_σ has the explicit form

$$F_{L'_\sigma}(y) = P(h_\sigma(L) \leq y) = P\left(L \leq h_\sigma^{-1}(y)\right) = F_L\left(h_\sigma^{-1}(y)\right),$$

and the density is $f_{L'_\sigma}(y) = \frac{f_L\left(h_\sigma^{-1}(y)\right)}{h'_\sigma\left(h_\sigma^{-1}(y)\right)} = \frac{f_L\left(h_\sigma^{-1}(y)\right)}{\sigma\left(F_L\left(h_\sigma^{-1}(y)\right)\right)}$ by use of (23). The quantile function

$$F_{L'_\sigma}^{-1} = h_\sigma \circ F_L^{-1}. \tag{25}$$

is obtained by inversion.

Example 18. Figure 1 contains the densities of both distortions, L_σ and L'_σ, for the standard normal distribution. The distortion function chosen in this example is $\sigma(u) = 0.7 + 0.9u^2$. This example reveals that the mode, as well as the tails of the random variables L_σ and L'_σ differ significantly; the tails of L'_σ are heavier. The mode $m(L'_\sigma)$ of L'_σ is even left of L's mode, $m(L'_\sigma) < m(L) < m(L_\sigma)$, although their mean satisfies $\mathbb{E}(L) \le \mathbb{E}(L'_\sigma) = \mathbb{E}(L_\sigma)$.

Opposite perspectives. The latter formula (25) reveals that L'_σ has distorted outcomes, distorted by h_σ, but the probabilities are unchanged. So L'_σ can be considered as alternative to (4), doing exactly the opposite of L_σ (cf. (4)): L'_σ has the same probabilities as L, but the outcomes are distorted by h_σ whereas L_σ has the same outcomes as L, but the probabilities are distorted by τ_σ. However, both, L_σ and L'_σ, have the same expected value

$$\mathbb{E}(L_\sigma) = \pi_\sigma(L) = \mathbb{E}(L'_\sigma).$$

Explicit distances. As the cumulative distribution function is available for L_σ and L'_σ as elaborated, explicit expressions are available for selected distances of random variables. An explicit representation of the Kolmogorov–Smirnov distance for example is

$$\sup_{y\in\mathbb{R}} \left|F_L(y) - \tau_\sigma\left(F_L\left(h_\sigma(y)\right)\right)\right|,$$

and the Wasserstein distance of order r (cf. [30]) has the explicit formula

$$\left(\int_0^1 \left|h_\sigma\left(F_L^{-1}\left(\tau_\sigma(y)\right)\right) - F_L^{-1}(y)\right|^r \sigma(u)\,\mathrm{d}u\right)^{1/r}.$$

5.2 Actuarial Applications

Actuarial concerns have been addressed at various points in the paper, however, we stress again that π, π_σ and in particular CTE constitute premium principles to price individual contracts. For a given loss distribution with monotone (increasing, or decreasing) loss function L (note that this is almost always the case in life insurance), the function $\pi_\sigma(L) = \int_0^1 F_L^{-1}(u)\,\sigma(u)\,\mathrm{d}u$ can be given in closed form.

Example 19. Considering the simple life expectancy,[6] i.e., the random variable $L(k) = k$ (which is strictly increasing), then $F_L^{-1}({}_kq_x) = k$ and

$$\pi_\sigma(L) = \int_0^1 F_L^{-1}(u)\,\sigma(u)\,\mathrm{d}u = \sum_{k=0}^{\infty} k \cdot \int_{{}_kq_x}^{{}_{k+1}q_x} \sigma(u)\,\mathrm{d}u$$

is the distorted life expectancy.

[6] Note that the life expectancy is an annuity with an interest rate of 0 %, so we have chosen an annuity as a representative example for a typical life insurance contract. Considering the life expectancy allows moreover to exclude the interest rate in order to simplify the presented results.

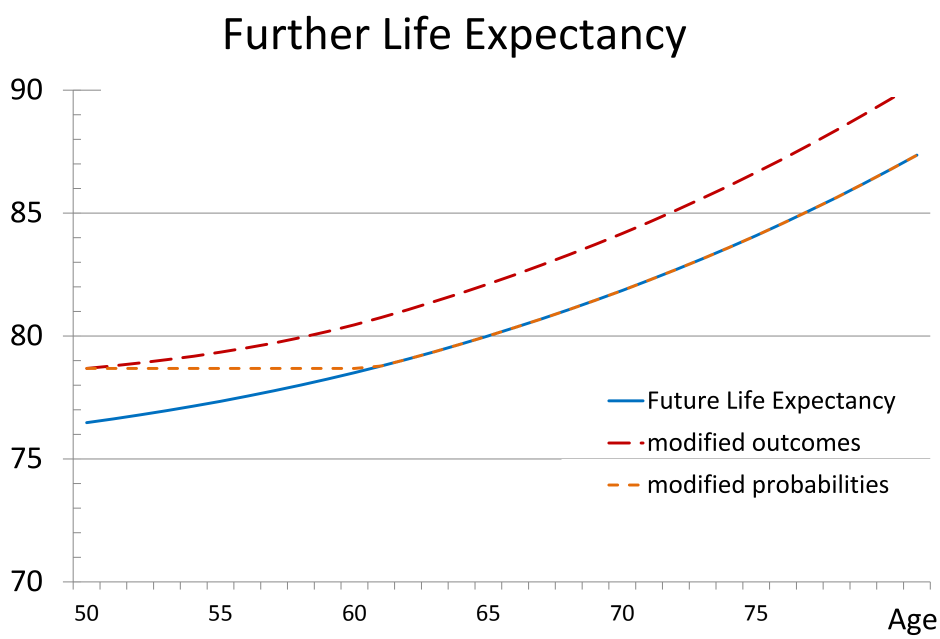


Figure 2: Further life expectancies based on distorted outcomes, and based on distorted probabilities. The distortion employed is the conditional tail expectation at the level of 10 %, $\mathsf{CTE}_{10\%}$.

Distorted probabilities. Following (3) one may consider $\int_{_k q_x}^{_{k+1} q_x} \sigma(u) =: {}_k\tilde{p}_x \cdot \tilde{q}_{x+k}$ as a probability of a new life table (indicated by the tilde), and use this new life table to compute premiums, as well as reserves. This is illustrated in Figure 2. It is visible in this chart that the modified life table increases the life expectancy by approximately 2 years initially, but the increasing effect disappears at the age representing the quantile (here, at the age of 60 years for $\alpha = 10\,\%$, considering a person with an initial age of 50). For this reason it is appropriate to use $\pi_\sigma(L)$ as a premium, but it is *not* desirable to use the new life table to compute reserves. The reserves loose the safety loading by employing the new life table, whenever the age exceeds the quantile.

Distorted outcomes. As already indicated it is natural to use the distorted outcomes instead of distorted probabilities in actuarial practice. To compute premiums the above discussion applies equally well, and an explicit form is available to compute the premium. For the exposed case of life expectancy the result is

$$\pi_\sigma(L) = \int_0^1 F_L^{-1}(u)\,\sigma(u)\,\mathrm{d}u = \sum_{k=0} h_\sigma(k) \cdot {}_k p_x\, q_{x+k}.$$

The big advantage of distorted outcomes is that the reserves can be handled with the same ingredients as the premium, that is with the same probabilities and the same function h_σ: L simply needs to be replaced by $L'_\sigma = h_\sigma(L)$. It is evident in Figure 2 that the safety loading is preserved over time.

Distorted premiums, interpreted as distorted outcomes, are thus a reliable premium principle which provide not only premiums, but also reserves in a correct and time-consistent way. The distorted premium principle π_σ to compute the reserves can be applied by the actuary easily, and along with the related outcomes distorted by h_σ.

6 Concluding remarks

This article outlines new descriptions of distorted premium principles. Distorted premium principles constitute a basic class of premium principles, as every premium satisfying sufficiently strong axioms can be built by involving just elementary distortions.

The first representation derived is described as a supremum and based on conjugate duality. The convex conjugate function is formulated in terms of second order stochastic dominance constrains.

The other representation, which is a further central result of this article, is described as an infimum and can be considered as the opposite formulation. This alternative description makes distorted premiums eligible for successful use in actuarial applications, as the reserve process is easily available for concrete insurance contracts and, above all, the process of reserves is consistent over time. The results thus make distorted premiums eligible for extended actuarial use.

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Appendix

For reference and for the sake of completeness we list the following elementary result for affine linear transformations of the convex conjugate function.

Lemma 20. *The convex conjugate of the function* $g(x) := \alpha + \beta x + \gamma \cdot f(\lambda x + c)$ *for* $\gamma > 0$ *and* $\lambda \neq 0$ *is*

$$g^*(y) = -\alpha - c\,\frac{y-\beta}{\lambda} + \gamma \cdot f^*\left(\frac{y-\beta}{\lambda\gamma}\right).$$

Proof. Just observe that

$$
\begin{aligned}
g^*(y) &= \sup_x \; yx - g(x) \\
&= \sup_x \; yx - \alpha - \beta x - \gamma \cdot f(\lambda x + c) \\
&= \sup_x \; y\frac{x-c}{\lambda} - \alpha - \beta\frac{x-c}{\lambda} - \gamma \cdot f(x) \\
&= -\alpha - c\frac{y-\beta}{\lambda} + \sup_x \; x\frac{y-\beta}{\lambda} - \gamma \cdot f(x) \\
&= -\alpha - c\frac{y-\beta}{\lambda} + \gamma \cdot \sup_x \; x\frac{y-\beta}{\lambda\gamma} - f(x) \\
&= -\alpha - c\frac{y-\beta}{\lambda} + \gamma \cdot f^*\left(\frac{y-\beta}{\lambda\gamma}\right),
\end{aligned}
\tag{26}
$$

where we have replaced x by $\frac{x-c}{\lambda}$ in (26). □