Quantitative Asset

and Risk Management

Exercises[1]

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1. Interest Rates

1.1 Suppose you were given 1 cent in year 0. Given an interest rate of 1%, what is your wealth today? And for an interest rate of 2%?

1.2 Give d, r, δ and v for $i = 5\%$, for $\delta = 3\%$ and for $v = 94\%$.

1.3 Give $d^{(12)}$, $i^{(12)}$ and $d^{(\infty)}$, $i^{(\infty)}$ for $i = 3\%$.

1.4 Verify $\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$.

1.5 What is the internal rate of return given this following, annual cash-flow: -1,000, 100, 100, 100, 100, 100, 100, 100 and 200? And reverse, that is 200, 100, 100, 100, 100, 100, 100, 100 and -1000?

1.6 Give the internal rate of return for the Austrian bond ISIN AT0000383864

- straight forward,
- taking withholding tax into account.

1.7 As above for the

- Finnish FI4000018049 and
- the US bond ISIN US912828JH40. How will the return vary, if the currency jumps/ drops +/- 10%?

1.8 A payment of € 100 is due every month. What is the respective present value? Compute this value explicitly for some realistic interest rates.

1.9 Verify the formulae

$$\ddot{a}_n = \frac{1-v^n}{1-v},$$
$$\ddot{a}_n^{(m)} = \frac{1}{m}\frac{1-v^n}{1-v^{\frac{1}{m}}}$$

and evaluate

$$a_n := \sum_{j=1}^{\infty} v^j.$$

1.10 Compare $\ddot{a}_n^{(m)} = \frac{1}{m}\frac{1-v^n}{1-v^{\frac{1}{m}}}$ with $\ddot{a}_n - \frac{m-1}{2m}(1-v^n)$ for some representative settings.

1.11 Verivy

$$\sum_{i=0}^{n-1} i \cdot v^i = \frac{nv^{n+1} + v - nv^n - v^{n+1}}{(1-v)^2}.$$

Derive, that

$$\sum_{i=0}^{\infty} i \cdot v^i = \frac{v}{(1-v)^2} < \infty.$$

Interpret this formala as annuity. – What is the related annual payment? Aren't you surprised that this quantity is still finite?

1.12 A person adds these following amounts to his saving book:

time	1.1.2010	7.7.2010	1.2.2011	1.10.2012	13.10.2015	1.4.2016
amount	€ 1,000	€ 1,000	€ 1,500	€ 2,000	€ 3,000	€ 500

The savings' book face amount is € 10,000 at 1.10.2016. What is the APR?

1.13 How does withholding tax impact the internal rate of return? Explain, given the example above and another.

1.14 Suppose the force of interest will evolve in time as $\delta(t) = 3\% + 1\% * t$. What would you be willing to pay for a cero-coupon bond which will pay € 10,000 in 2 years time?

1.15 The Austrian state subsidizes contributions (!) to *Bausparen* currently with 9%. What is the impact to the internal yield, given a 6 year's term?

1.16 The Austrian state subsidizes contributions to *Zukunftsvorsorge* currently with 9%. What is the impact to the internal yield, given a person at age 30, saving till retirement?

1.17 A savings book pays these following amounts in case of early withdrawal:

time	$<$ 1 year	1- 2 years	2- 3 years	3- 4 years	4- 5 years	5 years
annual interest rate	0.1%	0.2%	0.3%	0.3%	0.4%	5%

Suppose the probabilities for an early withdrawal is 5% per year. What is the APR the bank will actually need to cover the savings book?

1.18 Verify that $v^n + d \cdot \ddot{a}_n = 1$.

1.19 Compute the duration for any of the examples above (if applicable).

1.20 What is (an)

- Asset (Aktiva),
- security (Wertpapier),
- bond (Anleihe),
- stock (Aktie),
- equity (Stammkapital),
- hybrid instrument (Wandelanleihe),
- warrant,
- derivative,
- option,
- forward,
- future,
- swap,
- liquidity and
- completeness?

1.21 Give a realistic example of a 10 years loan and compute the internal yield.

- An initial fee is due at the beginning of a loan. How does an increase of the initial fee of 1% – approximately – impact the yield and the installment?
- How does an increase of interest rate by 1% increase the installment?

Compare and discuss the results.

1.22 Give the present value of an annuity which is paid m-times a year in the amount of $\frac{1}{m}$.

1.23 A loan of € 100.000 is to be repaid in the next $n = 10$ years at an interest of 5% and 7%. What is the respective installment?

Verify that the outstanding amount after k years is $a_{n-k} = \frac{1}{v^k}(a_n - a_k)$, and interpret this equation!

1.24 The estimated probability of a loan to default in the next year is $q = 3\%$, the estimate for all subsequent years is $i = 4\%$. Give the probability for a default within the

- next two years
- next 10 years.

1.25 What is the price (lump-sum) of a (single) payment which is due in future, in t years? What is the duration? Suppose the payment is due *only* if another, independing event occurs, its probability being ${}_tp$. Again, give the price and the duration?

1.26 (Catastrophy bond) An annual event (earthquake, eg.) occures every year with probability p. An annual payment is due if and only if the specified event never occured before. What is the price (lump-sum) you are willing to pay for this investment, given a constant interest rate i?

2. Optimium and Extrema

2.1 Find the optimum of $x \mapsto 17 - 9x - 3x^2 + x^3$, when $x \in [-10, 4]$.

2.2 Find the optimum of $(x, y) \mapsto 28 - x^2 - y^4$, provided that $x + y = 2$.

2.3 Find the maximum of $(x, y) \mapsto 28 + 11x + 4.2y$, subject to $y - x \leq 2$ and $8x + 2y \leq 17$.

2.4 What is the Lagrangian of a problem with restrictions?

3. Probability Distributions

3.1 Explain the probability triple $(\Omega, \Sigma, \mathbb{P})$.

3.2 Explain the role of a random variable $X : \Omega \to \mathbb{S}$ and the notation $\mathbb{P}[X \in A] := \mathbb{P}[\{\omega : X(\omega) \in A\}]$. (In all our examples, $\mathbb{S} = \mathbb{R}$)

3.3 Verify that $\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

3.4 Compute the expectation $\mathbb{E}[X]$ and variance $\mathbb{V}ar[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$ for the random Variable satisfying $\mathbb{P}[X = 1] = p$ and $\mathbb{P}[X = 0] = 1 - p$.

3.5 Explain $\mathbb{E}[g(X)]$ for some examples, eg. for

$$\begin{aligned} g :& \mathbb{R} \to \mathbb{R} \\ & x \mapsto 1, \end{aligned}$$

$$\begin{aligned} g :& \mathbb{R} \to \mathbb{R} \\ & x \mapsto x \end{aligned}$$

and

$$\begin{aligned} g :& \mathbb{R} \to \mathbb{R} \\ & x \mapsto x^2. \end{aligned}$$

How can we denote $\mathbb{E}[g(X)]$ for discrete (continuous, respectively) random variables?

3.6 Let X_i be independent copies of the random variable from the previous example. Define $Z := \sum_{i=1}^n X_i$ and compute $\mathbb{E}[Z]$ and $\mathbb{V}ar[Z]$ for this resulting random variable.

3.7 Show that Z's distribution (the previous example) is

$$\mathbb{P}[Z \leq k] = \sum_{i \leq k} \binom{n}{i} p^i (1-p)^{n-i}$$

for $0 < p < 1$ and $n \in \mathbb{N}$; explain, using an appropriate plot.

3.8 Explain and motivate the formula for the convolution for the random variable introduced above.

3.9 Given the random variable Z from above, state

$$\mathbb{E}[g(X)]$$

in explicit terms.

3.10 For discrete random variables set $p_k := \mathbb{P}[X = k]$. Discuss and explain

$$\begin{aligned} \mathbb{E}[g(X)] =& \sum_{k=0}^{n} g_k p_k \\ =& g(0) + \sum_{k=0}^{n-1} (g_{k+1} - g_k) \sum_{j=k+1}^{n} p_j \end{aligned}$$

for $\mathbb{P}[X \geq 0] = 1$.

3.11 For continuous random variables: Discuss, explain, and compare with the latter example

$$\begin{aligned}\mathbb{E}\left[g\left(X\right)\right] &= \int g\left(x\right) f\left(x\right) \mathrm{d}x \\ &= \int g\left(x\right) \mathrm{d}\mathbb{P}\left[X \leq x\right] \\ &= g\left(a\right) + \int_a^\infty g'\left(x\right) \mathbb{P}\left[X > x\right] \mathrm{d}x\end{aligned}$$

if $\mathbb{P}\left[X \geq a\right] = 1$.

3.12 Give the general density of a Normal (Gaussian) Distribution and compute the expectation value and variance. Why is the Gaussian distribution of outstanding importance?

3.13 Let g^{-1} denote the g's inverse function, that is $g^{-1}\left(g\left(x\right)\right) = x$. Verify, that $g\left(g^{-1}\left(y\right)\right) = y$ and prove that

$$\left(g^{-1}\right)'\left(y\right) = \frac{1}{g'\left(g^{-1}\left(y\right)\right)}.$$

3.14 (Transformations of densities) Let f_X be the density of a random variable X, and f_Y the density of the random variable $Y := g\left(X\right)$. Show that $f_Y\left(y\right) = f_X\left(g^{-1}\left(y\right)\right) \cdot \left|\left(g^{-1}\right)'\left(y\right)\right| = \frac{f_X\left(g^{-1}\left(y\right)\right)}{\left|g'\left(g^{-1}\left(y\right)\right)\right|}$.

3.15 Let X be a Gaussian Random variable, $Y := e^X$ $\left(g\left(x\right) := e^x\right)$ is the log-normal distribution.

Give the density.

Discuss, that $\mathbb{E}\left[Y\right] = e^{\mu + \frac{1}{2}\sigma^2}$ and $\mathbb{V}\left[Y\right] = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}$.

3.16 A Stock was observed on these following dates with the following closings:

time t	1.7.	1.8.	1.9.	25.10	31.10.	7.11.	18.11.	3.12.	7.1.	15.1.	28.2
$S_t/$ €	43.12	44.15	40.30	39.30	37.70	38.21	39.22	40.31	39.13	40.10	41.24

What is the stock's volatility?

3.17 Simplify the formula to estimate the volatility for a stock for observations at the end of every month.

3.18 Find a homepage which gives the volatility of a stock in addition and compare your result with the value given.

3.19 Give the marginal probabilities of this following bivariate probability distribution

1%	3%	0%	1%
2%	5%	0	2%
5%	1%	1%	9%
2%	3%	2%	2%
1%	7%	3%	1%
4%	2%	20%	15%
1%	1%	2%	4%

Give some (at least two) other bivariate probability distribution, which has the same marginal distribution.

Is there a "natural" one?

3.20

Y_i	-10.1	-7.2	-3.3	-1.8	2.8	3.1	3.2	3.7	4.1	5.1	8.2
$\mathbb{P}\left[Y = Y_i\right]$	0.02	0.03	0.10	0.12	0.20	.11	.07	.03	.08	.14	.10

Giver the Value at risk for $\alpha = 10\%$, and the Average value at risk!

3.21 Give the cdf. and pdf for both distributions of X and Y. Moreover, compute the expectation, variance and in particular the covariance of both following bivariate distributions:

$\mathbb{P}\begin{bmatrix} X = x_i, \\ Y = y_i \end{bmatrix}$	$x_1 = 3$	$x_2 = 4$	$x_3 = 6$
$y_1 = 2$	10 %	10 %	0 %
$y_2 = 5$	0 %	30 %	0 %
$y_3 = 7$	0 %	10 %	40 %

and

$\mathbb{P}\begin{bmatrix} X = x_i, \\ Y = y_i \end{bmatrix}$	$x_1 = 3$	$x_2 = 4$	$x_3 = 6$
$y_1 = 2$	0 %	0 %	20 %
$y_2 = 5$	0 %	10 %	20 %
$y_3 = 7$	10 %	40 %	0 %

3.22 Give another bivariate distribution such that the covariance in example 3.21 vanishes (is 0).

4. Evaluate Options

4.1 Compute, by explicitely appling the relevant formulae, the price of an european call and put option given the following ingredients:

- Stock today= € 100,
- risk free interest rate= 4%,
- volatility= 40%,
- strike: € 80.

4.2 Find a webpage (an example) on the internet which gives the Greeks and verify them!