A step towards a unified treatment of continuous and discrete time control problems

Volker Mehrmann
Fakultät für Mathematik
TU Chemnitz-Zwickau
D-09107 Chemnitz, FRG
(0371)-531-8367 (2659 secr.)
(0371)-531-2657 (fax)
mehrmann@mathematik.tu-chemnitz.de

20.10.94

Abstract

In this paper we introduce a new approach for a unified theory for continuous and discrete time (optimal) control problems based on the generalized Cayley transformation. We also relate the associated discrete and continuous generalized algebraic Riccati equations. We demonstrate the potential of this new approach by proving a new result for discrete algebraic Riccati equations. But we also discuss where this new approach as well as all other approaches still is non-satisfactory. We explain a discrepancy observed between the discrete and continuous cse and show that this discrepancy is partly due to the consideration of the wrong analogues. We also present an idea for a metatheorem that relates general theorems for discrete and continuous control problems.

1 Introduction

For given matrices $Q, A, E \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$, $C \in \mathbf{C}^{p\times n}$, $R \in \mathbf{C}^{m\times m}$ B with full column rank, C with full row rank, Q Hermitian, and R Hermitian positive definite, the *generalized continuous algebraic Riccati equation* has the form

$$C^*QC + A^*XE + E^*XA - (B^*XE)^*R^{-1}(B^*XE) = 0, \tag{1}$$

while the corresponding *generalized discrete time Riccati equation* takes the form

$$-E^*XE + A^*XA + C^*QC - (B^*XA)^*(R + B^*XB)^{-1}(B^*XA) = 0, \tag{2}$$

where * denotes the conjugate transpose.
It is well known, e.g. [16, 17, 18, 19], that the solutions of the algebraic Riccati equations (1) and (2) can be used to obtain solutions to linear quadratic optimal control problems and optimal filter problems. See also the forthcoming book [15]. In the continuous time case this is the problem to minimize the cost functional

$$\frac{1}{2}\int_{t_0}^{\infty} [y(t)^*Qy(t) + u(t)^*Ru(t)]\,dt \tag{3}$$

subject to the dynamics

$$\begin{array}{rcl} E\dot{x}(t) & = & Ax(t) + Bu(t), \\ x(t_0) & = & x^0, \\ y(t) & = & Cx(t). \end{array} \tag{4}$$

In the discrete time case one considers the problem of minimizing the cost functional

$$\frac{1}{2}\sum_{k=0}^{\infty} [y_k^*Qy_k + u_k^*Ru_k] \tag{5}$$

subject to the dynamics

$$\begin{array}{rcl} Ex_{k+1} & = & Ax_k + Bu_k, \\ x_0 & = & x^0, \\ y_k & = & Cx_k. \end{array} \tag{6}$$

It is also well known, e.g. [16, 17, 18, 19], that the solutions of the algebraic Riccati equations can be obtained via the computation of deflating subspaces of the following pencils. In the continous time case the pencil is of the form

$$\begin{array}{rcl} \lambda\mathcal{E}_c - \mathcal{H}_c & := & \lambda\begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} A & BR^{-1}B^* \\ C^*QC & -A^* \end{bmatrix} \\ & =: & \lambda\begin{bmatrix} E_c & 0 \\ 0 & E_c^* \end{bmatrix} - \begin{bmatrix} F_c & G_c \\ H_c & -F_c^* \end{bmatrix}. \end{array} \tag{7}$$

and in the discrete time case the pencil is of the form

$$\begin{array}{rcl} \lambda\mathcal{E}_d - \mathcal{A}_d & := & \lambda \begin{bmatrix} E & -BR^{-1}B^* \\ 0 & A^* \end{bmatrix} - \begin{bmatrix} A & 0 \\ C^*QC & E^* \end{bmatrix} \\ & := & \lambda \begin{bmatrix} E_d & -G_d \\ 0 & F_d^* \end{bmatrix} - \begin{bmatrix} F_d & 0 \\ H_d & E_d^* \end{bmatrix}. \end{array} \tag{8}$$

In the case that $E = I$, it is well known that $\mathcal{H}_c = \begin{bmatrix} F_c & G_c \\ H_c & -F_c^* \end{bmatrix}$ is a Hamiltonian matrix and that $\lambda\mathcal{E}_d - \mathcal{A}_d$ is a symplectic pencil.

Definition 1 *Let* $J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$.
a) A pencil $\lambda\mathcal{E}_c - \mathcal{A}_c \in \mathbf{C}^{2n,2n}$ *is called Hamiltonian iff* $\mathcal{E}_c J \mathcal{A}_c^* = -\mathcal{A}_c J \mathcal{E}_c^*$. *The set of Hamiltonian pencils in* $\mathbf{C}^{2n,2n}$ *is denoted by* $\mathcal{H}_{2n}^p$.
b) A matrix $\mathcal{H}_c \in \mathbf{C}^{2n,2n}$ *is called Hamiltonian iff* $(\mathcal{H}_c J)^* = \mathcal{H}_c J$. *The Lie Algebra of Hamiltonian matrices in* $\mathbf{C}^{2n,2n}$ *is denoted by* $\mathcal{H}_{2n}$.
c) A pencil $\lambda\mathcal{E}_d - \mathcal{A}_d \in \mathbf{C}^{2n,2n}$ *is called symplectic iff* $\mathcal{E}_d J \mathcal{E}_d^* = \mathcal{A}_d J \mathcal{A}_d^*$. *The set of symplectic pencils in* $\mathbf{C}^{2n,2n}$ *is denoted by* $\mathcal{S}_{2n}^p$.
d) A matrix $\mathcal{S}_d \in \mathbf{C}^{2n,2n}$ *is called symplectic iff* $\mathcal{S}_d J \mathcal{S}_d^* = J$. *The Lie group of symplectic matrices in* $\mathbf{C}^{2n,2n}$ *is denoted by* $\mathcal{S}_{2n}$.

If $\mathcal{E}_c$ or $\mathcal{E}_d$ are invertible, in a) or c) respectively, then $\mathcal{H}_c = \mathcal{E}_c^{-1}\mathcal{A}_c$ is Hamiltonian and $\mathcal{S}_d = \mathcal{E}_d^{-1}\mathcal{A}_d$ is symplectic. Note further that in general a pencil of the form (7) is not a Hamiltonian pencil, since a pencil of the form

$$\lambda \begin{bmatrix} E & 0 \\ 0 & E^* \end{bmatrix} - \begin{bmatrix} F & G \\ H & -F^* \end{bmatrix} \tag{9}$$

is Hamiltonian if and only if

$$GE^*, HE \text{ are Hermitian and } FE = EF. \tag{10}$$

Analogously a pencil of the form

$$\lambda \begin{bmatrix} E & -G \\ 0 & F^* \end{bmatrix} - \begin{bmatrix} F & 0 \\ H & E^* \end{bmatrix} \tag{11}$$

is symplectic if and only if (10) holds.
It is also well known that the spectra of Hamiltonian pencils or symplectic pencils have a certain symmetry, e.g. [18, 19]. Namely if λ is a finite eigenvalue of a Hamiltonian pencil or matrix, then also $-\bar{\lambda}$ is an eigenvalue, i.e. the eigenvalues lie symmetric with respect to the imaginary axis. For symplectic pencils or matrices, if λ is an eigenvalue then also $\bar{\lambda}^{-1}$ is an eigenvalue, i.e. the eigenvalues lie symmetric with respect to the unit circle.
Based on this property one is immediately lead to consider transformations that relate the Lie algebra of Hamiltonian matrices and the Lie group of symplectic

matrices. This topic is well studied in classical group theory [25]. There are many such transformations and they are based on classical results from complex analysis how to construct mappings that map the open right half complex plane to the outside of the unit disk, the open left half to the inside of the unit disk and the imaginary axis to the unit circle. One such mapping, the one we will study here, is the so called *Cayley transformation*

$$y = \mathcal{C}_{\lambda_1}(z) = (1 - \lambda_1 z)^{-1}(\lambda_1 + z), \tag{12}$$

where the *shiftpoint* $\lambda_1 \neq \pm i$ is any complex number of modulus one. The inverse transformation is

$$z = \mathcal{C}^{-}_{\lambda_1}(y) = (y - \lambda_1)(\lambda_1 y + 1)^{-1}. \tag{13}$$

Note that in (12) and (13) we cannot use $\lambda_1 = \pm i$, since then the transformation maps everything to one point.
It is obvious that both transformations (12) and (13) have poles and hence the transformations are not continuous at these poles. This property will create difficulties and we will discuss these in detail.
A matrix version of the Cayley transformation can be used to relate Hamiltonian and symplectic matrices, e.g. [5, 19] or more generally discrete and continuous control problems. This is a well known and widely used fact, e.g. [24, 2, 6, 23, 18, 19, 22]. Consider the matrix transformations

$$\mathcal{C}_{\lambda_1} : \mathcal{S}_{2n} \rightarrow \mathcal{H}_{2n},\ Y = \mathcal{C}_{\lambda_1}(Z) = (I - \lambda_1 Z)^{-1}(\lambda_1 I + Z), \tag{14}$$

and the inverse transformation

$$\mathcal{C}^{-}_{\lambda_1} : \mathcal{H}_{2n} \rightarrow \mathcal{S}_{2n},\ Z = \mathcal{C}^{-}_{\lambda_1}(Y) = (Y - \lambda_1 I)(I + \lambda_1 Y)^{-1}. \tag{15}$$

Again both mappings are not continuous at the poles but we will show in the next section that we can make the mappings continuous by considering them as mappings between Hamiltonian and symplectic pencils:

$$\mathcal{C}^{p}_{\lambda_1} : \mathcal{S}^{p}_{2n} \rightarrow \mathcal{H}^{p}_{2n},\ \lambda\mathcal{E}_c - \mathcal{A}_c = \mathcal{C}^{p}_{\lambda_1}(\lambda\mathcal{E}_d - \mathcal{A}_d) = \lambda(\mathcal{E}_d - \lambda_1\mathcal{A}_d) - (\lambda_1\mathcal{E}_d + \mathcal{A}_d), \tag{16}$$

and the inverse transformation

$$\mathcal{C}^{p-}_{\lambda_1} : \mathcal{H}^{p}_{2n} \rightarrow \mathcal{S}^{p}_{2n},\ \lambda\mathcal{E}_d - \mathcal{A}_d = \mathcal{C}^{p-}_{\lambda_1}(\lambda\mathcal{E}_c - \mathcal{A}_c) = \lambda(\lambda_1 A_c + \mathcal{E}_c) - (\mathcal{A}_c - \lambda_1\mathcal{E}_d). \tag{17}$$

We will discuss this generalized Cayley transformation for matrix pencils in detail in Section 2. When we study this generalized transformation, which is continuous also at the poles of the original Cayley transformation, we obtain a new analogue between discrete and continuous time control systems, which explains some of the well known discrepancies between the discrete and continuous case. We will show that the analogy should be between the Riccati equations

$$-X_d + F_d^* X_d F_d + H_d + F_d^* X_d (I - G_d X_d)^{-1} F_d = 0, \tag{18}$$

in the discrete case and

$$A_c^* H_c A_c + A_c^* X_c + X_c A_c - A_c^* X_c G_c X_c A_c = 0, \tag{19}$$

with F_d, G_d, G_c, H_d, H_c as in (7), (8) and $A_c = F_c^{-1}$.
Based on the generalized Cayley transformation we will give explicite formulas in Section 2 that relate specific parametrizations of Hamiltonian and symplectic pencils. These formulas are then used to show that standard assumptions in control theory, like controllability and observability in Section 3 as well as semidefiniteness of blocks of Hamiltonian or symplectic matrices in Section 4 are directly related for discrete and continuous systems.
This relationship then allows us to present in Section 5 a metatheorem that states that whenever the Cayley transformation transforms the assumptions and the statement of a theorem for discrete or continuous control problems, then either side can be proved via the other and thus we obtain a unified treatment. This is essentially a folklore result, but there are examples in the literature, e.g. [8, 26] that show that the analogy between continuous and discrete time problems is not complete. An example for such a result is given in Section 6, where the existence of arbitrary solutions for algebraic Riccati equations based on deflating subspaces is discussed. The differences occur for several reasons. One reason is that the standard Cayley transformation has poles, where it is not continuous. Another reason is that in the pencil formulation we have to consider deflating subspaces to compute the solution of the algebraic Riccati equation. In the pencil case we have that, in contrast to the case of Hamiltonian or symplectic matrices, not every Lagrangian subspace leads to a solution of the Riccati equation, since eigenvectors to infinite eigenvalues cannot be used. This leads to differences between continuous and discrete algebraic Riccati equations, since the continuous algebraic Riccati equation is associated to a Hamiltonian matrix, while the discrete equation is associated to a symplectic pencil. Based on the pencil formulation and the new analogy between discrete and continuous Riccati equations, we observe that the same restriction occurs for the analogous continuous time algebraic Riccati equation.
Unfortunately the new approach creates more open questions, since (19) actually is a generalized algebraic Riccati equation for which the theory is not complete. We will give some examples and pose some open questions.
It is the main purpose of this paper to introduce the new unifying approach. We will demonstrate its potential by proving a new result for the discrete time case, and it is clear that this approach can be used to simplify many proofs for known results, but we refrain here from doing so.

2 The Cayley Transformation

In this section we develop the basic properties of the Cayley transformation and how it can be used to relate Hamiltonian and symplectic pencils. We begin with a well known but key Lemma [17].

Lemma 2
a) Let $\lambda\mathcal{E}_d - \mathcal{A}_d$ be a symplectic pencil. Assume that $\lambda_1 \in \mathcal{C} \setminus \{i, -i\}$ with $|\lambda_1| = 1$.
Then

$$\lambda\mathcal{E}_c - \mathcal{A}_c := \lambda(\mathcal{E}_d - \lambda_1\mathcal{A}_d) - (\lambda_1\mathcal{E}_d + \mathcal{A}_d) \tag{20}$$

is a Hamiltonian pencil.
b) Let $\lambda\mathcal{E}_c - \mathcal{A}_c$ be a Hamiltonian pencil and $\lambda_1 \in \mathcal{C} \setminus \{i, -i\}$ with $|\lambda_1| = 1$. Then

$$\lambda\mathcal{E}_d - \mathcal{A}_d = \lambda(\lambda_1\mathcal{A}_c + \mathcal{E}_c) - (\mathcal{A}_c - \lambda_1\mathcal{E}_c) \tag{21}$$

is a symplectic pencil.

Proof. a) Since $\lambda\mathcal{E}_d - \mathcal{A}_d$ is symplectic we have

$$\lambda_1\mathcal{E}_d J\mathcal{E}_d^* - \mathcal{E}_d J\mathcal{A}_d^* + \mathcal{A}_d J\mathcal{E}_d^* - \bar{\lambda}_1\mathcal{A}_d J\mathcal{A}_d^* + \bar{\lambda}_1\mathcal{E}_d J\mathcal{E}_d^* - \mathcal{A}_d J\mathcal{E}_d^* + \mathcal{E}_d J\mathcal{A}_d^* - \lambda_1\mathcal{A}_d J\mathcal{A}_d^* = 0.$$

Equivalently we have

$$(\lambda_1\mathcal{E}_d + \mathcal{A}_d)J(\mathcal{E}_d^* - \bar{\lambda}_1\mathcal{A}_d^*) + (\mathcal{E}_d - \lambda_1\mathcal{A}_d)J(\bar{\lambda}_1\mathcal{E}_d^* + \mathcal{A}_d^*) = 0$$

which proves a).
b) $\mathcal{E}_d J\mathcal{E}_d^* - \mathcal{A}_d J\mathcal{A}_d^* = (\lambda_1\mathcal{A}_c + \mathcal{E}_c)J(\lambda_1\mathcal{A}_c + \mathcal{E}_c)^* - (\mathcal{A}_c - \lambda_1\mathcal{E}_c)J(\mathcal{A}_c - \lambda_1\mathcal{E}_c)^*$.
Since $|\lambda_1| = 1$ we have that $\mathcal{E}_d J\mathcal{E}_d^* - \mathcal{A}_d J\mathcal{A}_d^* = \lambda_1(\mathcal{A}_c J\mathcal{E}_c^* + \mathcal{E}_c J\mathcal{A}_c^*) + \bar{\lambda}_1(\mathcal{A}_c J\mathcal{E}_c^* + \mathcal{E}_c J\mathcal{A}_c^*)$, which proves b). □
Note that in Lemma 2 no assumption is made that excludes λ_1 to be an eigenvalue of the pencil that is transformed. This means that the pencil formulation allows to consider the Cayley transformation also at the poles. Away from the poles we have the following result.

Corollary 3
a) Let $\lambda\mathcal{E}_d - \mathcal{A}_d$ be a symplectic pencil. Assume that $\lambda_1 \in \mathcal{C} \setminus \{i, -i\}$ with $|\lambda_1| = 1$ and that $det(\mathcal{E}_d - \lambda_1\mathcal{A}_d) \neq 0$.
Then

$$\mathcal{H}_c := (\mathcal{E}_d - \lambda_1\mathcal{A}_d)^{-1}(\lambda_1\mathcal{E}_d + \mathcal{A}_d) \tag{22}$$

is a Hamiltonian matrix.
b) Let $\lambda\mathcal{E}_c - \mathcal{A}_c$ be a Hamiltonian pencil. Assume that $\lambda_1 \in \mathcal{C} \setminus \{i, -i\}$ with $|\lambda_1| = 1$ and that $det(\lambda_1\mathcal{A}_c + \mathcal{E}_c) \neq 0$.
Then

$$\mathcal{S}_d := (\lambda_1\mathcal{A}_c + \mathcal{E}_c)^{-1}(\mathcal{A}_c - \lambda_1\mathcal{E}_c) \tag{23}$$

is a symplectic matrix.

Proof. The result follows directly from Lemma 2 by taking inverses. □
Lemma 2 and Corollary 3 give the relationship between symplectic pencils or matrices and Hamiltonian pencils or matrices. Now, since the Cayley transformation is one-to-one and, as we will show, in the pencil version also continuous, it can be used to jump back and forth between symplectic pencils and Hamiltonian pencils. In many cases, however, they have a special structure, like for example in the applications from control theory. These applications lead to specific parametrizations of symplectic and Hamiltonian pencils or matrices. The most important of these parametrizations are described in the following Lemma:

Lemma 4
a) Let $\mathcal{S}_d = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$, *with blocks* $S_{ij} \in \mathbf{C}^{n\times n}$, *be symplectic and suppose that* S_{22} *is invertible. Then* $\mathcal{S}_d$ *can be factored as a product of three symplectic matrices*

$$\mathcal{S}_d = \begin{bmatrix} I & G_d \\ 0 & I \end{bmatrix} \begin{bmatrix} F_d & 0 \\ 0 & F_d^{-*} \end{bmatrix} \begin{bmatrix} I & 0 \\ H_d & I \end{bmatrix} = \begin{bmatrix} F_d + G_d F_d^{-*} H_d & G_d F_d^{-*} \\ F_d^{-*} H_d & F_d^{-*} \end{bmatrix}. \tag{24}$$

The blocks are given by

$$F_d = S_{22}^{-*},\ G_d = S_{12} S_{22}^{-1},\ H_d = S_{22}^{-1} S_{21} \tag{25}$$

and the pencil

$$\lambda \mathcal{E}_d - \mathcal{A}_d := \lambda \begin{bmatrix} I & -G_d \\ 0 & F_d^* \end{bmatrix} - \begin{bmatrix} F_d & 0 \\ H_d & I \end{bmatrix} \tag{26}$$

is symplectic. An analogous result can be formulated if S_{11} *is invertible.*
b) Let $\lambda \mathcal{E}_c - \mathcal{A}_c$ *be a Hamiltonian pencil and suppose that* $\mathcal{E}_c$ *is invertible, then this pencil is equivalent to the pencils*

$$\lambda \tilde{\mathcal{E}}_c - \tilde{\mathcal{A}}_c := \lambda \begin{bmatrix} A_c & 0 \\ 0 & -A_c^* \end{bmatrix} - \begin{bmatrix} I & A_c G_c \\ -A_c^* H_c & I \end{bmatrix} \tag{27}$$

and

$$\lambda I - \mathcal{H}_c := \lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} F_c & G_c \\ H_c & -F_c^* \end{bmatrix}, \tag{28}$$

where $F_c = A_c^{-1}$.

Proof. For a) see [19], p. 119. Part b) is trivial by (10). □
On first site part b) of Lemma 20 looks a complete triviality. Why should one write the pencil in the form (27). The reason is that this representation as well as the one in part a) gives us the continuity in the pencil formulation of the Cayley transformation. Actually this representation of symplectic matrices as symmetric pencils allows a compactification of the symplectic group. This can

be seen as follows: Let $\{F_d^i\}$ be a sequence of nonsingular matrices converging to a singular matrix F_d. Then for all i

$$\lambda \mathcal{E}_d^i - \mathcal{A}_d^i := \lambda \begin{bmatrix} I & -G_d \\ 0 & (F_d^i)^* \end{bmatrix} - \begin{bmatrix} F_d^i & 0 \\ H_d & I \end{bmatrix} \tag{29}$$

is a symplectic pencil and this also holds in the limit. Similarly for all i

$$\mathcal{S}_d^i = \begin{bmatrix} I & G_d \\ 0 & I \end{bmatrix} \begin{bmatrix} (F_d^i) & 0 \\ 0 & (F_d^i)^{-*} \end{bmatrix} \begin{bmatrix} I & 0 \\ H_d & I \end{bmatrix}. \tag{30}$$

is symplectic, while the limit does not exist. A similar property is obtained for Hamiltonian pencils in the form (27) if we consider a sequence of nonsingular matrices $\{A_c^i\}$ converging to a singular matrix A_c. While the limit in (27) exists and is still a Hamiltonian pencil, the limit in (28) does not exist.
In view of this observation we may conclude that the analogous continuous time control problem correponding to

$$x_{k+1} = F_d x_k + B_d u_k \tag{31}$$

that one should consider to obtain a more unified theory is not

$$\dot{x} = F_c x + B_c u \tag{32}$$

but

$$A_c \dot{x} = x + A_c B_c u. \tag{33}$$

The latter now represents a descriptor system while the first one does not. Clearly if A_c is invertible then the two systems (32) and (33) are equivalent.
The corresponding Riccati equations are of the form

$$-X_d + F_d^* X_d F_d + H_d + F_d^* X_d (I - G_d X_d)^{-1} F_d = 0, \tag{34}$$

and

$$A_c^* H_c A_c + A_c^* X_c + X_c A_c - A_c^* X_c G_c X_c A_c = 0, \tag{35}$$

with G_d, G_c, H_d, H_c as in (7), (8). We will come back to these two Riccati equations later.
In our next Lemma we give explicit formulas for the relationship of the blocks of Hamiltonian pencils and the blocks of the corresponding symplectic pencils in the parametrizations given in Lemma 4 if the Cayley transformation is used to transform from one to the other.

Lemma 5
a) Let

$$\lambda \mathcal{E}_d - \mathcal{A}_d := \lambda \begin{bmatrix} I & -G_d \\ 0 & F_d^* \end{bmatrix} - \begin{bmatrix} F_d & 0 \\ H_d & I \end{bmatrix} \tag{36}$$

be a symplectic pencil. Let $\lambda_1 \in \mathcal{C} \setminus \{+i,-i\}$ *with* $|\lambda_1| = 1$. *Suppose that the matrices* $I - \lambda_1 F_d$ *and* $(\lambda_1 I + F_d) - G_d(\bar{\lambda}_1 I - F_d)^{-*} H_d$ *are nonsingular. Then the Hamiltonian pencil* $\lambda \mathcal{E}_c - \mathcal{A}_c := \lambda(\mathcal{E}_d - \lambda_1 \mathcal{A}_d) - (\lambda_1 \mathcal{E}_d + \mathcal{A}_d)$ *is equivalent to the Hamiltonian pencil*

$$\lambda \begin{bmatrix} A_c & 0 \\ 0 & -A_c^* \end{bmatrix} - \begin{bmatrix} I & A_c G_c \\ -A_c^* H_c & I \end{bmatrix} \tag{37}$$

where the blocks satisfy the following formulas:

$$\begin{aligned} A_c &= (I + G_d \tilde{H}_d)(\mathcal{C}_d - \bar{\lambda}_1 G_d \tilde{H}_d)^{-1} && (38)\\ &= (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}(I + \tilde{G}_d H_d) && (39)\\ -A_c^* H_c &= (\lambda_1 + \bar{\lambda}_1)(\mathcal{C}_d^* - \lambda_1 \tilde{H}_d G_d)^{-1} \tilde{H}_d && (40)\\ A_c G_c &= -(\lambda_1 + \bar{\lambda}_1)(\mathcal{C}_d - \lambda_1 \tilde{G}_d H_d)^{-1} \tilde{G}_d. && (41) \end{aligned}$$

Here we have set $\tilde{G}_d := (I - \lambda_1 F_d)^{-1} G_d (I - \lambda_1 F_d)^{-*}$, $\tilde{H}_d := (I - \lambda_1 F_d)^{-*} H_d (I - \lambda_1 F_d)^{-1}$ *and* $\mathcal{C}_d := (I - \lambda_1 F_d)^{-1}(\lambda_1 I + F_d)$. *If furthermore* $F_c = A_c^{-1}$ *exists then we have*

$$\begin{aligned} F_c &= (\mathcal{C}_d - \bar{\lambda}_1 G_d \tilde{H}_d)(I + G_d \tilde{H}_d)^{-1} = (I + \tilde{G}_d H_d)^{-1}(\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d) && (42)\\ H_c &= -(\lambda_1 + \bar{\lambda}_1)\tilde{H}_d (I + G_d \tilde{H}_d)^{-1} && (43)\\ G_c &= -(\lambda_1 + \bar{\lambda}_1)(I + \tilde{G}_d H_d)^{-1} \tilde{G}_d. && (44) \end{aligned}$$

b) Let

$$\lambda \mathcal{E}_c - \mathcal{A}_c = \lambda \begin{bmatrix} A_c & 0 \\ 0 & -A_c^* \end{bmatrix} - \begin{bmatrix} I & A_c G_c \\ -A_c^* H_c & I \end{bmatrix} \tag{45}$$

be a Hamiltonian pencil. Let $\lambda_1 \in \mathcal{C} \setminus \{+i,-i\}$ *with* $|\lambda_1| = 1$ *be such that the matrices* $A_c + \lambda_1 I$ *and* $(A_c + \lambda_1 I) + A_c G_c (A_c + \lambda_1 I)^{-*} A_c^* H_c$ *are nonsingular. Then the symplectic pencil* $\lambda \mathcal{E}_d - \mathcal{A}_d := \lambda(\lambda_1 \mathcal{A}_c + \mathcal{E}_c) - (\mathcal{A}_c - \lambda_1 \mathcal{E}_c)$ *is equivalent to the symplectic pencil*

$$\lambda \begin{bmatrix} I & -G_d \\ 0 & F_d^* \end{bmatrix} - \begin{bmatrix} F_d & 0 \\ H_d & I \end{bmatrix} \tag{46}$$

with blocks

$$\begin{aligned} F_d &= (\mathcal{C}_c + \bar{\lambda}_1 G_c \tilde{H}_c)(I + G_c \tilde{H}_c)^{-1} = (I + \tilde{G}_c H_c)^{-1}(\mathcal{C}_c + \bar{\lambda}_1 \tilde{G}_c H_c) && (47)\\ H_d &= -(\lambda_1 + \bar{\lambda}_1)\tilde{H}_c (I + G_c \tilde{H}_c)^{-1} && (48)\\ G_d &= -(\lambda_1 + \bar{\lambda}_1)(I + \tilde{G}_c H_c)^{-1} \tilde{G}_c, && (49) \end{aligned}$$

where $\mathcal{C}_c := (I - \lambda_1 A_c)(\lambda_1 I + A_c)^{-1}$, $\tilde{G}_c := (\lambda_1 I + A_c)^{-1} A_c G_c A_c^* (\lambda_1 I + A_c)^{-*}$ *and* $\tilde{H}_c := (\lambda_1 I + A_c)^{-*} A_c^* H_c A_c (\bar{\lambda}_1 I + A_c)^{-1}$.

Proof.
a) Let us assume first that λ_1 is chosen such that $\mathcal{E}_d - \lambda_1 \mathcal{A}_d$ is nonsingular. Then we can form $\mathcal{H}_c := (\mathcal{E}_d - \lambda_1 \mathcal{A}_d)^{-1}(\lambda_1 \mathcal{E}_d + \mathcal{A}_d)$ and from (36) we directly obtain

$$\begin{aligned}
\mathcal{H}_c &= \begin{bmatrix} F_c & G_c \\ H_c & -F_c^* \end{bmatrix} \\
&= \begin{bmatrix} I - \lambda_1 F_d & -G_d \\ -\lambda_1 H_d & F_d^* - \lambda_1 I \end{bmatrix}^{-1} \begin{bmatrix} \lambda_1 I + F_d & -\lambda_1 G_d \\ H_d & \lambda_1 F_d^* + I \end{bmatrix} \\
&= \begin{bmatrix} I & -(I - \lambda_1 F_d)^{-1} G_d \\ -\lambda_1 (F_d^* - \lambda_1 I)^{-1} H_d & I \end{bmatrix}^{-1} \times \\
&\times \begin{bmatrix} (I - \lambda_1 F_d)^{-1}(\lambda_1 I + F_d) & -(I - \lambda_1 F_d)^{-1} \lambda_1 G_d \\ (F_d^* - \lambda_1 I)^{-1} H_d & (F_d^* - \lambda_1 I)^{-1}(\lambda_1 F_d^* + I) \end{bmatrix} \\
&=: \begin{bmatrix} I & -\hat{G} \\ \hat{H} & I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{C}_d & -\lambda_1 \hat{G} \\ -\bar{\lambda}_1 \hat{H} & -\mathcal{C}_d^* \end{bmatrix} \\
&= \left(\begin{bmatrix} I & 0 \\ \hat{H} & I \end{bmatrix} \begin{bmatrix} I & -\hat{G} \\ 0 & I + \hat{H}\hat{G} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathcal{C}_d & -\lambda_1 \hat{G} \\ -\bar{\lambda}_1 \hat{H} & -\mathcal{C}_d^* \end{bmatrix} \\
&= \begin{bmatrix} I & \hat{G}(I + \hat{H}\hat{G})^{-1} \\ 0 & (I + \hat{H}\hat{G})^{-1} \end{bmatrix} \begin{bmatrix} \mathcal{C}_d & -\lambda_1 \hat{G} \\ -\hat{H}(\bar{\lambda}_1 I + \mathcal{C}_d) & -\mathcal{C}_d^* + \lambda_1 \hat{H}\hat{G} \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{C}_d - \hat{G}(I + \hat{H}\hat{G})^{-1}\hat{H}(\bar{\lambda}_1 I + \mathcal{C}_d) & \hat{G}(-\lambda_1 I - (I + \hat{H}\hat{G})^{-1}(\mathcal{C}_d^* - \lambda_1 \hat{H}\hat{G}) \\ -(I + \hat{H}\hat{G})^{-1}\hat{H}(\bar{\lambda}_1 I + \mathcal{C}_d) & (I + \hat{H}\hat{G})^{-1}(-\mathcal{C}_d^* + \lambda_1 \hat{H}\hat{G}) \end{bmatrix},
\end{aligned}$$

where we have used the abbreviations $\hat{H} := (I - \lambda_1 F_d)^{-*} H_d$, $\hat{G} := (I - \lambda_1 F_d)^{-1} G_d$, $\mathcal{C}_d := (I - \lambda_1 F_d)^{-1}(\lambda_1 I + F_d)$. We have $\bar{\lambda}_1 I + \mathcal{C}_d = (I - \lambda_1 F_d)^{-1}(\lambda_1 + \bar{\lambda}_1)$, $\hat{G}\hat{H} = \tilde{G}_d H_d$ and $\hat{H}\hat{G} = \tilde{H}_d G_d$. Using these formulas we obtain that

$$\begin{aligned}
A_c &= F_c^{-1} = (I + \hat{H}\hat{G})^*(\mathcal{C}_d^* - \lambda_1 \hat{H}\hat{G})^{-*} = (I + G_d \tilde{H}_d)(\mathcal{C}_d - \bar{\lambda}_1 G_d \tilde{H}_d)^{-1} \\
&= [(\mathcal{C}_d + \bar{\lambda}_1 I)(I + G_d \tilde{H}_d)^{-1} - \bar{\lambda}_1 I]^{-1} = (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}(I + \tilde{G}_d H_d)
\end{aligned}$$

It follows immediately that

$$-A_c^* H_c = (\mathcal{C}_d^* - \lambda_1 \hat{H}\hat{G})^{-1}\hat{H}(\bar{\lambda}_1 I + \mathcal{C}_d) = (\mathcal{C}_d^* - \lambda_1 \tilde{H}_d G_d)^{-1}\tilde{H}_d(\lambda_1 + \bar{\lambda}_1).$$

For the other block we obtain

$$\begin{aligned}
A_c G_c &= (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}(I + \tilde{G}_d H_d)\hat{G}(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_d^* + \lambda_1 I) \\
&= (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}(I + \hat{G}\hat{H})\hat{G}(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_d^* + \lambda_1 I) \\
&= (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}\hat{G}(\mathcal{C}_d^* + \lambda_1 I),
\end{aligned}$$

which gives the required formula. The formulas for G_c, H_c follow analogously. Now if λ_1 is such that $\mathcal{E}_d - \lambda_1 \mathcal{A}_d$ is singular, then we take a sequence of shiftpoints

λ_1^i converging to λ_i that satisfy the assumptions and for which we have that $\mathcal{E}_d - \lambda_1^i \mathcal{A}_d$ is nonsingular. Then by continuity it follows that the formulas (38)–(41) also hold in the limiting case, since we have assumed that $\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d$ and $I - \lambda_1 F_d$ are nonsingular.
b) Again assume first that λ_1 is such that $\lambda_1 \mathcal{A}_c + \mathcal{E}_c$ is nonsingular. Then we have the symplectic matrix

$$\begin{aligned}
\mathcal{S}_d &:= \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} := (\lambda_1 \mathcal{A}_c + \mathcal{E}_c)^{-1}(\mathcal{A}_c - \lambda_1 \mathcal{E}_c) \\
&= \begin{bmatrix} \lambda_1 I + A_c & \lambda_1 A_c G_c \\ -\lambda_1 A_c^* H_c & \lambda_1 I - A_c^* \end{bmatrix}^{-1} \begin{bmatrix} I - \lambda_1 A_c & A_c G_c \\ -A_c^* H_c & \lambda_1(\bar{\lambda}_1 I + A_c^*) \end{bmatrix} \\
&= \begin{bmatrix} I & \lambda_1(\lambda_1 I + A_c)^{-1} A_c G_c \\ -(\bar{\lambda}_1 I + A_c^*)^{-1} A_c^* H_c & (\bar{\lambda}_1 I + A_c^*)^{-1}(I - \bar{\lambda}_1 A_c^*) \end{bmatrix}^{-1} \times \\
&\times \begin{bmatrix} (\lambda_1 I + A_c)^{-1}(I - \lambda_1 A_c) & (\lambda_1 I + A_c)^{-1} A_c G_c \\ -\bar{\lambda}_1(\bar{\lambda}_1 I + A_c^*)^{-1} A_c^* H_c & I \end{bmatrix}
\end{aligned}$$

Set $\mathcal{C}_c := (I - \lambda_1 A_c)(\lambda_1 I + A_c)^{-1}$, $\hat{G} := (\lambda_1 I + A_c)^{-1} A_c G_c$, $\hat{H} := (\bar{\lambda}_1 I + A_c^*)^{-1} A_c^* H_c$. Then

$$\begin{aligned}
\mathcal{S}_d &:= \begin{bmatrix} I & \lambda_1 \hat{G} \\ -\hat{H} & \mathcal{C}_c^* \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{C}_c & \hat{G} \\ -\bar{\lambda}_1 \hat{H} & I \end{bmatrix} \\
&= \left(\begin{bmatrix} I & 0 \\ -\hat{H} & I \end{bmatrix} \begin{bmatrix} I & \lambda_1 \hat{G} \\ 0 & (\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G}) \end{bmatrix} \right)^{-1} \begin{bmatrix} \mathcal{C}_c & \hat{G} \\ -\bar{\lambda}_1 \hat{H} & I \end{bmatrix} \\
&= \begin{bmatrix} \mathcal{C}_c - \lambda_1 \hat{G}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}\hat{H}(\mathcal{C}_c - \bar{\lambda}_1 I) & \hat{G}(I - \lambda_1(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}(I + \hat{H}\hat{G}) \\ (\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}\hat{H}(\mathcal{C}_c - \bar{\lambda}_1 I) & (\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}(I + \hat{H}\hat{G}) \end{bmatrix}
\end{aligned}$$

Now $\mathcal{C}_c - \bar{\lambda}_1 I = -(\lambda_1 + \bar{\lambda}_1) A_c(\lambda_1 I + A_c)^{-1}$, $\hat{H}\hat{G} = \tilde{H}_c G_c$ and $\hat{G}\hat{H} = \tilde{G}_c H_c$ and thus applying Lemma 4 we obtain

$$\begin{aligned}
F_d &= S_{22}^{-*} = (\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^*(I + \hat{H}\hat{G})^{-*} = (\mathcal{C}_c + \bar{\lambda}_1 G_c \tilde{H}_c)(I + G_c \tilde{H}_c)^{-1} \\
&= (I + \tilde{G}_c H_c)^{-1}(\mathcal{C}_c + \bar{\lambda}_1 \tilde{G}_c H_c)
\end{aligned}$$

For the other blocks we obtain

$$\begin{aligned}
G_d &= S_{12} S_{22}^{-1} \\
&= \hat{G}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* - \lambda_1 I)(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G}) \\
&= \hat{G}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* - \lambda_1 I)[(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* - \lambda_1 I) + \lambda_1 I] \\
&= \hat{G}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}[(\mathcal{C}_c^* - \lambda_1 I)(I + \hat{H}\hat{G})^{-1} + \lambda_1 I](\mathcal{C}_c^* - \lambda_1 I) \\
&= \hat{G}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* + \lambda_1 \hat{H}\hat{G})(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* - \lambda_1 I) \\
&= \hat{G}(I + \hat{H}\hat{G})^{-1}(\mathcal{C}_c^* - \lambda_1 I) \\
&= -(\bar{\lambda}_1 + \lambda_1)(I + \tilde{G}_c H_c)^{-1} \tilde{G}_c,
\end{aligned}$$

$$
\begin{aligned}
H_d &= S_{22}^{-1} S_{21} = (I + \hat{H}\hat{G})^{-1} \hat{H}(\mathcal{C}_c - \bar{\lambda}_1 I) \\
&= -(I + \tilde{H}_c G_c)^{-1} \tilde{H}_c(\bar{\lambda}_1 + \lambda_1)
\end{aligned}
$$

If λ_1 is such that $\lambda_1 \mathcal{A}_c + \mathcal{E}_c$ is singular, then we construct a sequence of numbers $\{\lambda_1^i\}$ converging to λ_1 and satisfying the assumptions such that $\lambda_1^i \mathcal{A}_c + \mathcal{E}_c$ is nonsingular. Then the resulting formulas are valid, hence by continuity they also hold in the limit, since we have assumed that all the occuring inverses exist. □

In this section we have given explicite formulas that relate special parametrizations of Hamiltonian and symplectic pencils via the Cayley transformation.

In the transformations we have excluded some points as shiftpoints but we have not excluded the poles of the Cayley transformation. The reason for the exclusion of some shiftpoints is that we wish to have the parametrization in Lemma 5, which relates to the algebraic Riccati equations. As we have seen in Lemma 4 we do not need these assumptions to relate general symplectic and Hamiltonian pencils but it is an interesting **open question** to study such pencils that do not have these specific parametrizations, their algebraic structure and what applications there are that belong to such problems.

In the next section we now discuss how properties like controllability and observability are transformed under the Cayley transformation.

3 Controllability conditions

In order to obtain a unifying theory for discrete and continuous control problems using the Cayley transformation, we have to analyse how typical assumptions are transformed via the Cayley transformation. In this section we discuss how conditions like controllability, observability, stabilizability and detectability for continuous and associated discrete time systems are related.

We first give definitions of these conditions using the Hautus criteria, e.g. [7, 19].

Definition 6

i) A pair of matrices (A, B), $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$ is called *controllable* iff rank $\begin{bmatrix} \lambda I - A, & B \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}$;

ii) A pair of matrices (A, C), $A \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{p\times n}$ is called *observable* iff rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}$;

iii) A pair of matrices (A, B), $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$ is called *c-stabilizable* iff rank $\begin{bmatrix} \lambda I - A, & B \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}, Re(\lambda) \geq 0$;

iv) A pair of matrices (A, C), $A \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{p\times n}$ is called *c-detectable* iff rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}, Re(\lambda) \geq 0$;

v) A pair of matrices (A, B), $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$ is called *d-stabilizable* iff rank $\begin{bmatrix} \lambda I - A, & B \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}, |\lambda| \geq 1$;

vi) A pair of matrices (A, C), $A \in \mathbf{C}^{n\times n}$, $C \in \mathbf{C}^{p\times n}$ is called *d-detectable* iff rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathcal{C}, |\lambda| \geq 1$.

Our next Lemma gives a relationship between the rank conditions for discrete and continuous systems.

Lemma 7
Consider the symplectic pencil (36) and the associated Hamiltonian matrix pencil via the generalized Cayley transformation given by (37). Here λ_1 satisfies the assumptions of Lemma 5 a). Then

$$\operatorname{rank} \begin{bmatrix} \lambda A_c - I, & A_c G_c \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mu I - F_d, & G_d \end{bmatrix} \tag{50}$$

and

$$\operatorname{rank} \begin{bmatrix} \lambda A_c - I \\ A_c^* H_c \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mu I - F_d \\ H_d \end{bmatrix} \tag{51}$$

for all $\lambda, \mu \in \mathcal{C}$, which are related via

$$\mu = \bar{\lambda}_1 \frac{\lambda - \lambda_1}{\bar{\lambda}_1 + \lambda} \tag{52}$$

b) Consider a Hamiltonian pencil and the associated symplectic pencil as in (46), where λ_1 satisfies the assumptions of Lemma 5 b). Then

$$\operatorname{rank} \begin{bmatrix} \lambda I - F_d, & G_d \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mu A_c - I, & A_c G_c \end{bmatrix} \tag{53}$$

and

$$\operatorname{rank} \begin{bmatrix} \lambda I - F_d \\ H_d \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \mu A_c - I \\ A_c^* H_c \end{bmatrix} \tag{54}$$

for all $\lambda, \mu \in \mathcal{C}$, which are related via

$$\lambda = \frac{\mu + \lambda_1}{1 - \lambda_1 \mu} \tag{55}$$

Proof. a) Using the formulas (38)–(41) we obtain

$$\begin{aligned}
& \operatorname{rank}[\lambda A_c - I, A_c G_c] \\
= \;& \operatorname{rank}[\lambda(\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}(I + \tilde{G}_d H_d) - I, -(\lambda_1 + \bar{\lambda}_1)(\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1} \tilde{G}_d] \\
= \;& \operatorname{rank}(\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d)^{-1}[\lambda(I + \tilde{G}_d H_d) - (\mathcal{C}_d - \bar{\lambda}_1 \tilde{G}_d H_d), \tilde{G}_d] \begin{bmatrix} I & 0 \\ 0 & -(\lambda_1 + \bar{\lambda}_1) I \end{bmatrix}) \\
= \;& \operatorname{rank}[\lambda I - \mathcal{C}_d + (\lambda + \lambda_1)\tilde{G}_d H_d), \tilde{G}_d] \\
= \;& \operatorname{rank}[\lambda I - \mathcal{C}_d, \tilde{G}_d] \\
= \;& \operatorname{rank}[\lambda(I - \lambda_1 F_d) - (\lambda_1 I + F_d), G_d (I - \lambda_1 F_d)^{-*}] \\
= \;& \operatorname{rank}[(\lambda - \lambda_1) I - (\lambda\lambda_1 + 1) F_d, G_d] \\
= \;& \operatorname{rank}[\mu I - F_d, G_d]
\end{aligned}$$

The other parts are proven analogously. □

We see in this Lemma that we have to be careful with the rank equalities if $\lambda = -\bar{\lambda}_1$ in a) and $\mu = \bar{\lambda}_1$ in b), since then we are again at the poles of the Cayley transformation and exactly in these points we may lose the controllability properties. Let us consider an example:

Example 1 Consider the continuous time system given by

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_c = H_c = I$$

Then $\text{rank}[\lambda A_c - I, A_c G_c] = 2$ for all λ but the matrices obtained from the generalized Cayley transformation with $\lambda_1 = 1$ are

$$F_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \; G_d = H_d = -0.4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and clearly this system is not controllable, since $\begin{bmatrix} \mu I - F_d, & G_d \end{bmatrix}$ has a rank drop at $\mu = 1$ corresponding to $\lambda = \infty$.

Nonetheless we can use this Lemma to obtain the following equivalence results for the case that A_c^{-1} exists:

Theorem 8
Consider a symplectic pencil and the associated Hamiltonian matrix given by the formulas in Lemma 5 a) or b) and assume that $F_c = A_c^{-1}$ exists.

Then we have the following equivalences:

i) $[F_d, G_d]$ is controllable if and only $[F_c, G_c]$ is controllable;

ii) $[F_d, H_d]$ is observable if and only $[F_c, H_c]$ is observable;

iii) $[F_d, G_d]$ is d-stabilizable if and only $[F_c, G_c]$ is c-stabilizable;

iv) $[F_d, H_d]$ is d-detectable if and only $[F_c, H_c]$ is c-detectable.

Proof. The proof of i) and ii) follows direct from Lemma 7. For iii) and iv) observe that the relationship between λ and μ in (52),(55) is just the scalar Cayley transformation, hence the spectra are transformed accordingly. □

It is well known, e.g.[3, 19] that the concepts defined in Definition 6 cannot be applied directly to descriptor systems

$$E\dot{x} = Ax + Bu \tag{56}$$

even if they have the special form (33). This is the reason why we have assumed that A_c^{-1} exists in the previous Lemma and this is also the reason for the difficulties described in Example 1.

For descriptor systems several different concepts have to be considered, [3, 4]. Define the conditions

(**C1**) $\text{rank}[\lambda E - A, B] = n$ for all $\lambda \in \mathcal{C}$

(**C2**) $\text{rank}[E, AS_\infty, B] = n$, where the columns of S_∞ span the right nullspace of E.

A descriptor system (56) that satisfies conditions (**C1**) and (**C2**) is called *strongly controllable*. It is obvious how corresponding conditions like *strong observability*, *strong stabilizability* and *strong detectability* are defined, see [3, 4]. Condition (**C1**) describes the controllability of the finite eigenvalues and (**C2**) describes the controllability of the infinite eigenvalues. If (**C2**) holds then there exists a feeedback that makes the system regular and of index at most one, which means that the system behaves essentially like a lower dimensional standard system, see [3, 4].

We see that Lemma 7 only relates the properties of the finite eigenvalues. In order to get a unified theory we have to add for the continuous time systems the assumption that (**C2**) holds, even for special descriptor systems like (33). For such special systems, however, it is easy to characterize when (**C2**) holds. To see this assume that A_c is in Jordan canonical form and that system (33) is partitioned as

$$\begin{bmatrix} J_1 & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} J_1 B_1 \\ N B_2 \end{bmatrix} u, \tag{57}$$

where N contains all Jordan blocks to zero eigenvalues of A_c. We see immediately that (**C2**) holds if and only if $N = 0$, i.e. if the matrix pencil $\lambda A_c - I$ is of index at most one. If (**C2**) does not hold, then the solution behaviour of the algebraic Riccati equation has not been characterized completely, see [11], but in view of the new analogy we have constructed, there is some hope that an approach like that discussed in [27, 26] for discrete time systems will lead to analogous results for the continuous time case. This topic is currently under investigation. If $N = 0$, however, then the solvability theory can be reduced to that for standard systems, see [11, 19]. In this case we restrict the system to the subspace corresponding to the finite eigenvalues. But we see here another difficulty. It may happen that the discrete system is controllable, while the corresponding continuous time problem has to be considered in a smaller dimensional subspace. Such a behaviour certainly creates difficulties for a unified theory.

So far we have related the controllability conditions for the parametrization that we have constructed. Usually one is, however, interested in controllability conditions for (A_c, B_c) or (A_d, B_d) repectively, and observability conditions for (A_c, C_c) or (A_d, C_d), respectively, where $G_c = \pm B_c B_c^*$, $G_d = \pm B_d B_d^*$, $H_c = \pm C_c C_c^*$, $H_d = \pm C_d C_d^*$ are full rank factorizations. It is clear that such full rank factorizations only exist if the matrices G_c, G_d, H_c, H_d are semidefinite. We study the question when this is the case in the next section.

4 Semidefiniteness of blocks

We have seen in the introduction that if our Hamiltonian or symplectic pencil arises from optimal control problems, then the blocks G_d and G_c are semidefinite. In this section we now discuss the question under which conditions this property is retained under the Cayley transformation, i.e. when G_d and G_c or H_d and H_c, respectively, are both semidefinite in the formulas of Lemma 5. Here we assume that the shiftpoints are chosen so that we are not at the pole of the Cayley transformation. The reason is that in the continuous time case the symmetry of the blocks is not directly displayed. In the case that the pencil $\lambda A_c - I$ is of index one, it is enough to have the symmetry and semidefiniteness in the range of A_c, see [19].

Lemma 9
a) Consider the symplectic pencil (36) and the associated Hamiltonian pencil (37) with λ_1, A_c, G_c, H_c as in Lemma 5 a) and λ_1 such that $(\mathcal{E}_d - \lambda_1 \mathcal{A}_d)^{-1}$ exists.
Assume that $G_d = B_d R_d^{-1} B_d^*$ with B_d of full column rank and that R_d is positive definite. Then G_c is semidefinite if and only if the Popov function

$$\Psi_d(\lambda_1) := R_d + B_d^*(I - \lambda_1 F_d)^{-*} H_d (I - \lambda_1 F_d)^{-1} B_d \tag{58}$$

is definite.
b) Consider a Hamiltonian pencil as in (27) and the associated symplectic pencil (46) with λ_1, A_d, G_d, H_d as in Lemma 5 b) and λ_1 such that $(\mathcal{E}_c + \lambda_1 \mathcal{A}_c)^{-1}$ exists.
Assume that $G_c = B_c R_c^{-1} B_c^*$ with B_c of full column rank and that R_c is positive definite. Then G_d is semidefinite if and only if the Popov function

$$\Psi_c(\lambda_1) := R_c + B_c^*(A_c + \lambda_1 I)^{-*} A_c^* H_c A_c (A_c + \lambda_1 I)^{-1} B_c \tag{59}$$

is definite.
c) Consider the symplectic pencil (36) and the associated Hamiltonian pencil (37) with λ_1, A_c, G_c, H_c as in Lemma 5 a) and λ_1 such that $(\mathcal{E}_d - \lambda_1 \mathcal{A}_d)^{-1}$ exists.
Assume that $H_d = C_d^* C_d$ with C_d of full row rank. Then H_c is semidefinite if and only if the Popov function

$$\Phi_d(\lambda_1) := I + C_d (I - \lambda_1 F_d)^{-1} G_d (I - \lambda_1 F_d)^{-*} C_d^* \tag{60}$$

is definite.
d) Consider a Hamiltonian pencil as in (27) and the associated symplectic pencil (46) with λ_1, A_d, G_d, H_d as in Lemma 5 b) and λ_1 such that $(\mathcal{E}_c + \lambda_1 \mathcal{A}_c)^{-1}$ exists.
Assume that $H_c = C_c^* C_c$ with C_c of full row rank. Then H_d is semidefinite if and only if the Popov function

$$\Phi_c(\lambda_1) := I + C_c (A_c + \lambda_1 I)^{-1} A_c G_c A_c^* (A_c + \lambda_1 I)^{-*} C_c^* \tag{61}$$

is definite.

Proof. a) From (44) we have

$$\begin{aligned} G_c &= -\tilde{G}_d(I + H_d\tilde{G}_d)^{-1}(\lambda_1 + \bar{\lambda}_1) \\ &= -2Re(\lambda_1)(I - \lambda_1 F_d)^{-1} B_d R_d^{-1} B_d^*(I - \lambda_1 F_d)^{-*} \times \\ & \quad [I + H_d(I - \lambda_1 F_d)^{-1} B_d R_d^{-1} B_d^*(I - \lambda_1 F_d)^{-*}]^{-1} \end{aligned}$$

Let

$$(I - \lambda_1 F_d)^{-1} B_d = Q_d \begin{bmatrix} U_d \\ 0 \end{bmatrix}$$

be a QR-factorization. Then

$$\begin{aligned} & G_c \\ = \;& -2Re(\lambda_1) Q_d \begin{bmatrix} U_d \\ 0 \end{bmatrix} R_d^{-1} \begin{bmatrix} U_d \\ 0 \end{bmatrix}^* Q_d^* \left(I + H_d Q_d \begin{bmatrix} U_d \\ 0 \end{bmatrix} R_d^{-1} \begin{bmatrix} U_d \\ 0 \end{bmatrix}^* Q_d^* \right)^{-1} \\ = \;& -2Re(\lambda_1) Q_d \begin{bmatrix} U_d R_d^{-1} U_d^* & 0 \\ 0 & 0 \end{bmatrix} \left(I + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} U_d R_d^{-1} U_d^* & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} Q_d^*, \end{aligned}$$

where $\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} := Q_d^* H_d Q_d$ is partitioned conformally with $\begin{bmatrix} U_d R_d^{-1} U_d^* & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} G_c &= -2Re(\lambda_1) Q_d \begin{bmatrix} U_d R_d^{-1} U_d^* (I + H_{11} U_d R_d^{-1} U_d^*)^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q_d^* \\ &= -2Re(\lambda_1) Q_d \begin{bmatrix} U_d R_d^{-1/2} (R_d + U_d^* H_{11} U_d)^{-1} R_d^{-1/2} U_d^* & 0 \\ 0 & 0 \end{bmatrix} Q_d^* \\ &= -2Re(\lambda_1) Q_d \begin{bmatrix} U_d \\ 0 \end{bmatrix} R_d^{-1/2} (R_d + \begin{bmatrix} U_d^* & 0 \end{bmatrix} Q_d^* H_d Q_d \begin{bmatrix} U_d \\ 0 \end{bmatrix})^{-1} \times \\ & \qquad R_d^{-1/2} \begin{bmatrix} U_d^* & 0 \end{bmatrix} Q_d^* \\ &= -2Re(\lambda_1)(I - \lambda_1 F_d)^{-1} B_d R_d^{-1/2} \times \\ & \quad (R_d + B_d^*(I - \lambda_1 F_d)^{-*} H_d (I - \lambda_1 F_d)^{-1} B_d)^{-1} R_d^{-1/2} (I - \lambda_1 F_d)^{-1} B_d)^*. \end{aligned}$$

Now since B_d has full column rank, it follows that G_c is semidefinite if and only if the middle term $R_d + B_d^*(I - \lambda_1 F_d)^{-*} H_d (I - \lambda_1 F_d)^{-1} B_d$ is definite, which finishes the proof.
The proof of the other parts follow the same line of arguments as in a). □
The necessity to study the Popov functions Ψ, Φ for values on the unit circle has already been observed in several places, see [14, 12, 13, 15, 20, 21, 26]. Here it shows up naturally in order to relate the semidefiniteness of the blocks.
As a consequence we immediately obtain the following corollary:

Corollary 10
a) Consider the symplectic pencil (36) and the associated Hamiltonian matrix (37) with λ_1, A_c, G_c, H_c *as in Lemma 5 a) and* λ_1 *such that* $(\mathcal{E}_d - \lambda_1 \mathcal{A}_d)^{-1}$ *exists.*
If G_d *and* H_d *are both positive semidefinite, then* G_c *and* H_c *are semidefinite.*
b) Consider a Hamiltonian pencil as in (45) and the associated symplectic pencil (46) with λ_1, A_d, G_d, H_d *as in Lemma 5 b).*
If G_c *and* H_c *are positive semidefinite, then* G_d *and* H_d *are semidefinite.*

Proof. Clear from Lemma 9. □

5 A Metatheorem

Based on the constructions of the previous sections, we are now able to present a metatheorem that relates results for a large class of discrete and continuous time control problems. In principle the existence of such a metatheorem is a folklore result and it has been widely used to construct analogous results for discrete and continuous problems. The major reason why such a result has not been explictely formulated yet is probably that in some cases a discrepancy between the two problems shows up. We will discuss one such discrepancy in Section 6. but we believe that in the pencil formulation these discrepancies are much better understood and can also be partly removed.
We assume in this section that the necessary parametrizations exist and we obtain the following metatheorem:

A Metatheorem *Suppose that* (A^c) *is a set of assumptions for a continuous time control system and* (A^d) *is a corresponding set of assumptions for the corresponding discrete system. Let* (B^c) *be an assertion for the continuous time system and* (B^d) *for the corresponding discrete time system. Then we have the following implication diagram*

$$\begin{array}{ccccc} & (A^c) & \Longrightarrow & (B^c) & \\ \text{Cayley} & \Updownarrow & & \Updownarrow & \text{Cayley} \\ & (A^d) & \Longrightarrow & (B^d) & \end{array} . \qquad (62)$$

With other words, we can prove $(A^c) \Rightarrow (B^c)$ if for the corresponding discrete time system obtained via the generalized Cayley transformation $(A^d) \Rightarrow (B^d)$, and vice versa, provided the Cayley transformation gives a proper transformation for the assumptions and the assertion. We will demonstrate the use of this metatheorem in the next section.

6 Invariant Subspaces

In this section we discuss deflating subspaces of Hamiltonian and symplectic pencils. Such invariant subspaces are used for the computation of solutions of the algebraic Riccati equations, e.g. [18, 19].
Part a) of the following result is due to Wimmer [28] and generalizes previous results in [9, 10], part b) was conjectured by Wimmer.

Theorem 11
a) Consider the Hamiltonian matrix

$$\mathcal{H}_c = \begin{bmatrix} F_c & B_c R_c^{-1} B_c^* \\ H_c & -F_c^* \end{bmatrix} \tag{63}$$

with B_c *of full column rank and* R_c *positive definite. Let* $U, V \in \mathbf{C}^{n\times n}$ *with* $V^*U = U^*V$ *be such that*

$$\operatorname{rank} \begin{bmatrix} U \\ V \end{bmatrix} = n \tag{64}$$

and the columns of $\begin{bmatrix} U \\ V \end{bmatrix}$ *span an invariant subspace of* $\mathcal{H}_c$*, i.e.*

$$\mathcal{H}_c \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} \Lambda, \tag{65}$$

with $\Lambda \in \mathbf{C}^{n\times n}$*. If* $[F_c, B_c]$ *is controllable then* U *is nonsingular and* $X_c = VU^{-1}$ *is a solution of the algebraic Riccati equation*

$$F_c^* X_c + X_c F_c + H_c - X_c B_c R_c^{-1} B_c^* X_c = 0. \tag{66}$$

b) Consider the symplectic pencil

$$\lambda \mathcal{E}_d - \mathcal{A}_d = \lambda \begin{bmatrix} F_d & 0 \\ H_d & I \end{bmatrix} - \lambda \begin{bmatrix} I & -B_d R_d^{-1} B_d^* \\ 0 & F_d^* \end{bmatrix} \tag{67}$$

Let $U, V \in \mathbf{C}^{n\times n}$ *with* $V^*U = U^*V$ *be such that*

$$\operatorname{rank} \begin{bmatrix} U \\ V \end{bmatrix} = n \tag{68}$$

and the columns of $\begin{bmatrix} U \\ V \end{bmatrix}$ *span a deflating subspace of* $\lambda \mathcal{E}_d - \mathcal{A}_d$ *not containing eigenvectors to infinite eigenvalues, i.e.*

$$\mathcal{E}_d \begin{bmatrix} U \\ V \end{bmatrix} = \mathcal{A}_d \begin{bmatrix} U \\ V \end{bmatrix} \Lambda, \tag{69}$$

with $\Lambda \in \mathbf{C}^{n\times n}$*. Suppose there exists* $\lambda_1 \in \mathcal{C}$*,* $|\lambda_1| = 1$*, such that* $I - \lambda_1 F_d$ *and* $\lambda_1 I + F_d - G_d(\bar{\lambda}_1 - F_d)^{-*} H_d$ *are nonsingular and*

$$\Psi_d(\lambda_1) := R_d + B_d^*(F_d - \lambda_1 I)^{-*} H_d (F_d - \lambda_1 I)^{-1} B_d \text{ is definite.} \tag{70}$$

If $[F_d, B_d]$ *is controllable then* U *is nonsingular and* $X_d = VU^{-1}$ *solves the discrete algebraic Riccati equation*

$$-X_d + F_d^* X_c F_d + F_d^* X_d (I - B_d R_d^{-1} B_d^* X_d)^{-1} F_d = 0. \tag{71}$$

Proof. a) See Wimmer [28]. We give the proof here for completeness. From (65) we obtain

$$\begin{aligned} A_c U - B_c R_c^{-1} B_c^* V &= U\Lambda & (72)\\ H_c U - A_c^* V &= V\Lambda. & (73) \end{aligned}$$

In a first step we show that $\ker U$ is A_c-invariant. Let $z \in \mathcal{C}^n \backslash \{0\}$ and $Uz = 0$. Multiplying (72) from the left by $z^* V^*$ and from the right by z, we obtain

$$z^*(V^* A_c U - V^* B_c R_c^{-1} B_c^* V) z = z^* V^* U \Lambda z = z^* U^* V \Lambda z.$$

This implies that $z^* V^* B_c R_c^{-1} B_c^* V z = 0$ and, since by (64) $z^* V^* \neq 0$, and since R_c is positive definite and B_c has full column rank, we obtain

$$B_c R_c^{-1} B_c^* V z = 0, \tag{74}$$

which implies $U\Lambda z = 0$ from (72).
Suppose now that $\ker U$ is not empty. Then by the previous observations $\ker U$ contains an eigenvector z of Λ, i.e. $\Lambda z = \lambda z$. By (73) we obtain $-A_c^* V z = V \Lambda z$ and then from (72) we obtain $z^* V^* [A_c + \bar{\lambda} I, B_c] = 0$. The controllability of $[A_c, B_c]$ implies that $z^* V^* = 0$ which contradicts assumption (64). The rest is well known, e.g. [19, 15].
b) Applying the Cayley transformation with λ_1 such that (70) holds we obtain a Hamiltonian matrix with the same invariant subspace $\begin{bmatrix} U \\ V \end{bmatrix}$. By Lemma 8 and Lemma 9 this Hamiltonian matrix satisfies the conditions of a). Thus part a) gives the required conclusion. □
Observe that we cannot allow generalized eigenvectors to infinite eigenvalues in the deflating subspace spanned by the columns of $\begin{bmatrix} U \\ V \end{bmatrix}$ in part b). If such an eigenvector would be included, then we have $\mathcal{E}_c \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} \Lambda_1$, and $\mathcal{A}_c \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} \Lambda_2$, but Λ_2 would not be invertible and hence we could not obtain the subspace equation in the form (69). This leads to a discrepancy in

the solvability theory for discrete and continuous algebraic Riccati equations. In the standard case (A_c invertible) we obtain all Hermitian solutions of (66) by taking in (65) any Lagrangian invariant subspace corresponding to any set of n eigenvalues, e.g. [1]. So the number of different Hermitian solutions is equal to the number of different Lagrangian subspaces. In the discrete time case, if F_d is singular, all Lagrangian subspaces that contain eigenvectors to infinite eigenvalues have to be excluded, so in general there are not as many solutions. as in the continuous time case. If we consider, however, Hamiltonian pencils with A_c singular, we have the same difficulty, that we have to exclude eigenvectors to infinite eigenvalues. In this case there are always infinitely many solutions, see [11], but in the case of index 1 systems we can restrict ourselves to the range of A_c and apply the standard theory. The situation becomes more complicated though, since zero and infinite eigenvalues of the Hamiltonian pencil will always be mapped to the shiftpoint, hence the symplectic matrix or pencil has multiple eigenvalues on the unit circle. This always creates difficulties, too.
It is currently under investigation what the solvability results are for the case of Hamiltonian pencils. If the Hamiltonian pencil is of index at most one, see [19].
To illustrate these observations consider the following examples.

Example 2 [8, 27].
Consider a symplectic pencil of the form (46) with

$$F_d = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; G_d = B_d B_d^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; H_d = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

The spectrum of the pencil is $0, \infty, -1.5 \pm .5\sqrt{5}$, i.e. F_d is singular. The discrete algebraic Riccati equation has exactly two solutions

$$X_1 = \begin{bmatrix} 1 & 2 \\ 2 & 2+\sqrt{5} \end{bmatrix}, \; X_2 = \begin{bmatrix} 1 & 2 \\ 2 & 2-\sqrt{5} \end{bmatrix}.$$

but none of them is negative semidefinite. On the other hand (F_d, B_d) is controllable. In the standard case of a continuous time system, this property would assure the existence of a negative semidefinite solution.
We can apply the generalized Cayley transformation and obtain

$$F_c = \begin{bmatrix} 0.4 & 0.2 \\ -0.6 & -0.8 \end{bmatrix}, H_c = \begin{bmatrix} -0.2 & -0.6 \\ -0.6 & -1.8 \end{bmatrix}, G_c = \begin{bmatrix} -0.2 & -0.2 \\ -0.2 & -0.2 \end{bmatrix}.$$

The corresponding continuous algebraic Riccati equation with these coefficient matrices has four different solutions among them one positive and one negative semidefinite. Thus the transformed continuous system has a different solution behaviour. The reason is that on the discrete side we are at the pole of the Cayley transformation, which leads to infinite eigenvalues, which have to be

treated differently than finite eigenvalues. In other words, the Metatheorem cannot be applied, since the set of Lagrangian subspaces correponding to finite eigenvalues is bigger in the continuous time case then in the discrete case. We observe a similar behaviour if we go the other direction
Consider the Hamiltonian pencil (27) with

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; G_c = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \; H_c = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

where A_c is not invertible. The generalized Cayley transformation yields

$$F_d = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix}, \; H_d = \begin{bmatrix} -0.4 & 0 \\ 0 & 0 \end{bmatrix}, \; G_d = \begin{bmatrix} -0.4 & 0 \\ 0 & 0 \end{bmatrix}.$$

and the symplectic pencil is equivalent to a symplectic matrix

$$\mathcal{S}_d = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with a double eigenvalue at 1. Thus the discrete system has no infinite eigenvalues, but it is not controllable anymore.

Other examples can be constructed, where properties are destroyed when the Cayley transformation is considered at the poles. A MATLAB code for the generalized Cayley transformation is available from the author.

7 Conclusion

We have presented a new approach towards a unified theory for the discrete and continuous control problems. A generalized Cayley transformation is constructed for the transformation between Hamiltonian and symplectic pencils. This generalization leads to the observation that the analogous continuous problem to a discrete Riccati equation with singular system matrix F_d is a generalized continuous Riccati equation arising from a descriptor system. The solvability theory for the generalized algebraic Riccati equation is not completely settled, but the constructed analogy gives hope for a complete theory in the near future, using the ideas from discrete equations. Well known discrepancies between the discrete and continuous situation are analyzed under this new analogy and it is demonstrated that they are related to deflating subspaces for symplectic pencils containing eigenvectors to infinite eigenvalues.

8 Acknowlegdement

We thank H. Wimmer for initiating this research by presenting examples where there was no analogy between the discrete and continuous case. We also thank C. Rost and L. Elsner and D. Happel for illuminating discussions.

References

[1] G.S. Ammar and V. Mehrmann. On Hamiltonian and symplectic Hessenberg forms. *Linear Algebra Appl.*, 149:55–72, 1991.

[2] A.Y. Barraud. Produit étoile et fonction signe de matrice—application à l'équation de Riccati dans le cas discret. *R.A.I.R.O. Automatique/Systems Analysis and Control*, 14:55–85, 1980.

[3] A. Bunse-Gerstner, V. Mehrmann, and N.K. Nichols. Regularization of descriptor systems by derivative and proportional state feedback. *SIAM J. Matrix Anal. Appl.*, 13:46–67, 1992.

[4] A. Bunse-Gerstner, V. Mehrmann, and N.K. Nichols. Regularization of descriptor systems by output feedback. *IEEE Trans. Automat. Control*, AC-39:1742–1748, 1994.

[5] F.R. Gantmacher. *Theory of Matrices*, volume 1. Chelsea, New York, 1959.

[6] J.D. Gardiner and A.J. Laub. A generalization of the matrix-sign-function solution for algebraic Riccati equations. *Internat. J. Control*, 44:823–832, 1986. (see also *Proc. 1985 CDC*, pp. 1233–1235).

[7] M.C.J. Hautus. Controllability and observability conditions of linear autonomous systems. In *Proceedings of the Kon. Ned. Akad. Wetensch.*, volume 72 of *Ser A*, pages 443–448, 1969.

[8] E. Jonckheere. On the existence of a negative seimdefinite antistabilizing solution to the diescrete algebraic riccati equation. *IEEE Trans. Automat. Control*, AC-26:707–712, 1981.

[9] V. Kučera. The structure and properties of time optimal discrete linear control. *IEEE Trans. Automat. Control*, AC-16:375–377, 1971.

[10] V. Kučera. A review of the matrix Riccati equation. *Kybernetica* (Prague), 9:42–61, 1973.

[11] P. Kunkel and V. Mehrmann. Numerical solution of Riccati differential algebraic equations. *Linear Algebra Appl.*, 137/138:39–66, 1990.

[12] P. Lancaster, A.C.M. Ran, and L. Rodman. Hermitian solutions of the discrete algebraic Riccati equation. *Internat. J. Control*, 44:777–802, 1986.

[13] P. Lancaster, A.C.M. Ran, and L. Rodman. An existence and monotonicity theorem for the discrete algebraic Riccati equation. *Lin. Multilin. Alg.*, 20:353–361, 1987.

[14] P. Lancaster and L. Rodman. Solutions of the continuous and discrete–time algebraic Riccati equations: A review. In S. Bittanti, editor, *Lecture Notes of the Workshop on "The Riccati Equation in Control, Systems and Signal"*, pages 78–82, Bologna, Italy, 1989. Pitagora Editrice.

[15] P. Lancaster and L. Rodman. *The algebraic Riccati equation.* Oxford University Press, Oxford, 1994.

[16] A.J. Laub. A Schur method for solving algebraic Riccati equations. *IEEE Trans. Automat. Control*, AC-24:913–921, 1979. (see also *Proc. 1978 CDC (Jan. 1979)*, pp. 60-65).

[17] A.J. Laub. Algebraic aspects of generalized eigenvalue problems for solving Riccati equations. In C.I. Byrnes and A. Lindquist, editors, *Computational and Combinatorial Methods in Systems Theory*, pages 213–227. Elsevier (North-Holland), 1986.

[18] A.J. Laub. Invariant subspace methods for the numerical solution of Riccati equations. In S. Bittanti, A.J. Laub, and J.C. Willems, editors, *The Riccati Equation*, pages 163–196. Springer-Verlag, Berlin, 1991.

[19] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution.* Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, July 1991.

[20] B.P. Molinari. Equivalence relations for the algebraic Riccati equation. *SIAM J. Cont. Optim.*, 11:272–285, 1973.

[21] B.P. Molinari. The stabilizing solution of the discrete algebraic Riccati equation. *IEEE Trans. Automat. Control*, 20:396–399, 1975.

[22] R. Ober. Balanced parametrizations of classes of linear systems. *SIAM J. Cont. Optim.*, 29:1251–1287, 1991.

[23] R. Ober and S. Montgomery-Smith. Bilinear trasnformation of infinite dimensional state space systems and balanced realizations of nonratoinal transfer functions. *SIAM J. Cont. Optim.*, 28:438–465, 1990.

[24] T. Pappas, A.J. Laub, and N.R. Sandell. On the numerical solution of the discrete-time algebraic Riccati equation. *IEEE Trans. Automat. Control*, AC-25:631–641, 1980.

[25] H. Weyl. *The Classical Groups.* Princeton University Press, Princeton, N.J.,, 1973. 8th Printing.

[26] H.K. Wimmer. Hermitian solutions of the discrete-time algebraic Riccati equation. Technical report, Math. Institut, Univ. Würzburg, D-97074 Würzburg, 1994.

[27] H.K. Wimmer. On the existence of a least and negative semidefinite solution of the discrete-time algebraic Riccati equation. *J. Math. Syst. Estimat. Control*, to appear, 1994.

[28] H.K. Wimmer. private communication. Technical report, Math. Institut, Univ. Würzburg, D-97074 Würzburg, 1994.