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Eigenvalue Decay Bounds for Solutions of Lyapunov Equations: The Symmetric Case*

Thilo Penzl

Abstract

We present two new bounds for the eigenvalues of the solutions to a class of continuous-time and discrete-time Lyapunov equations. These bounds hold for Lyapunov equations with symmetric coefficient matrices and right-hand side matrices of low rank. They reflect the fast decay of the nonincreasingly ordered eigenvalues of the solution matrix.

Keywords: Lyapunov equation; eigenvalue decay; eigenvalue bound.

1 Introduction

In this note we mainly focus on the continuous-time algebraic Lyapunov equation (CALE)

$$AX + XA^T = -BB^T \tag{1}$$

where $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$. We denote the open left half of the complex plane and that of the real axis by $\mathbb{C}_-$ and $\mathbb{R}_-$, respectively. The matrix A is assumed to be c-stable, i.e., its spectrum $\sigma(A)$ is a subset of $\mathbb{C}_-$. It is well-known that under this assumption a unique solution $X \in \mathbb{R}^{n,n}$ exists, which is symmetric and positive semidefinite [15]. Consequently, its eigenvalues are real and nonnegative. Throughout this paper eigenvalues of symmetric matrices are arranged in a nonincreasing order, e.g., $\lambda_1(X) \geq \ldots \geq \lambda_n(X)$. It follows from the uniqueness of the solution that $X \neq 0$ and $\lambda_1(X) > 0$ if $B \neq 0$. In the sequel we consider the special case of the CALE (1) where the right-hand side has a very low rank. More precisely, we assume that $m \ll n$. Such CALEs arise from large dynamical systems with a relatively small number of input and output variables. Numerical experiments (e.g., [13]) show that the eigenvalues of their solution matrices tend to decay very fast. The purpose of this paper is to give some insight into this phenomenon. In particular, we are interested in upper bounds for the expression $\lambda_{k+1}(X)/\lambda_1(X)$ with $k = 1, 2, \ldots, n-1$. This issue is closely related to the computation of the best low-rank approximation to the symmetric, positive semidefinite matrix X, because

$$\min_{\tilde{X} \in \mathbb{R}^{n,n},\, \operatorname{rank} \tilde{X} \leq k} \frac{\|X - \tilde{X}\|}{\|X\|} = \frac{\lambda_{k+1}(X)}{\lambda_1(X)}; \tag{2}$$

see [3, Theorem 2.5.3.], for example. Here, $\|\cdot\|$ denotes the spectral norm of a matrix. Note that $\|X\| = \lambda_1(X)$. A motivation for the investigation of the eigenvalue decay and the corresponding matrix approximation problem is given by several numerical methods for the solution of very large Lyapunov equations [14, 6, 7, 4, 5, 13, 10] which are based on similar low-rank approximations.

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In the last two decades a relatively large number of bounds for solutions of Lyapunov equations have been derived; see [8] for a recent survey. Unfortunately, these bounds are generally useless if $m < n$. The only existing nontrivial bound for this case is that by Mori et al [11]. It is based on the integral representation of the CALE solution, e.g., [9]. However, it is impossible to compute this bound analytically. Its numerical computation is involved and very expensive. Thus, it does not give much insight into the eigenvalue decay problem.

In the remainder of this paper we provide eigenvalue decay bounds for CALEs and their discrete-time counterparts with symmetric coefficient matrices. These bounds are very inexpensive to compute and reflect the fast decay of the eigenvalues. Furthermore, a few comments on the unsymmetric case are made.

2 The symmetric, continuous-time case

In this section we consider the special case where the coefficient matrix A of the CALE (1) is symmetric and negative definite. This case is interesting because several applications involve dynamical systems with a symmetry structure, which in turn lead to symmetric CALEs. For example, such systems arise from the semidiscretization of parabolic PDEs (for example, diffusion problems) and the simulation of circuits, e.g., [1].

The following theorem provides an eigenvalue decay bound, that depends only on the condition number of the coefficient matrix. The condition number is defined as $\kappa(A) = \|A\|\|A^{-1}\|$. For symmetric, negative definite matrices A this is equivalent to $\kappa(A) = \lambda_n(A)/\lambda_1(A)$.

Theorem 1 *Let $A \in \mathbb{R}^{n,n}$ be a symmetric, negative definite matrix with the condition number $\kappa = \kappa(A)$, $B \in \mathbb{R}^{n,m}$ a nonzero matrix, and $\lambda_i(X)$ with $i = 1, \ldots, n$ the nonincreasingly ordered eigenvalues of X. Then,*

$$\frac{\lambda_{mk+1}(X)}{\lambda_1(X)} \leq \left(\prod_{j=0}^{k-1} \frac{\kappa^{(2j+1)/(2k)} - 1}{\kappa^{(2j+1)/(2k)} + 1} \right)^2 \tag{3}$$

for $1 \leq mk < n$.

The following elementary proof is based on the construction of a rank-mk-approximation X_k to the solution X of the CALE. For this purpose we apply k steps of the ADI iteration [12, 17]. The ADI shift parameters are chosen in a manner which is not optimal, but allows us to compute an upper bound for $\|X - X_k\|/\|X\|$. Finally, (2) is applied to find the eigenvalue decay bound.

Proof: Let $k > 1$ be an arbitrary but fixed number such that $mk < n$. First, we introduce the rational functions

$$s_p(t) := \frac{p - t}{p + t} \quad \text{and} \quad s_{\{p_1,\ldots,p_k\}}(t) := \prod_{i=1}^{k} s_{p_i}(t),$$

where $p, p_i, t \in \mathbb{R}_-$. Next, we consider the sequence of ADI iterates $\{X_i\}_{i=0}^{\infty}$ generated by an initial matrix X_0 and

$$X_i = s_{p_i}(A) X_{i-1} s_{p_i}(A) - 2p_i (A + p_i I)^{-1} B B^T (A + p_i I)^{-1}. \tag{4}$$

It is easily verified that the solution X is a stationary point of this mapping (i.e., $X_{i-1} = X$ implies $X_i = X$), which gives

$$X - X_i = s_{p_i}(A)(X - X_{i-1})s_{p_i}(A).$$

We choose $X_0 = 0$ and obtain recursively

$$X - X_k = s_{\{p_1,\ldots,p_k\}}(A) X s_{\{p_1,\ldots,p_k\}}(A). \tag{5}$$

Due to (4), $\operatorname{rank} X_i \le \operatorname{rank} X_{i-1} + m$ and, consequently, $\operatorname{rank} X_k \le km$. This, together with (2), results in the left inequality of

$$\frac{\lambda_{mk+1}(X)}{\lambda_1(X)} \le \frac{\|X - X_k\|}{\|X\|} \le \|s_{\{p_1,\ldots,p_k\}}(A)\|^2, \tag{6}$$

whereas the right inequality is an immediate consequence of (5). The norm term on the right-hand side can be estimated as

$$\begin{aligned} \|s_{\{p_1,\ldots,p_k\}}(A)\| &= \max\left|\sigma(s_{\{p_1,\ldots,p_k\}}(A))\right| && (7)\\ &= \max\{|s_{\{p_1,\ldots,p_k\}}(\lambda)| \ : \ \lambda \in \sigma(A)\} \\ &\le \max\{|s_{\{p_1,\ldots,p_k\}}(\lambda)| \ : \ \lambda \in [\beta,\alpha]\}, && (8) \end{aligned}$$

with $\beta := \lambda_n(A)$ and $\alpha := \lambda_1(A)$. Observe that (7) is valid because $s_{\{p_1,\ldots,p_k\}}(A)$ is a symmetric matrix.

Before we continue estimating (8), we briefly study the behavior of the rational function $s_p(t)$. Basic analysis reveals that $s_p(t)$ is monotonically increasing in $\mathbb{R}_-$ for any $p \in \mathbb{R}_-$ and $|s_p(t)| < 1$ for any $p, t \in \mathbb{R}_-$. Moreover, let $\tilde\alpha, \tilde\beta \in \mathbb{R}$ be two arbitrary numbers with $\tilde\beta < \tilde\alpha < 0$, and define $\tilde\kappa := \tilde\beta/\tilde\alpha$ and $\tilde p := -(\tilde\alpha\tilde\beta)^{1/2}$. Then,

$$0 < -s_{\tilde p}(\tilde\beta) = s_{\tilde p}(\tilde\alpha) = \frac{\sqrt{\tilde\kappa} - 1}{\sqrt{\tilde\kappa} + 1} < 1$$

and, because of the monotonicity,

$$|s_{\tilde p}(t)| \le \frac{\sqrt{\tilde\kappa} - 1}{\sqrt{\tilde\kappa} + 1} \quad \text{for} \quad t \in [\tilde\beta, \tilde\alpha]. \tag{9}$$

Now we return to (8), where we choose the parameters $p_1, \ldots, p_k$. To this end, we first set $t_0 := \alpha$ and $t_i := t_0(\beta/\alpha)^{i/k} = t_0\kappa^{i/k}$ for $i = 0, \pm1, \pm2, \ldots$ This is a geometric sequence with $t_{i+1} < t_i$ and $t_k = \beta$, which forms a partitioning of $[\beta, \alpha]$ into k subintervals, i.e., $[\beta, \alpha] = \cup_{i=1}^k [t_i, t_{i-1}]$. Next, we define the parameters p_i as the geometric center points of the subintervals $[t_i, t_{i-1}]$. Note that

$$p_i := -\sqrt{t_i t_{i-1}} = -\sqrt{t_{i+j} t_{i-1-j}}$$

for any $j = 0, 1, 2, \ldots$ For brevity, we introduce the auxiliary variables

$$\kappa_{k,j} := \frac{t_{i+|j|}}{t_{i-1-|j|}} = \kappa^{(2|j|+1)/k}$$

and

$$r_{k,j} := \frac{\sqrt{\kappa_{k,j}} - 1}{\sqrt{\kappa_{k,j}} + 1} = \frac{\kappa^{(2|j|+1)/(2k)} - 1}{\kappa^{(2|j|+1)/(2k)} + 1} \tag{10}$$

for $j = 0, \pm 1, \pm 2, \ldots$ It follows from (9), where we set $\tilde{\beta} = t_{i+j}$ and $\tilde{\alpha} = t_{i-1-j}$, that

$$|s_{p_i}(t)| \leq r_{k,j} \quad \text{for } \ t \in [t_{i+j}, t_{i-1+j}] \subseteq [t_{i+|j|}, t_{i-1-|j|}]$$

with $i = 1, \ldots, k$ and $j = 0, \pm 1, \pm 2, \ldots$ Note that the right-hand side of the inequality does not depend on i. Multiplying the inequalities for $i = 1, \ldots, k$ and assuming that $t \in [t_l, t_{l-1}]$, where $l = 1, \ldots, k$ determines an arbitrary subinterval of $[\beta, \alpha]$, leads to the left inequality of

$$\left| s_{\{p_1,\ldots,p_k\}}(t) \right| \leq \prod_{i=1}^{k} r_{k,l-i} \leq \prod_{j=0}^{k-1} r_{k,j}. \tag{11}$$

The right inequality holds because

$$0 < r_{k,0} < r_{k,-1} = r_{k,1} < r_{k,-2} = r_{k,2} < \ldots,$$

which can be verified by considering the monotonic dependence of $r_{k,j}$ on $\kappa_{k,j}$ and that of $\kappa_{k,j}$ on j. Since the right term in (11) does not depend on l, it is an upper bound for any $t \in [\beta, \alpha]$. Finally, combining the relations (6), (8), (10), and (11) leads to (3). □

In view of the proof it should be mentioned that Wachspress [16] proposed a procedure for computing parameters $p_1, \ldots, p_k$ which are optimal in the sense of minimizing the right term in (8). However, this procedure is restricted to the special case where k is a power of 2 and no bound for the minimal value of this term is given.

The right-hand side of (3) is monotonically increasing in κ, i.e., enlarging the condition number of the coefficient matrix results in a slower decay of the bound. Note further that the right-hand side of (3) does not depend on m. Roughly speaking, this means that the minimal rank of an approximation to the CALE solution is about proportional to the rank of the right-hand side, if the coefficient matrix and the desired accuracy of this approximation are fixed.

Assuming that the condition number of A is given, our bound is very inexpensive to compute. However, it is not easy see how the right-hand side in (3) depends on κ and k. For this reason we illustrate this dependence in Figure 1. It reveals a fast eigenvalue decay for magnitudes of κ which are typical for many applications.

Finally, we investigate the tightness of our bound using the following test example.

Example 1 [5, Example 4.1] In this example the so-called controllability Gramian of a dynamical system is computed by solving the CALE (1). The underlying system arises from the discretization of a one-dimensional heat transfer problem. The matrices are given as

$$A = \frac{1}{h} \begin{bmatrix} -1 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{h} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

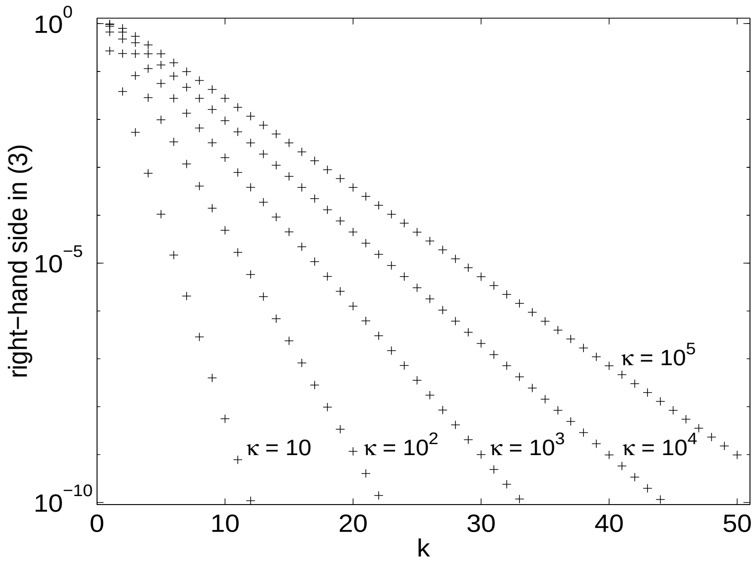

Figure 1: The right-hand side term in (3) as a function of κ and k.

where $h = 1/(n + 1)$, $n = 100$, and $\kappa \approx 1.6 \cdot 10^4$.

In Figure 2 the eigenvalue ratio $\lambda_{mk+1}(X)/\lambda_1(X)$ and the eigenvalue decay bound are plotted for this example. In other words, we compare the left-hand side and the right-hand side of (3) in this figure.

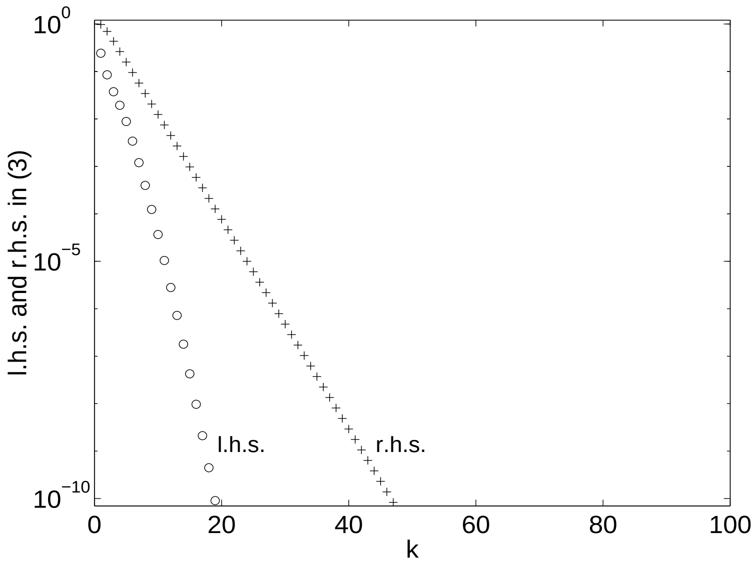

Figure 2: Example 1. Eigenvalue decay bound (right-hand side in (3)) versus actual eigenvalue ratios (left-hand side in (3)).

3 Notes on the unsymmetric, continuous-time case

We do not provide bounds for the general case, where A is not necessarily symmetric. However, we want to discuss some aspects of this case in this section.

The following lemma reveals that it is impossible to find nontrivial eigenvalue decay bounds (as well as trace or determinant bounds etc.) for CALEs, which **only** depend on the eigenvalues or the Jordan structure of A. A matrix pair (A, B) with $A \in \mathbb{R}^{n,n}$ and $B \in \mathbb{R}^{n,m}$ is called controllable if rank $\left[\begin{array}{cc} A - \lambda I & B \end{array} \right] = n$ for all $\lambda \in \mathbb{C}$.

Lemma 1 *Let $\tilde{A} \in \mathbb{R}^{n,n}$ be a c-stable matrix for which a matrix $\tilde{B} \in \mathbb{R}^{n,1}$ exists such that the pair $(\tilde{A}, \tilde{B})$ is controllable. Then, for any symmetric, positive definite matrix $X \in \mathbb{R}^{n,n}$ a nonsingular transformation matrix $T \in \mathbb{R}^{n,n}$ and a nonzero matrix $B \in \mathbb{R}^{n,1}$ exist such that X is the solution of (1) where $A = T\tilde{A}T^{-1}$.*

Proof. Because of the stability and controllability assumptions a unique solution $\tilde{X}$ of

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = -\tilde{B}\tilde{B}^T \tag{12}$$

exists, which is symmetric and positive definite. This is an immediate consequence of [2, Thm. 4]. We define $T := X^{1/2}\tilde{X}^{-1/2}$, which corresponds to $X = T\tilde{X}T^T$ and $B := T\tilde{B}$. Setting $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$, and $\tilde{X} = T^{-1}XT^{-T}$ in (12) shows that X is indeed the solution of (1). □

Next, we consider a test example depending on a parameter which determines the dominance of the skew-symmetric part of A.

Example 2 In this example with dimensions $n = 2d+1$ and $m = 1$ the matrices in the CALE are defined as

$$A = \mathrm{diag}(-1, A_1, A_2, \ldots, A_d) \quad \text{and} \quad B = \left[\begin{array}{cccc} 1 & 1 & \ldots & 1 \end{array} \right]^T,$$

where

$$A_j = \left[\begin{array}{cc} -1 & jt/d \\ -jt/d & -1 \end{array} \right]$$

for $j = 1, \ldots, d$ and $t > 0$ is a real parameter. The eigenvalues of the unsymmetric blockdiagonal matrix A are located in an interval parallel to the imaginary axis or, more precisely, $\mathrm{Re}\,\lambda_i(A) = -1$ and $-t \le \mathrm{Im}\,\lambda_i(A) \le t$ holds for $i = 1, \ldots, n$. We have generated three test examples by choosing $d = 50$ and $t = 10$, 100, 1000. The condition numbers of the resulting matrices A are $\kappa(A) = \|A\|\|A^{-1}\| \approx 10$, 100, 1000, respectively.

Figure 3 shows the eigenvalue ratios $\lambda_{k+1}(X)/\lambda_1(X)$ as a function of k for the three different values of t. This figure reveals two facts. First, the bound (3) does not hold for unsymmetric CALEs, which follows from a comparison with Figure 1. Second, the increasing dominance of the skew-symmetric part of A and the imaginary parts of the eigenvalues of A tends to slow down the decay of the eigenvalues of X. This can even lead to nearly constant eigenvalues (see case $t = 1000$ in Figure 3).

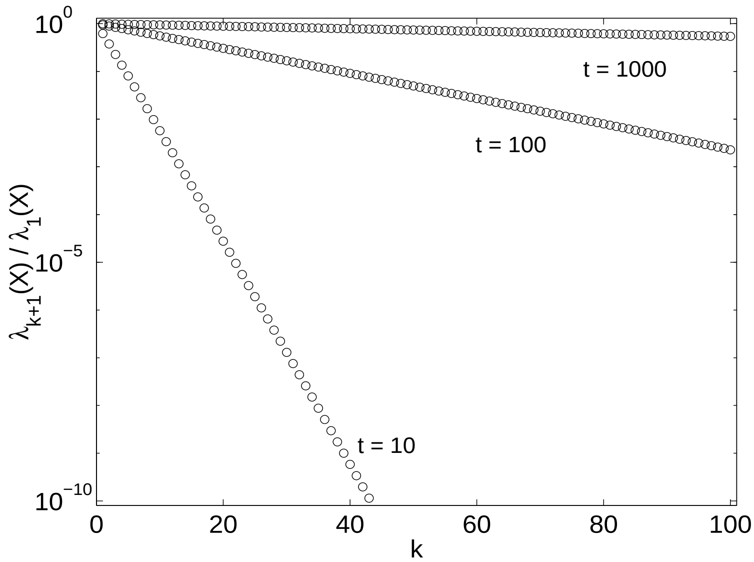

Figure 3: Example 2. Eigenvalue ratio $\lambda_{k+1}(X)/\lambda_1(X)$ as a function of k for three different values of t.

4 The symmetric, discrete-time case

Using the Cayley transformation an eigenvalue decay bound for the discrete-time case can be derived from the continuous-time case. Here, we consider the discrete-time algebraic Lyapunov equation (DALE)

$$DXD^T - X = -EE^T \tag{13}$$

with $D \in \mathbb{R}^{n,n}$ and $E \in \mathbb{R}^{n,m}$. The matrix D is assumed to be d-stable, i.e., $\rho(D) < 1$, where $\rho(D) = \max|\sigma(D)|$ is the spectral radius of D. This guarantees the existence of a unique solution, which is symmetric and positive semidefinite. An eigenvalue decay bound for DALEs is given by the following corollary, which is derived from Theorem 1.

Corollary 1 *If $D \in \mathbb{R}^{n,n}$ is a d-stable, symmetric matrix and $E \in \mathbb{R}^{n,m}$ is a nonzero matrix, then (3) with*

$$\kappa = \frac{(\lambda_n(D) - 1)(\lambda_1(D) + 1)}{(\lambda_n(D) + 1)(\lambda_1(D) - 1)}$$

holds for the nonincreasing eigenvalues of the solution X of (13).

Proof: The proof is based on the inverse Cayley transformation $f(t) = (t-1)/(t+1)$. We define $A = f(D)$ and $B = \sqrt{2}(D+I)^{-1}E$. Then it can be proved that (1) and (13) are equivalent, i.e., both Lyapunov equations have the same solution X. From $\rho(D) < 1$ it follows that A is c-stable. Moreover, A is symmetric. Consequently, (3) holds with $\kappa = \kappa(A)$. Since A is a matrix function in D and $f(t)$ is monotonically increasing for $t \in (-1, 1)$, the relation $\lambda_i(A) = f(\lambda_i(D))$ holds for $i = 1, \ldots, n$. We obtain

$$\kappa = \kappa(A) = \frac{\lambda_n(A)}{\lambda_1(A)} = \frac{f(\lambda_n(D))}{f(\lambda_1(D))} = \frac{(\lambda_n(D) - 1)(\lambda_1(D) + 1)}{(\lambda_n(D) + 1)(\lambda_1(D) - 1)},$$

which completes the proof. □

5 Conclusions

In this paper we have presented eigenvalue decay bounds for the solutions to a class of CALEs and DALEs. These bounds are restricted to equations with symmetric coefficient matrices and right-hand side matrices of low rank. Under these assumptions our bounds reveal the fast decay of the eigenvalues of the solution. This implies that the solution matrices of large Lyapunov equations tend to be ill-conditioned or even **numerically** singular although, under mild conditions, they can be proved to be positive definite. This in turn explains the numerical difficulties of some algorithms, which involve the solutions of Lyapunov equations or their inverses, such as certain methods for partial stabilization of systems. Another consequence of the fast eigenvalue decay is the fact, that the solution matrices can usually be approximated very well by a product of low-rank Cholesky factors. The decay rate of our bounds is determined by the condition number of the coefficient matrix in the continuous-time case and by a simple expression of its extremal eigenvalues in the discrete-time case. Unfortunately, we are not able to provide decay bounds which also hold for unsymmetric Lyapunov equations.

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