Placing Plenty of Poles is Pretty Preposterous

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Abstract

We discuss the pole placement problem for single-input or multi-input control models of the form $\dot{x} = Ax + Bu$. This is the problem of determining a linear state feedback of the form $u = Fx$ such that in the closed-loop system $\dot{x} = (A + BF)x$, the matrix $A + BF$ has a prescribed set of eigenvalues. We analyze the conditioning of this problem and show that it is an intrinsically ill-conditioned problem, and especially so when the system dimension is large. Thus even the best numerical methods for this problem may yield very bad results.

On the other hand, we also discuss the question of whether one really needs to solve the pole placement problem. In most circumstances what is really required is stabilization or that the poles are in a specified region of the complex plane. This related problem may have much better conditioning. We demonstrate this via the example of stabilization.

1 Introduction

Consider a linear control system model of the form

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. We discuss the state-feedback problem, i.e., choosing a feedback matrix $F \in \mathbb{R}^{m \times n}$ such that in the closed-loop

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system

$$\dot{x} = (A + BF)x, \quad x(0) = x_0 \tag{2}$$

the spectrum of $A + BF$ is a prescribed set of eigenvalues $\mathcal{P} := \{\lambda_1, \ldots, \lambda_n\}$. Equivalently, the transfer matrix $(sI - A - BF)^{-1}B$ has poles at $\lambda_1, \ldots, \lambda_n$.

Throughout this paper we assume that this problem is solvable for any specified set of poles, which is equivalent to the system (1) being *controllable*, i.e., $\text{Rank}[\lambda I - A, B] = n$ for all complex numbers λ.

Many algorithms have been proposed for this problem, but as we show in the sequel, the pole placement problem is, in general, an intrinsically ill-conditioned problem. Hence, all known methods may yield bad results, even if they are numerically stable. It is a well-known but often overlooked fact in numerical analysis [10, 28, 29] that one has to distinguish between the conditioning of a problem and the stability of an algorithm. In general, these two concepts have nothing to do with each other. Nevertheless, they are sometimes confused.

A problem is ill conditioned if small perturbations to the data can yield large changes in the solution, while a numerical algorithm is (backwards) unstable if the computed solution is the exact solution of a problem that is far away from the original problem [28]. As a consequence, one can guarantee that the computed solution of a problem is close to the exact solution only if the problem is well conditioned and the algorithm used is stable. In all other cases it can be expected that the computed solution is far from the exact solution.

Many approaches to analyzing the conditioning of the pole placement problem have been proposed; see, for example, [2, 5, 6, 16, 18, 24]. We survey the current state of the research in this area in Sections 2 and 3 and demonstrate that the pole placement problem is, in general, very ill conditioned, especially if the dimension of the system becomes large. This means that small perturbations can lead to drastic changes in the placed eigenvalues. This is disastrous not only because of the rounding errors committed in the computation of the feedback, but also because of the fact that the data and the model to which pole placement is applied are usually noisy and corrupted by measurement or modeling errors. So we cannot expect, in general, that the poles that are placed have anything to do with the actual modes of the practical problem.

Much effort has been devoted in recent years to devising numerically stable algorithms for the solution of the pole placement problem. Recent state-of-the-art algorithms and software, together with discussion of numerical sensitivity issues and numerical experimentation, should be consulted in [20]. It is the purpose of this paper to discuss the pole placement problem and not pole placement algorithms. The reason for this is that often the pole placement problem is only used as a substitute problem to solve another problem, like the stabilization of a system or the movement of poles into a specified region of the complex plane. These problems often have much better conditioning [12].

So the first question that somebody who wants to solve a pole placement problem should ask ought to be: Is this really the problem I want to solve? We are not aware of any realistic application where one truly wants to have the poles exactly at specific positions.

But even if one really wants to solve the pole placement problem, one usually has to specify further conditions since, except for the single-input case ($m = 1$), the solution is not unique. In the multi-input problem, there are many different ways to resolve the non-uniqueness in the pole placement problem. Usually a certain specified cost function is minimized to make the solution unique or at least locally unique. It is clear that one wants to obtain a closed-loop system that is robust to perturbations, not only because of rounding errors in the computations but also because of modeling errors and noise. To obtain such a solution, the cost function that is minimized in [14] is the condition number of the eigenvector matrix of the closed-loop system, since it is well known that this is a measure of the sensitivity of the eigenvalues under perturbations [23]. Another cost function that has been discussed in the context of stabilization, e.g. [12], is the distance to instability, i.e., the smallest perturbation that makes the system unstable. Analogously, if the issue is to place the poles into a certain region in the complex plane, then maximizing the distance to the boundary of this region would be appropriate. Effective algorithms for these latter minimization problems are not known but one can usually get quite good results if one solves a linear-quadratic optimal control problem instead [12, 13]. Another measure that is often minimized is the norm of the feedback matrix F, since in many cases a large norm of the feedback matrix is responsible for bad solutions of the pole placement problem [15]. But even if the norm of the feedback is small the resulting closed-loop system may be very ill conditioned.

Each of these measures has the disadvantage that to obtain the optimum, the complexity of the methods increases greatly. A compromise was introduced in [27], whereby the minimization of the norm of the feedback matrix is carried out locally at each step of the pole placement method.

Despite all these difficulties, pole placement is used frequently as a substitute for other potentially better-conditioned problems.

2 The Single-Input Pole Placement Problem

In this section, we discuss the following single-input pole placement problem:

Given a linear system

$$\dot{x} = Ax + bu, \quad x(0) = x_0 \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, find a vector $f \in \mathbb{R}^n$ such that the closed-loop system

$$\dot{x} = (A + bf^T)x \tag{4}$$

has a prescribed set of poles $\mathcal{P} := \{\lambda_1, \ldots, \lambda_n\}$, i.e., the spectrum of $A+bf^T$ is $\mathcal{P}$. Since we want to have a real closed-loop system, we assume that the set $\mathcal{P}$ is closed under complex conjugation. We assume throughout this section that the pair (A, b) is controllable, so it is known that a unique solution exists.

The perturbation analysis for this problem has been the subject of many publications [2, 5, 6, 14, 16, 18]. The most recent and most complete first-order perturbation result for this problem was given by Sun [24]. We state this result here for completeness. To do so, we first introduce some notation.

For a given solution to the pole placement problem (3), let the eigendecomposition of the closed-loop system matrix be given by

$$A + bf^T = X\Lambda X^{-1}, \tag{5}$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Let $X =: [x_1, \ldots, x_n]$ and $X^{-1} =: Y =: [y_1, \ldots, y_n]$. (Note that we are assuming that the poles to be assigned are pairwise distinct; we thus have a complete set of eigenvectors.) Now let $a = \text{vec}(A)$ and $\lambda = \begin{bmatrix} \lambda_1, \ldots, \lambda_n \end{bmatrix}^T$, where the vec operator forms a vector of length n^2 by successively stacking the n columns of A on top of each other. Then the Jacobians of the mapping from the data (a, b, λ) to the solution of the pole placement problem (see [24]) are given by

$$(-W_f^{-1}W_a, -W_f^{-1}W_b, -W_f^{-1}W_\lambda)$$

respectively, where

$$\begin{aligned} W_f: &= (Y\,\mathrm{diag}(\frac{1}{y_1^T b},\ldots,\frac{1}{y_n^T b}))^{-1} \\ W_a: &= (D_1(X)X^{-1},\ldots,D_n(X)X^{-1}) \\ W_b: &= \mathrm{diag}(f^T x_1,\ldots,f^T x_n)X^{-1} \\ W_\lambda: &= -I_n \end{aligned}$$

and

$$D_i(X) := \mathrm{diag}(x_{i1},\ldots,x_{in}) \tag{6}$$

and I_n is the identity matrix of size n.

Theorem 1 *(Corollary 3.5 in [24]) Given a controllable system (3) and a set $\mathcal{P} = \{\lambda_1,\ldots,\lambda_n\}$ (closed under complex conjugation), where $\lambda_i \neq \lambda_j$ for $i \neq j$, suppose that A and b are slightly perturbed to $\tilde{A}$ and $\tilde{b}$. Suppose further that $\mathcal{P}$ is slightly perturbed to $\tilde{\mathcal{P}} = \{\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n\}$ (also closed under complex conjugation). Let $f, \tilde{f}$ be the solutions to the pole placement problem for (3) with the data A, b, λ and $\tilde{A}, \tilde{b}, \tilde{\lambda}$, respectively (the definition for $\tilde{\lambda}$ being obvious). Then for any consistent matrix norm $\|\cdot\|$ and vector norm $\|\cdot\|$ consistent with it, we have*

$$\begin{aligned} \|\tilde{f} - f\| &\leq \delta_f + \mathrm{O}\left(\left\|\begin{bmatrix}\tilde{a}\\ \tilde{b}\\ \tilde{\lambda}\end{bmatrix} - \begin{bmatrix}a\\ b\\ \lambda\end{bmatrix}\right\|^2\right) \\ &\leq \Delta_f + \mathrm{O}\left(\left\|\begin{bmatrix}\tilde{a}\\ \tilde{b}\\ \tilde{\lambda}\end{bmatrix} - \begin{bmatrix}a\\ b\\ \lambda\end{bmatrix}\right\|^2\right). \end{aligned}$$

Here

$$\begin{aligned} \delta_f &= \|\Phi(\tilde{a} - a) + \Psi(\tilde{b} - b) + \Theta(\tilde{\lambda} - \lambda)\| \\ \Delta_f &= \|\Phi\|\|\tilde{a} - a\| + \|\Psi\|\|\tilde{b} - b\| + \|\Theta\|\|\tilde{\lambda} - \lambda\| \end{aligned}$$

with

$$\begin{aligned} \Theta &= -W_f^{-1}W_\lambda = Y\,\mathrm{diag}(\frac{1}{y_1^T b},\ldots,\frac{1}{y_n^T b}) \qquad (7) \\ \Phi &= -W_f^{-1}W_a = -\Theta(D_1(X)X^{-1},\ldots,D_n(X)X^{-1}) \\ \Psi &= -W_f^{-1}W_b = -\Theta\,\mathrm{diag}(f^T x_1,\ldots,f^T x_n). \end{aligned}$$

Note that the assumption that the closed-loop poles λ_j are distinct is a necessary assumption to have a chance for a well-conditioned problem at all, since it is well known that multiple eigenvalues are very sensitive to perturbations; see e.g. [10], [29].

Based on this result, in [24] the following group of condition numbers for the pole placement problem is obtained (in the Euclidean vector norm and the associated matrix spectral norm $\|\cdot\|_2$):

$$\kappa_A(f) := \|\Phi\|_2 \quad (8)$$

$$\kappa_b(f) := \|\Psi\|_2 \quad (9)$$

$$\kappa_\lambda(f) := \|\Theta\|_2. \quad (10)$$

We can only expect good results from a finite precision algorithm for the single-input pole placement problem if the data are such that all these condition numbers are small. However, our examples show that when the system dimension is bigger than about 15, the condition numbers usually are very large. (Note that the condition numbers in (8)–(10) are based on the spectral decomposition of the closed-loop system $A + bf^T$, hence even the computed condition numbers may not be accurate when $A + bf^T$ is very ill conditioned.)

The numerical examples in this paper were performed in Matlab version 4.1 on an HP 715-33 workstation, with machine epsilon $\epsilon \approx 2.22 \times 10^{-16}$.

Example 1 In a first test we took random examples. The elements of the system matrices A, b were random numbers uniformly distributed in $(-100, 100)$. The system matrix A was of increasing dimension up to 35. The real parts of the assigned poles were random numbers uniformly distributed in $(-100, 0)$ and the imaginary parts of the assigned poles were uniformly distributed in $(-100, 100)$. Sets of poles to be assigned for each system consisted of the maximal number of complex conjugate pairs. This means that when the system order was even, there was no real pole to be assigned and when the system order was odd, there was only one real pole to be assigned. The Matlab code of [19] was used to perform the pole placement. For each specified system size varying from 1 to 35, a hundred pole placement tests were performed. The computed numerical values for the group of condition numbers in (8)–(10), as well as $\|f\|_2$ and cond(X), the condition number of the eigenvector matrix of $A + bf^T$, are shown in Table 1 for system sizes 5, 15, 25, and 35. The average of magnitudes (ave) is the sum of magnitudes divided by 100.

	K_λ	K_A	K_b	$\|f\|_2$	cond(X)
$n = 5$					
min	$1.5e-02$	$7.7e-02$	$1.8e-02$	$7.1e-01$	$8.4e+00$
ave	$3.9e-01$	$2.3e+03$	$5.3e+02$	$2.5e+01$	$2.9e+04$
max	$9.7e+00$	$1.9e+05$	$3.5e+04$	$6.0e+02$	$2.6e+06$
$n = 15$					
min	$4.6e-02$	$5.1e+03$	$2.2e+03$	$2.5e+00$	$2.3e+06$
ave	$1.4e+01$	$1.1e+11$	$2.3e+10$	$8.6e+02$	$2.2e+11$
max	$9.9e+02$	$6.4e+12$	$1.0e+12$	$6.0e+04$	$1.3e+13$
$n = 25$					
min	$1.5e-01$	$9.1e+10$	$1.7e+10$	$9.8e+00$	$2.7e+12$
ave	$5.4e+01$	$5.8e+13$	$1.1e+13$	$3.1e+03$	$4.6e+14$
max	$4.3e+03$	$7.1e+14$	$3.3e+14$	$2.5e+05$	$3.1e+15$
$n = 35$					
min	$3.6e-01$	$1.0e+13$	$4.8e+11$	$2.3e+01$	$8.3e+12$
ave	$4.7e+01$	$1.5e+14$	$2.3e+13$	$3.1e+03$	$8.5e+14$
max	$2.0e+03$	$1.2e+15$	$2.7e+14$	$1.4e+05$	$8.0e+15$

Table 1: Computed condition numbers in Example 1.

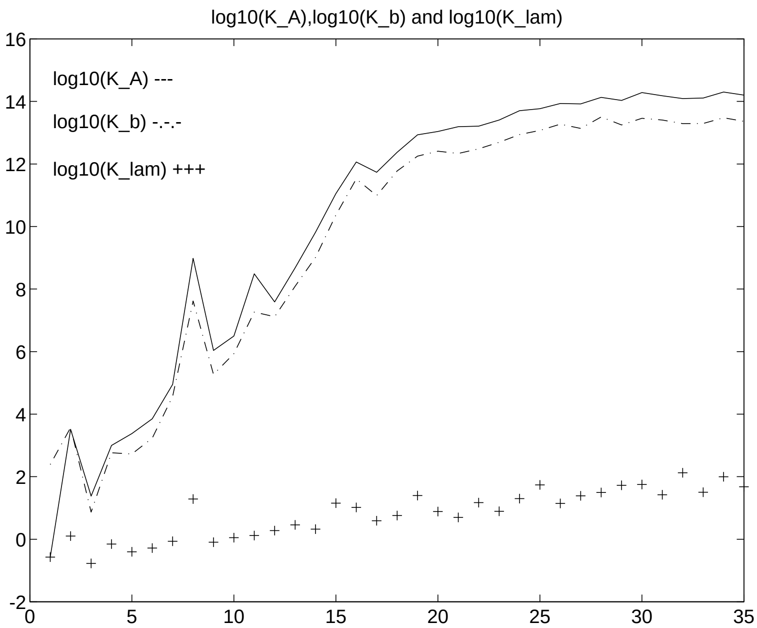

Figure 1: Condition numbers in Example 1, logarithmic scale.

The average of the group of condition numbers, $\|f\|_2$, and cond(X) for varying system sizes are shown in Figures 1 and 2.

This example demonstrates that as the system size increases, the pole placement problem becomes more ill conditioned. So does the closed-loop system $A + bf^T$.

Previously an extensive test was carried out in [18], where several pole placement methods were compared on random test problems with variable dimensions. The overall conclusion also from these tests is that the sensitivity of closed-loop eigenvalues increases drastically with the system size.

Note that random examples are usually well-conditioned problems, since the set of ill-conditioned problems is usually a lower dimensional variety. Hence, the probability that one is close to an ill-conditioned problem is small [7]. But here this is not the case, from which we may also infer that the single-input pole placement problem is itself ill conditioned.

Example 2 Let $A = 0.1 * \text{diag}(1, 2, \ldots, n)$, $b = \begin{bmatrix} 1 & 2 & \ldots & n \end{bmatrix}^T$. Suppose that we wish to assign the eigenvalues to be $-n, -(n-1), \ldots, -1$. Note that

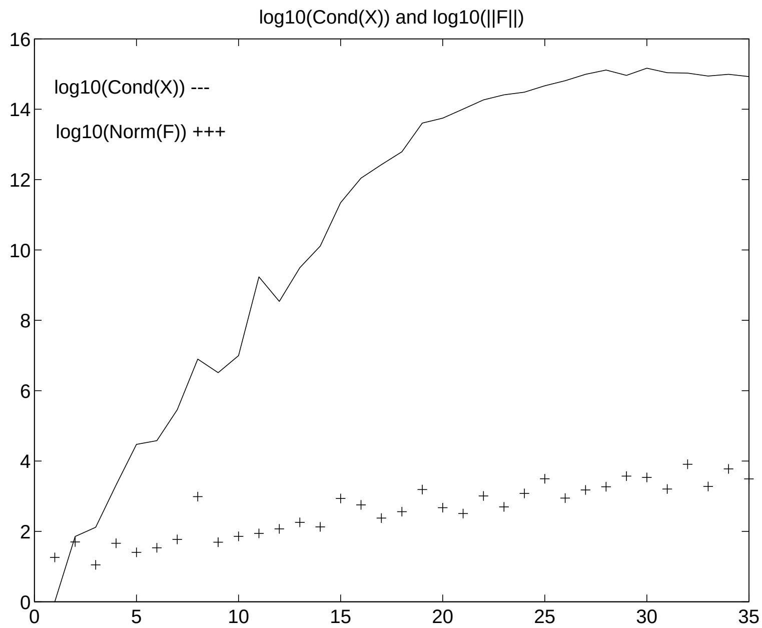

Figure 2: cond(X) and $\|F\|_2$ in Example 1, logarithmic scale.

the system was designed such that the desired poles are far away from the eigenvalues of A. We again used the Matlab code of [19]. The computed values for the group of condition numbers in (8)–(10), as well as $\|f\|_2$ and cond(X) for varying n up to 15 are shown in Figures 3 and 4.

In the case $n = 15$, only 8 eigenvalues were assigned by the pole placement code and 7 poles were detected as nearly uncontrollable.

A system that is nearly uncontrollable is certainly expected to be ill conditioned for the pole placement problem. But the ill conditioning may occur even if the system is far from being uncontrollable.

So far we have discussed first-order perturbation theory. For the single-input problem there is also another approach that we can take. We can express the solution of the single-input pole placement problem directly as the solution of a linear system.

Proposition 2 *Consider a controllable system (3) and a set* $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\}$ *(closed under complex conjugation), where* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *Let* f *be the solution of the single-input pole placement problem for (3), i.e., the*

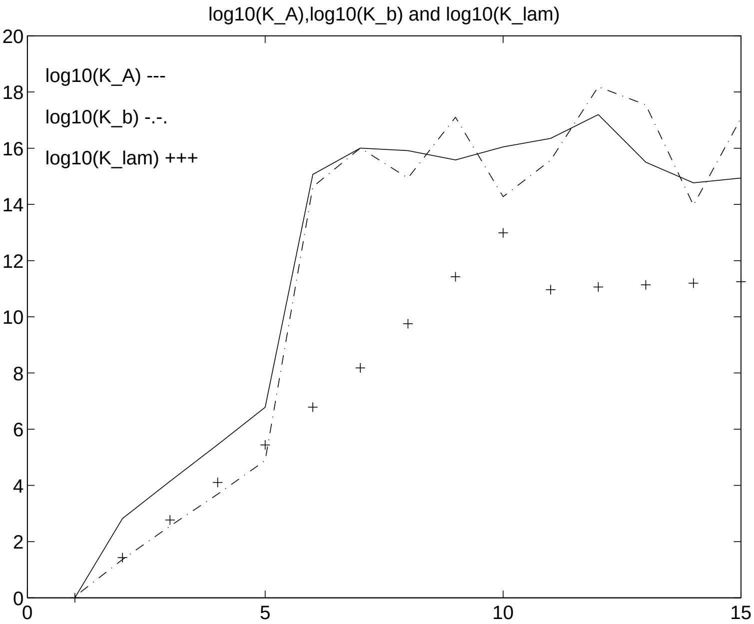

Figure 3: Condition numbers in Example 2, logarithmic scale.

spectrum of $A + bf^T$ *is* $\mathcal{P}$. *Suppose that* A *is diagonalizable and has the eigendecomposition* $A = Z\Sigma Z^{-1}$ *with* $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$. *Assume that* $\{\sigma_1, \ldots, \sigma_n\} \cap \mathcal{P} = \emptyset$. *Let* $\hat{b} = Z^{-1}b$ *and* $\hat{f}^T = f^T Z$. *Then* $\hat{f}$ *is the solution of the linear system* $CB\hat{f} = -e$, *where* e *is the vector of all ones,* $B = \mathrm{diag}(\hat{b}_1, \ldots, \hat{b}_n)$, *and* C *is the Cauchy matrix*

$$C := \begin{bmatrix} (\sigma_1 - \lambda_1)^{-1} & \ldots & (\sigma_n - \lambda_1)^{-1} \\ \vdots & \ddots & \vdots \\ (\sigma_1 - \lambda_n)^{-1} & \ldots & (\sigma_n - \lambda_n)^{-1} \end{bmatrix}. \tag{11}$$

Proof. Let λ_i be an eigenvalue of $A + bf^T$. Then $\det(A + bf^T - \lambda_i I) = 0$ or equivalently $\det(\Sigma + \hat{b}\hat{f}^T - \lambda_i I) = 0$. Using the Sherman-Morrison-Woodbury formula, e.g. [10], this is equivalent to $\hat{f}^T(\Sigma - \lambda_i I)^{-1}\hat{b} = -1$. We immediately obtain that

$$\hat{f}^T \,\mathrm{diag}(\hat{b}_1, \ldots, \hat{b}_n) \begin{bmatrix} (\sigma_1 - \lambda_1)^{-1} & \ldots & (\sigma_1 - \lambda_n)^{-1} \\ \vdots & \ddots & \vdots \\ (\sigma_n - \lambda_1)^{-1} & \ldots & (\sigma_n - \lambda_n)^{-1} \end{bmatrix} = -e^T$$

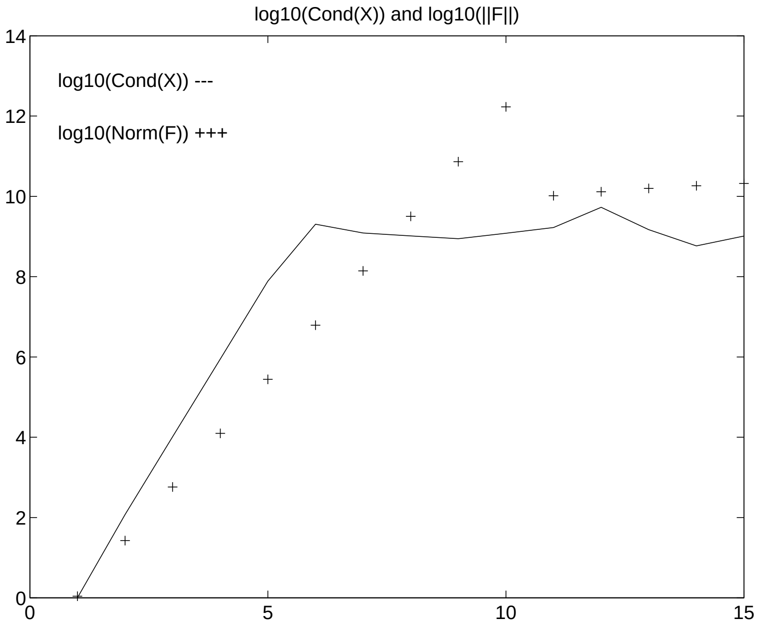

Figure 4: cond(X) and $\|F\|_2$ in Example 2, logarithmic scale.

and the result follows by transposition. □

Now it is a well-known result [11] that the solution of linear systems with Cauchy matrices is very ill conditioned if one considers general perturbations. The classical example of a Cauchy matrix is the well-known Hilbert matrix, which has a condition number that grows dramatically with the dimension of the problem; see [11]. We should also note that the condition number may be lower if one uses structured perturbation theory, i.e., the perturbations are considered only in the set of Cauchy matrices. For the solution of linear systems with Cauchy coefficient matrices, algorithms that are based completely on numerical computations in the set of Cauchy matrices have been discussed in, for example, [8, 9]. Thus, if we were to design a method in this way to solve the linear system (11), then we could guarantee that the feedback vector f is computed accurately. But to do so, we would need to compute the Jordan canonical form of the matrix A first, i.e., all eigenvalues and eigenvectors, which would then be used to construct the linear system (11). To get an accurate f we would need to solve the eigenvalue problem for A very accurately, which by itself may not be possible. Another problem would occur if we were to require that some of the

	cond(C)	$\|f\|_2$	cond(X)
$n = 5$	$1.2e + 08$	$2.7e + 05$	$7.7e + 07$
$n = 10$	$2.3e + 18$	$1.6e + 12$	$1.2e + 09$
$n = 15$	$6.3e + 17$	$2.0e + 10$	$1.0e + 09$

Table 2: Numerical results in Example 3

eigenvalues of A remain fixed or are close to the eigenvalues to be placed. In that case we could not apply this approach.

If we use another method that does not respect the Cauchy structure, and none of the well-known pole placement methods is constructed in such a way, we can expect large errors in f.

Thus, we can expect that the solution vector f is very inaccurate. Furthermore, additional inaccuracies arise if the eigenvector matrix X of the closed-loop system is ill conditioned or if $\|f\|_2$ is large.

Example 3 In this example we took A, b and the assigned eigenvalues as in Example 2. The results are displayed in Table 2. Here cond(C) is the spectral condition number of the Cauchy matrix, $\|f\|_2$ is the Euclidean norm of the feedback vector, and cond(X) is the spectral condition number of the eigenvector matrix of $A + bf^T$.

As a consequence of the previous discussion we draw the following conclusion:

> **The single-input pole placement problem is an intrinsically ill-conditioned problem, and the condition number increases drastically with the system dimension! Therefore, placing plenty of poles in a single-input problem is pretty preposterous!**

3 The Multi-Input Pole Placement Problem

In a multi-input system, the situation becomes a little better. We study the linear control system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{12}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $m > 1$ and B of full column rank. We discuss the problem of choosing a feedback matrix $F \in \mathbb{R}^{m\times n}$ such that the closed-loop system

$$\dot{x} = (A + BF)x, \quad x(0) = x_0 \tag{13}$$

has a prescribed set of pairwise distinct poles $\mathcal{P} := \{\lambda_1, \ldots, \lambda_n\}$, i.e., the spectrum of $A + BF$ is $\mathcal{P}$. Note that there is in general no unique solution to this problem.

As can be demonstrated, the ill conditioning decreases with m, the number of inputs, if the freedom in the solution is used to improve the sensitivity of the closed-loop system. But this decrease is still not significant compared with the increase in conditioning that accompanies increasing n. Similar to the single-input problem, several condition numbers have been derived in the literature. The most recent results are given in [16] and [24]. We cite the latter result for completeness; for a discussion of the differences see [24].

For a specific solution to the pole placement problem associated with (12), let the eigendecomposition of the closed-loop system be given by

$$A + BF = X\Lambda X^{-1} \tag{14}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. As before, let $X =: [x_1, \ldots, x_n]$ and $X^{-1} =: Y =: [y_1, \ldots, y_n]$ and let $F := [f_1, \ldots, f_m]^T$ with $f_i \in \mathbb{R}^n$. Let a and λ be as defined in the single-input case and let $b = \mathrm{vec}(B)$. Then the Jacobians of the mapping from the data (a, b, λ) to the solution of the pole placement problem (see [24]) are given by

$$(-W_f^+ W_a, -W_f^+ W_b, -W_f^+ W_\lambda)$$

respectively, where

$$\begin{aligned} W_f : &= \mathrm{diag}(S_1 X^T, \ldots, S_m X^T) \\ W_a : &= (D_1(X)X^{-1}, \ldots, D_n(X)X^{-1}) \\ W_b : &= \mathrm{diag}(T_1 X^{-1}, \ldots, T_n X^{-1}) \\ W_\lambda : &= -I_n \end{aligned}$$

with $S_j := diag(y_1^T b_j, \ldots, y_n^T b_j)$ and $T_i := \mathrm{diag}(f_i^T x_1, \ldots, f_i^T x_n)$, and $D_i(X)$ is defined in (6).

Theorem 3 *(Corollary 4.5 in [24]) Given a controllable system (12) and a set* $\mathcal{P} = \{\lambda_1, \ldots, \lambda_n\}$ *(closed under complex conjugation), where* $\lambda_i \neq \lambda_j$ *for* $i \neq j$*, suppose that* A *and* B *are slightly perturbed to* $\tilde{A}$ *and* $\tilde{B}$*. Suppose further that* $\mathcal{P}$ *is slightly perturbed to* $\tilde{\mathcal{P}} = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n\}$ *(also closed under complex conjugation). Let* F *be a solution to the pole placement problem for (12) with the data* $A, B, \mathcal{P}$*. Then there exists a solution* $\tilde{F}$ *to the problem with the perturbed data* $\tilde{A}, \tilde{B}, \tilde{\mathcal{P}}$ *(and obvious analogous definitions of* $\tilde{a}$, $\tilde{b}$, *and* $\tilde{\lambda}$*) such that for any consistent norm* $\|\cdot\|$ *and vector norm* $\|\cdot\|$ *consistent with it, we have*

$$\begin{aligned} \|\tilde{F} - F\| &\leq \delta_F + O\left(\left\| \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{\lambda} \end{bmatrix} - \begin{bmatrix} a \\ b \\ \lambda \end{bmatrix} \right\|^2\right) \\ &\leq \Delta_F + O\left(\left\| \begin{bmatrix} \tilde{a} \\ \tilde{b} \\ \tilde{\lambda} \end{bmatrix} - \begin{bmatrix} a \\ b \\ \lambda \end{bmatrix} \right\|^2\right). \end{aligned}$$

Here

$$\begin{aligned} \delta_F &= \|\Phi(\tilde{a} - a) + \Psi(\tilde{b} - b) + \Theta(\tilde{\lambda} - \lambda)\| \\ \Delta_F &= \|\Phi\|\|\tilde{a} - a\| + \|\Psi\|\|\tilde{b} - b\| + \|\Theta\|\|\tilde{\lambda} - \lambda\| \end{aligned}$$

with

$$\begin{aligned} \Theta &= +W_f^+ = \operatorname{diag}(S_1 X^T, \ldots, S_m X^T)^+ \\ \Phi &= -W_f^+ W_a = -\Theta(D_1(X)X^{-1}, \ldots, D_n(X)X^{-1}) \\ \Psi &= -W_f^+ W_b = -\Theta \operatorname{diag}(T_1 X^{-1}, \ldots, T_m X^{-1}) \end{aligned}$$

where $^+$ *denotes the Moore-Penrose pseudoinverse.*

The corresponding group of condition numbers is

$$\kappa_A(F) := \|\Phi\|_2 \quad (15)$$

$$\kappa_B(F) := \|\Psi\|_2 \quad (16)$$

$$\kappa_\lambda(F) := \|\Theta\|_2. \quad (17)$$

We see that the results are similar to the single-input case. The major difference is that, in general, it is not clear whether the solution to the problem that one obtains via a specific method is the one for which this perturbation result holds. In general, one can be much further away from the desired solution.

This is a difficulty with the multi-input problem that we discuss further in the sequel. In general, one uses the freedom of choice to minimize a robustness measure as is done, for example, in [14] or implicitly and locally in the implementation of [27]. To our knowledge an explicit perturbation theory for this modified problem has not been given. We re-examine this issue in Section 4. Let us now consider some numerical examples.

Example 4 In the fourth test we again used random examples. The elements of the system matrices A, B were random numbers uniformly distributed in $(-100, 100)$. The system matrix A was of increasing dimension up to 35. The matrix B always had 5 columns. The real parts of the desired poles were random numbers uniformly distributed in $(-100, 0)$ while the imaginary parts were uniformly distributed in $(-100, 100)$. The set of poles to be assigned for each system consisted of the maximal number of complex conjugate pairs. The Matlab code of [19] was used to perform the pole placement. For each specified system size varying from 1 to 35, one hundred pole placement tests were performed. The computed condition numbers in (15)–(17), as well as $\|F\|_2$ and $\text{cond}(X)$, are shown in Table 3 for system sizes 5, 15, 25, and 35. The average of magnitudes (ave) is the sum of magnitudes divided by 100.

The average of the group of condition numbers, $\|F\|_2$ and $\text{cond}(X)$ for varying system size are depicted in Figures 5 and 6.

As demonstrated by this example, when the system size is increased, the pole placement problem and the closed-loop system $A + BF$ become more ill conditioned. The ill conditioning increases less drastically as in the single-input case but for large enough system size the ill conditioning is equally bad. This happens even if the number of inputs m is equal to the number of outputs. Consider the following example.

Example 5 Let $A = \text{diag}(0.1, 0.2, \ldots, 0.1 * n)$ and

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 2 & 3 & 4 & 5 & \ldots & n & n-1 \\ 3 & 4 & 5 & \ldots & n & n-1 & n-2 \\ 4 & 5 & \ldots & n & n-1 & n-2 & n-3 \\ 5 & \ldots & n & n-1 & n-2 & n-3 & n-4 \end{bmatrix}^T .$$

	K_λ	K_A	K_b	$\|F\|_2$	cond(X)
$n = 5$					
min	$6.0e-03$	$8.0e-03$	$1.2e-02$	$1.7e+00$	$1.9e+00$
ave	$1.0e-02$	$2.1e-02$	$1.3e-01$	$1.7e+01$	$6.5e+00$
max	$2.2e-02$	$2.8e-01$	$1.2e+00$	$1.4e+02$	$6.8e+01$
$n = 15$					
min	$2.3e-05$	$1.2e+00$	$2.6e-01$	$2.6e+00$	$7.9e+03$
ave	$3.3e-04$	$4.0e+04$	$2.7e+04$	$1.5e+02$	$2.1e+11$
max	$1.4e-03$	$2.4e+06$	$2.3e+06$	$1.0e+04$	$1.8e+13$
$n = 25$					
min	$5.1e-08$	$2.2e+02$	$1.0e+02$	$3.7e+00$	$1.5e+09$
ave	$3.6e-06$	$1.1e+05$	$5.6e+04$	$1.1e+02$	$4.7e+14$
max	$5.9e-05$	$3.3e+06$	$1.1e+06$	$4.6e+03$	$2.1e+16$
$n = 35$					
min	$7.4e-10$	$2.7e+04$	$7.7e+03$	$3.7e+00$	$3.7e+13$
ave	$4.0e-06$	$3.4e+05$	$1.3e+05$	$9.3e+01$	$3.2e+15$
max	$3.3e-04$	$2.0e+06$	$9.8e+05$	$3.0e+03$	$2.6e+16$

Table 3: Computed condition numbers in Example 4.

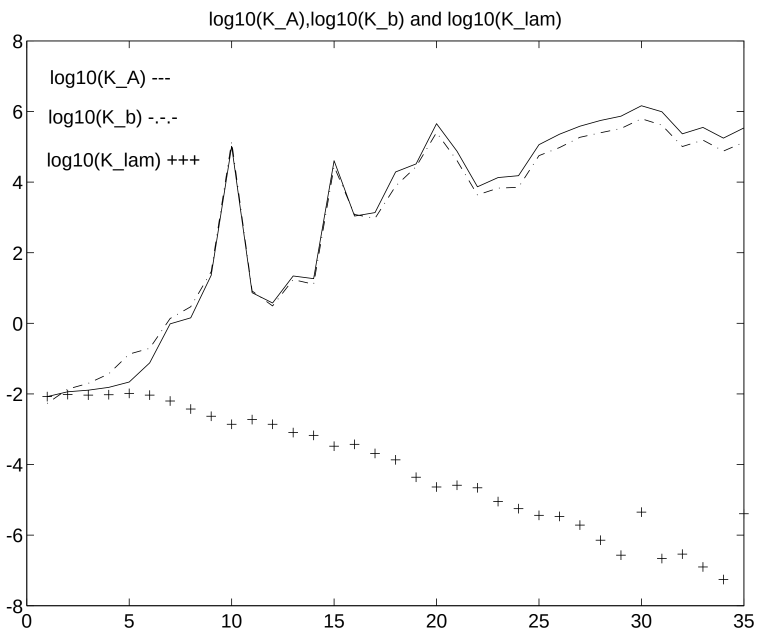

Figure 5: Condition numbers in Example 4, logarithmic scale.

Suppose we wish to assign the eigenvalues $-n, -(n-1), \ldots, -1$, which are obviously far away from the eigenvalues of A. Using the same procedures as before we obtained the following results for the group of condition numbers, $\|F\|_2$, and cond(X) with n varying from 5 to 20, in Figure 7 and Figure 8.

In the case $n = 20$, only 10 eigenvalues were assigned by the pole placement code of [19]. The code detected that the system (A, B) was almost uncontrollable.

It is clear that a system that is close to an uncontrollable system induces an ill-conditioned pole placement problem.

Often it is believed that one can improve the conditioning of the closed-loop system by choosing the poles. We carried out another experiment to show that for several groups of chosen poles, the conditioning of the closed-loop system was equally bad.

Example 6 Let $A = \text{diag}(1, 2, \ldots, n)$ and let B be a random $n \times 5$ matrix and assign the eigenvalues $-\alpha n, -\alpha(n-1), \ldots, -\alpha 1$. for $\alpha = 10, 1, 0.1$. In Tables 4–6, we give the results for this experiment.

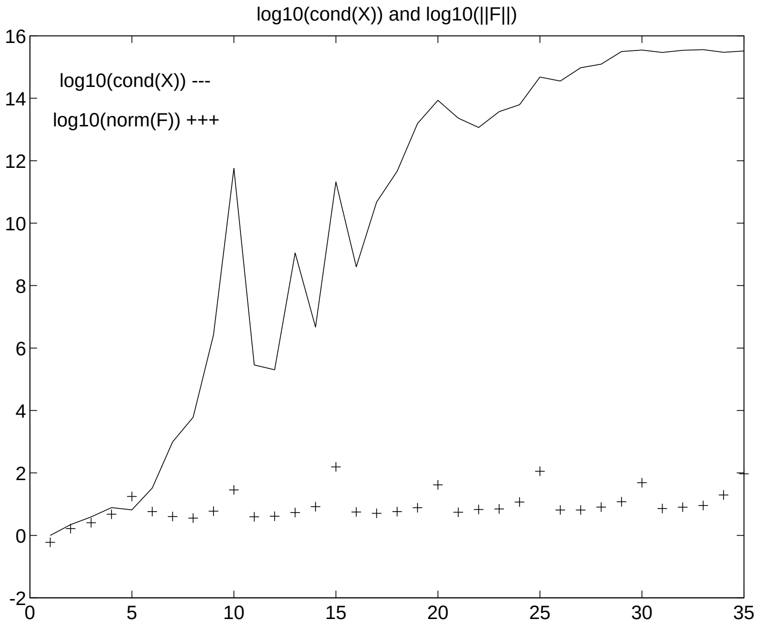

Figure 6: cond(X) and $\|F\|_2$ in Example 4, logarithmic scale.

We see from this experiment that the effect of these groups of poles has essentially no influence on the condition number of the closed-loop system.

As a consequence of the previous discussion we draw the following conclusion:

The multi-input pole placement problem is an intrinsically ill-conditioned problem, and the condition number

	K_λ	K_A	K_b	$\|F\|_2$	cond(X)
$n = 5$	$3.8e + 00$	3.8e+00	2.4e+02	1.1e+02	1.0e+00
$n = 10$	$2.1e + 01$	5.8e+02	3.0e+03	2.5e+03	7.7e+08
$n = 15$	$5.8e - 01$	6.6e+08	3.0e+03	3.3e+07	7.0e+11
$n = 20$	$9.2e + 00$	2.8e+10	8.7e+11	2.1e+06	1.7e+13

Table 4: Computed condition numbers for $\alpha = 10$ in Example 6

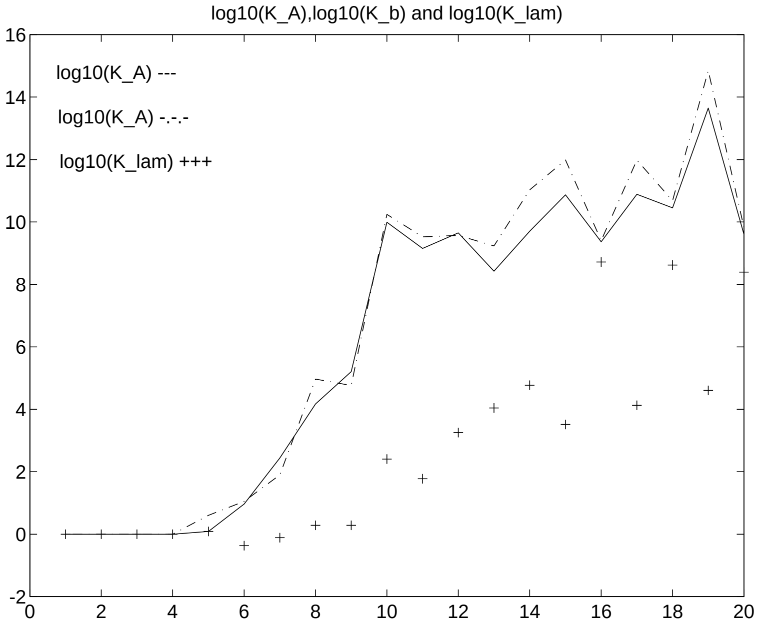

Figure 7: Condition numbers in Example 5, logarithmic scale.

	K_λ	K_A	K_b	$\|F\|_2$	cond(X)
$n = 5$	$1.9e + 00$	1.9e+00	1.1e+02	8.7e+02	1.0e+00
$n = 10$	$6.3e + 00$	6.7e+01	3.4e+02	9.9e+01	2.6e+08
$n = 15$	$7.8e - 02$	7.3e+07	1.1e+09	1.4e+04	2.7e+13
$n = 20$	$1.5e - 03$	2.3e+09	4.7e+10	8.2e+03	1.7e+14

Table 5: Computed condition numbers for $\alpha = 1$ in Example 6

	K_λ	K_A	K_b	$\|F\|_2$	cond(X)
$n = 5$	$1.3e + 01$	1.3e+01	7.7e+02	6.4e+01	1.0e+00
$n = 10$	$3.2e - 03$	4.0e+06	2.7e+07	7.1e+01	5.8e+14
$n = 15$	$2.1e - 03$	9.3e+07	3.3e+08	4.0e+03	2.8e+13
$n = 20$	$6.7e - 04$	4.0e+09	4.8e+10	9.4e+03	9.6e+13

Table 6: Computed condition numbers for $\alpha = 0.1$ in Example 6

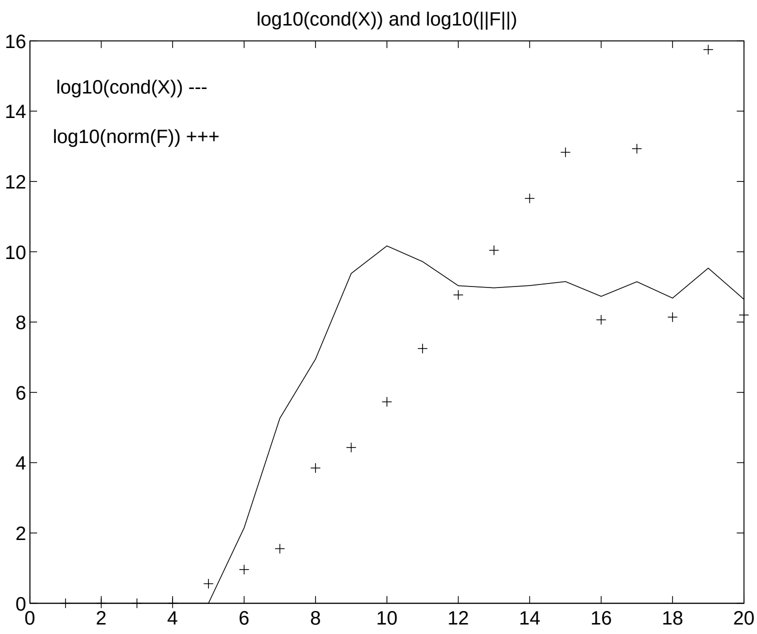

Figure 8: cond(X) and $\|F\|_2$ in Example 5, logarithmic scale.

increases drastically with the system dimension! Therefore, placing plenty of poles in a multi-input problem is pretty preposterous!

4 Robustness Measures and Stabilization

We have seen in Section 3 that the closed-loop system obtained via pole placement is sensitive to perturbation but that there is freedom in the choice of the feedback. So it is natural to use this freedom to minimize the sensitivity of the solution. Several approaches in this direction have been taken. In [14] the condition number of the closed-loop eigenvector matrix is minimized via an iterative procedure that minimizes one column at a time. A global minimization procedure for the same measure using general optimization methods was proposed in [4]. In the Schur-method-based pole placement algorithm of [27] the norm of the feedback matrix is minimized at each exchange step of the procedure. In view of the condition numbers shown in the last section, minimizing these measures is certainly a reasonable approach, although it seems more sensible to minimize a functional involving both $\|F\|_2$ and $\|X^{-1}\|$. So far, we do not know whether such an approach has been taken.

But as we have seen before, solving the pole placement problem may be only a substitute problem and if this is the case, then one should rather make the solution of the substitute problem robust against perturbations. We demonstrate this in the sequel with the problem of stabilization. A similar approach can be taken for the problem of moving the poles into a damped region of the complex plane. This topic is currently under investigation [13]. Still another approach is to assign only a few poles [21], which may be both more efficient as well as better conditioned.

In many cases pole placement is used simply to stabilize a system, i.e., it is used as a substitute for the problem of finding a feedback matrix F so that the closed-loop system

$$\dot{x} = (A + BF)x \tag{18}$$

is stable. Clearly, if we could successfully solve the pole placement problem then we would have a way to solve the stabilization problem. But as we have indicated before we cannot expect to solve the pole placement problem satisfactorily in finite precision arithmetic, since it is potentially so ill conditioned.

The best robustness measure for the stabilization problem is obviously the distance to instability, i.e., the smallest perturbation that makes the system unstable again. If this measure is large, then the system is robust against perturbations. In [25], for a given stable matrix A, this distance is defined as

$$\delta(A) := \min_{\mu \in \mathbb{R}} \sigma_n(A - \mu i I) \tag{19}$$

where σ_n denotes the smallest singular value. If $A + BF$ is diagonalizable and $A + BF = X \Lambda X^{-1}$ is an eigendecomposition of the closed-loop matrix, then a lower bound for the distance to instability for the closed-loop system is given by

$$\frac{1}{\mathrm{cond}_2(X)} \delta(\Lambda) \leq \delta(A + BF). \tag{20}$$

Thus, minimizing $\mathrm{cond}_2(X)$ maximizes a lower bound for the distance to instability. In view of this result, a pole placement method that minimizes $\mathrm{cond}(X)$ among all possible choices of feedback that assign the correct poles, which was introduced in [14], is a good approach. This method, however, is very costly and infeasible for large control problems. Furthermore, it happens quite often that the bound given by $\frac{1}{\mathrm{cond}(X)}$ is quite small even though the distance to instability is large.

In general, for the stabilization problem, we have the following open questions:

1. What is the stabilizing feedback of minimum norm?

2. What is the stabilizing feedback for which the condition number of the closed-loop eigenvector matrix is minimal?

3. What is the stabilizing feedback that maximizes the distance to instability?

We give partial answers to the first question; the other questions are essentially open problems. The basis for answering the first question is the well-known idea that one can stabilize a system via the solution of an appropriately chosen linear-quadratic optimal control problem of the form

$$J = \min_u \int_0^\infty (x^T Q x + u^T R u)\, dt \tag{21}$$

subject to (1) with appropriately chosen nonnegative definite matrix Q and positive definite matrix R.

The standard theory for such optimal control problems shows that if (A, B) is stabilizable, (a pair of matrices (A, B) is said to be *stabilizable* if Rank $(\lambda I - A, B) = n$ for all $\lambda \in \mathbb{C}$ with nonnegative real part), then the linear-quadratic optimal problem (21), subject to (1), has the unique solution

$$u = Fx = -R^{-1} B^T X x \tag{22}$$

where X is the unique nonnegative definite solution of the algebraic Riccati equation

$$A^T X + XA - XBR^{-1}B^T X + Q = 0 \tag{23}$$

for which the corresponding closed-loop system

$$\dot{x} = (A + BF)x = (A - BR^{-1}B^T X)x \tag{24}$$

is asymptotically stable.

The usual trick to do stabilization is to find the spectral decomposition of A and therefore stabilize only its unstable eigenvalues. This strategy, which is called partial stabilization, is helpful in reducing the norm of the feedback matrix [12, 26]. Thus in the following, A is restricted to having eigenvalues all of whose real parts are positive.

By finding the nonnegative definite stabilizing solution of the Riccati equation, the stabilization problem can be solved. But one still has the choice of the cost matrices R and $Q = C^T C$ and clearly these should be chosen so that the closed-loop system is insensitive to perturbations. At least it should be guaranteed that small perturbations do not make the system unstable again. Typically for this approach the cost matrix $Q = 0$ is chosen in which case the Riccati equation reduces to a Lyapunov equation for the inverse of X [1, 22, 26, 27] (assuming it exists). We now show that this choice of Q can be motivated from the fact, that it leads to a minimum norm feedback.

The following theorem is probably well known to some, but we do not know a reference.

Theorem 4 *If we consider the cost functional (21) as a function of Q then*

$$\min_{Q \geq 0} J(Q) = J(0).$$

Furthermore, suppose that $Re(\lambda) > 0$ for all eigenvalues λ of A, and suppose that

$$A^T X + XA - XBR^{-1}B^T X = 0 \tag{25}$$

has a nonsingular solution X. If $F = -R^{-1}B^T X$ is the corresponding feedback, then the eigenvalues of $A + BF$ are the negatives of the eigenvalues A.

Proof. The first part of the theorem follows trivially from a lemma of Willems [30]; see also [12]. For the second part observe that we can rewrite the Riccati equation (25) as

$$X(A + BF) = -A^T X.$$

Since X is nonsingular and all eigenvalues of A have positive real part, it follows that the eigenvalues of $A + BF$ are those of $-A^T$ and hence $A + BF$ is stable. □

Since we are looking for a nonsingular solution of the degenerate Riccati equation we can equivalently solve the Lyapunov equation

$$AY + YA^T = BR^{-1}B^T \tag{26}$$

where $X = Y^{-1}$ [1].

Although the value of the cost functional $J(Q)$ partially reflects the size of $\|F\|_2$, we are merely interested in minimal values for $\|F\|_2$ or $\|F\|_F$, where $\| \quad \|_F$ denotes the Frobenius norm. This is, in general, still an open problem and we present results only for the case that B is a nonsingular matrix or $(A + A^T)$ is positive definite. We begin with another lemma from [30].

Lemma 5 *Let X_i, $i = 1, 2$, be real symmetric solutions of the algebraic Riccati equations*

$$A^T X_i + X_i A - X_i B R^{-1} B^T X_i + Q_i = 0, \; i = 1, 2$$

respectively, and assume that all eigenvalues of $A - BR^{-1}B^T X_1$ have negative real part. Then $0 \leq Q_2 \leq Q_1$ implies $X_2 \leq X_1$.

Using this lemma we can prove the following theorem; see also [12].

Theorem 6 *Suppose that all eigenvalues of A have positive real part. Let B be square and nonsingular and let $R = (B^T B)^{1/2}$ be the positive square root of $B^T B$ (cf. [10]). Let X be the nonnegative definite stabilizing solution of the algebraic Riccati equation*

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

i.e., all eigenvalues of $A - BR^{-1}B^T X$ have negative real part. Then the minimum norm feedback matrix F taken over all positive semidefinite matrices Q occurs for $Q = 0$. It is given by $F = -R^{-1}B^T X$, where X is the positive definite stabilizing solution of the degenerate Riccati equation (25). Furthermore, the eigenvalues of $A + BF$ are the negatives of those of A.

Proof. Let X_1 and X_2 be the nonnegative definite stabilizing solutions of the Riccati equations

$$A^T X + XA - XBR^{-1}B^T X + Q_i = 0, \quad i = 1, 2$$

for $0 \leq Q_2 \leq Q_1$. Let $F_i = -R^{-1}B^T X_i$, $i = 1, 2$. Then Lemma 5 implies $X_2 \leq X_1$. Thus $\|X_2\|_2 \leq \|X_1\|_2$. Observe that $R^{-1}B^T$ is an orthogonal matrix and therefore $\|F_1\|_2 = \|X_1\|_2$ and $\|F_2\|_2 = \|X_2\|_2$. Thus $\|F_2\|_2 \leq \|F_1\|_2$ and the minimum of $\|F\|_2$ occurs at $Q = 0$. □

Theorem 7 *Suppose that all eigenvalues of A have positive real part and, moreover, $(A + A^T)$ is positive definite. Let X be the nonnegative definite, stabilizing solution of the algebraic Riccati equation*

$$A^T X + XA - XBB^T X + Q = 0$$

i.e., all eigenvalues of $A - BB^T X$ have negative real part. Then the minimum norm feedback F in Frobenius norm taken over all positive semidefinite matrices Q occurs for $Q = 0$.

Proof. Let X_1 and X_2 be the nonnegative definite stabilizing solutions of the Riccati equations

$$A^T X + XA - XBB^T X + Q_i = 0, \quad i = 1, 2$$

for $0 \leq Q_2 \leq Q_1$. Let $F_i = -B^T X_i$, $i = 1, 2$. Then Lemma 5 implies $X_2 \leq X_1$. Subtracting the equations $A^T X + XA - XBB^T X + Q_i = 0$, $i = 1, 2$ for the solutions X_1 and X_2, we obtain

$$A^T(X_1 - X_2) + (X_1 - X_2)A + (Q_1 - Q_2) = X_1 BB^T X_1 - X_2 BB^T X_2. \quad (27)$$

Note that in the case that $R = I$, we have $\|F_1\|_F^2 - \|F_2\|_F^2 = \text{Trace}(X_1 BB^T X_1) - \text{Trace}(X_2 BB^T X_2)$ [10]. It follows from (27) that

$$\begin{aligned} \|F_1\|_F^2 - \|F_2\|_F^2 &= \text{Trace}(A^T(X_1 - X_2) + (X_1 - X_2)A + (Q_1 - Q_2)) \\ &= \text{Trace}(A^T(X_1 - X_2) + (X_1 - X_2)A) + \text{Trace}(Q_1 - Q_2) \\ &= \text{Trace}((A^T + A)(X_1 - X_2)) + \text{Trace}(Q_1 - Q_2). \end{aligned}$$

Here the equality $\text{Trace}(AB) = \text{Trace}(BA)$ is used. Since $Q_1 \leq Q_2$, we have $\text{Trace}(Q_1 - Q_2) \geq 0$. On the other hand, since $(X_1 - X_2)$ is positive definite,

$$\text{Trace}(((A^T + A)(X_1 - X_2)) = \text{Trace}((X_1 - X_2)^{\frac{1}{2}}(A^T + A)(X_1 - X_2)^{\frac{1}{2}}),$$

where $(X_1 - X_2)^{\frac{1}{2}}$ denotes the positive square root of $(X_1 - X_2)$. Since $(A^T + A)$ is positive definite, it follows that $\text{Trace}((X_1 - X_2)^{\frac{1}{2}}(A^T + A)(X_1 - X_2)^{\frac{1}{2}}) \geq 0$ and $\text{Trace}(((A^T + A)(X_1 - X_2)) \geq 0$. Thus $\|F_2\|_2 \leq \|F_1\|_2$ and the minimum of $\|F\|_2$ occurs at $Q = 0$. □

It is natural to ask what happens when the minimization problem includes R. The answer is that minimizing the norm of feedback matrices among all $0 \leq Q$ is usually sufficient, since we can always scale the problem so that $\|R\|_2 = 1$ (see [3]). In fact, let $\alpha = \|R\|_2$. Then $\tilde{X} = X/\alpha$ satisfies the Riccati equation

$$A^T \tilde{X} + \tilde{X} A - \tilde{X} B \tilde{R}^{-1} B^T \tilde{X} + \tilde{Q} = 0$$

where $\tilde{R} = R/\alpha$ and $\tilde{Q} = Q/\alpha$. Observe that the feedback matrices produced by both Riccati equations are same, i.e.,

$$\tilde{F} = -\tilde{R}^{-1} B^T \tilde{X} = -R^{-1} B^T X = F.$$

Example 7 Consider the system given by

$$A = \begin{bmatrix} 0.1 & 1 & 10 & 0 & 0 \\ -1 & 0.1 & 0 & 10 & 0 \\ 0 & 0 & 2 & 1 & 10 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 5 & 4 & 3 \\ 4 & 5 & 4 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix},$$

and let $R = \alpha I$ and $Q = \beta I$. Table 7 shows the optimal stabilizing feedback $\|F\|_2$ as a function of α and β.

$\beta \setminus \alpha$	10^{-4}	10^{-2}	1	10^2	10^4
10^{-4}	9.80	6.41	6.01	5.98	5.98
10^{-2}	23.7	9.80	6.41	6.01	5.98
1	147	23.7	9.80	6.41	6.01
10^2	1397	147	23.7	9.80	6.41
10^4	10^4	1397	147	23.7	9.80

Table 7: $\|F\|_2$ for different values of α and β.

The Toeplitz structure of Table 7 is in accordance with our theoretical analysis: only one parameter plays a role. The minimum norm feedback matrix F with $\|F\|_2 = 5.9833$ occurs at $\beta = 0, \alpha = 1$ and $\Lambda(A + BF) = \{-5.0000, -0.1000 \pm 1.0000i, -2.0000 \pm 1.0000i\}$.

In this section we have discussed the minimization of the feedback F with respect to two different measures, the value of the cost functional $J(Q)$

and $\|F\|_2$. In the first case, and in special situations also in the second case, the optimal F is obtained for the choice $Q = 0$ in the cost functional.

The approach of applying the stabilization method based on linear-quadratic control can also be used for large control problems up to sizes of several thousands, provided that the number of unstable poles is small compared to the system size; see [12]. It is clear that this approach is superior to the pole placement approach for the problem of stabilization, since as we have seen, the pole placement problem becomes increasingly ill conditioned when the system size increases.

To demonstrate the superiority of this approach consider the stabilization algorithm proposed in [12] applied to one of the previous examples.

Example 8 For Example 6 in the case of $\alpha = 1$, the result obtained from the stabilization algorithm proposed in [12] is given in Table 8. Observe that the stabilization method and the pole placement method are comparable in this case, since the eigenvalues of the closed-loop systems are both $-n, -(n-1), \ldots, -1$.

	$\|F\|_2$	cond(X)
$n = 5$	$3.5e + 01$	$1.9e + 02$
$n = 10$	$5.3e + 02$	$2.2e + 04$
$n = 15$	$2.6e + 03$	$4.2e + 05$
$n = 20$	$1.5e + 04$	$7.0e + 06$

Table 8: $\|F\|_2$ and cond(X) for the stabilization method.

A comparison of Table 8 and Table 5 shows that the conditioning of the closed-loop system via the stabilization method is much better. It should be pointed out that in the case $n = 20$, the resulting closed-loop eigenvalues via the pole placement algorithm had no correct digits but those via the stabilization method had 7 valid digits. For further results in this direction, in particular for large sytems of several hundred states, see [12].

In this section we have demonstrated that for the problem of stabilization, the pole placement problem should not be considered as a substitute problem. A much better substitute (though not perfect) is the stabilization via the solution of a linear-quadratic control problem. In a similar way, one can avoid pole placement in other situations. For example, the construction of damped feedbacks via the solution of periodic Riccati equations is discussed in [13].

5 Conclusion

We have discussed the pole placement problem and given considerable evidence that this problem is intrinsically ill conditioned, i.e., even the best (numerically stable) algorithms for this problem may produce bad results. Furthermore, the ill conditioning increases with the dimension of the system. Hence, if it can be avoided, and to our knowledge this is usually the case, then one should replace the pole placement problem with alternative problems, such as stabilization or moving the poles only to certain regions of the complex plane. Conditioning of the latter problems is much better and hence better numerical results can be expected.

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