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Numerische Simulation auf massiv parallelen Rechnern

Arnd Meyer

Projected PCGM for Handling Hanging Nodes in Adaptive Finite Element Procedures

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Abstract

We consider an adaptive finite element method using so called 'hanging nodes' from subdividing an edge of the actual f.e. mesh and subdividing only one of the adjacent triangles (or quadrilaterals) at this edge. The solution method of the resulting linear system requires a solution belonging to a special subspace that generates a conformal f.e. function. We discuss the use of a special projection operator within the preconditioned conjugate gradient method and a favourable implementation of this projection in combination with the hierarchical f.e. basis for linear and quadratic elements.

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Author's addresses:

Arnd Meyer
TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz

http://www.tu-chemnitz.de/sfb393/

1 Introduction

We consider an adaptive finite element method for the numerical solution of partial differential equations, given in a weak formulation

$$\text{find } u \in \mathbb{V}_0 \text{ with } a(u,v) = \langle f, v\rangle \ \forall v \in \mathbb{V}_0. \tag{1.0.1}$$

In the simplest example of a Laplace equation, the space $\mathbb{V}_0$ is a subspace of $H^1(\Omega)$ with zero–Dirichlet type boundary conditions on $\partial\Omega$ and

$$a(u,v) = \int\limits_{\Omega} (\nabla u)\cdot(\nabla v)\, d\Omega \ .$$

An adaptiv f.e. solution for such a problem starts with a coarse conformal triangulation of Ω, where (1.0.1) is approximated from piecewise linear functions w.r.t. the given mesh. After having an approximate solution some error estimator (see i.e. [10, 1] for an overview) leads to an adaptive refinement, i.e. some of the actual triangles are subdivided into 4 equal subtriangles ('red' subdivision) due to the local large estimated error contribution.

This procedure can disturb the consistency of the mesh, if one triangle is refined on one side of an edge and on the other side not (compare fig.1). There are two possibilies to overcome this difficulty:

1. Usually a 'green' refinement is used for all these unconsistent triangles (τ_1 in fig.1). These 'green' triangles have to be removed before the next refinement steps are done, otherwise some angles could become very small. A generalization to quadrilaterals or bricks is complicate.

2. We accept the so called 'hanging nodes', but have to ensure that the finite element ansatz functions used on this nonconformal triangulation remain continuous piecewise linear (or piecewise quadratic) functions. This idea works on quadrilaterals or bricks as well.

This paper shall be concerned with the case 2. We will show, how the properties of the preconditioned conjugate gradient method can be used in such a way that a solution within this continuous subspace of the nonconformal (discontinuous) function space is guaranteed.

The advantage of such a proceeding is the following:
Either we assemble a stiffness matrix $\tilde{K}$ in the usual way:

for each element do
1) generate element matrix and right hand side
2) add these into the allocated arrays

or we work without any assembly of $\tilde{K}$ and carry out a matrix vector multiply within the PCGM element–by–element.
In both cases the effective stiffness matrix $\tilde{K}$ is defined from a nonconformal f.e. space in the presence of at least one 'hanging node'. This space, called $\mathbb{V}^{(non)}$ is spanned by the

basis functions $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n$ (on a mesh with n nodes), which are collected into a row vector $\tilde{\Phi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n)$ in the following. Each function $\tilde{\varphi}_i$ (nonzero only in triangles that contain the node i as vertex) is as usually the sum of the shape functions defined in the single triangles. Hence, if at least one 'hanging node' occurs in the actual mesh, some of the functions $\tilde{\varphi}_i$ are discontinuous (not in $H^1(\Omega)$).

Usually the 'hanging nodes' do not carry a degree of freedom, their values are defined from the values at neighboring nodes. This is equivalent to the definition of a continuous subspace $\mathbb{V}^{(con)} \subset \mathbb{V}^{(non)}$ with the smaller dimension ($n -$ #'hanging nodes') and we look for the finite element solution $u \in \mathbb{V}^{(con)}$ with

$$a(u, v) = \langle f, v \rangle \; \forall v \in \mathbb{V}^{(con)} \cap \mathbb{V}_0. \tag{1.0.2}$$

On the other hand, the stiffness matrix $\tilde{K}$ is defined with the basis $\tilde{\Phi}$ of $\mathbb{V}^{(non)}$ from $\tilde{K} = (a(\tilde{\varphi}_j, \tilde{\varphi}_i))_{i,j=1}^n$, so

$$u = \sum_{i=1}^{n} u_i \tilde{\varphi}_i = \tilde{\Phi}\underline{u}$$

is a continuous function in $\mathbb{V}^{(con)}$ only if some restrictions on the vector $\underline{u} \in \mathbb{R}^n$ are fulfilled. (Vectors in $\mathbb{R}^n$ are column vectors and are underlined to distinguish them from functions. From the definition of the basis $\tilde{\Phi}$ as row vector of the functions $\tilde{\varphi}_i$, a short abbreviation of the linear combination $u = \tilde{\Phi}\underline{u}$ is used throughout this paper.)

Note that $\underline{u} = (u_1, \ldots, u_n)^T$ contains expanding coefficients of u w.r.t. the basis considered. These coefficients u_i coincide with the values of the function u at node i only in the case of the nodal basis (i.e. $\tilde{\Phi}$). Later on we consider hierarchical basis functions as another basis in $\mathbb{V}^{(non)}$, then this is no longer true.

Using the basis $\tilde{\Phi}$, we transform (1.0.1) into a linear system

$$\tilde{K}\underline{u} = \underline{\tilde{b}},$$

but we have to solve

$$P^T \tilde{K} P \underline{u} = P^T \underline{\tilde{b}}$$

with $\underline{u} \in \underline{\mathbb{U}} = im\, P \subset \mathbb{R}^n$ and P the projection onto the subspace $\underline{\mathbb{U}}$ that leads to continuous functions:

$$u = \tilde{\Phi}\underline{u} \in \mathbb{V}^{(con)} \iff \underline{u} \in \underline{\mathbb{U}}$$

For using these ideas within an adaptive finite element method we have to investigate two basic features:

1. A special variant of PCGM has to be designed, that guarantees a solution within a prescribed subspace $\underline{\mathbb{U}} \subset \mathbb{R}^n$ working with the larger $(n \times n)$–matrix $\tilde{K}$. This is the goal of Chapter 2.

2. The projector P requires a cheap implementation which is simple for linear elements, but more complicate for quadratic ones. This is discussed in Chapters 3 and 4.

2 The Projected PCGM

2.1 The Basic Conjugate Gradient Method

Both algorithms, the PCGM and the projected PCGM are nothing but variants of the basic CG method, which is well–known from Hesteness/Stiefel [5] for a long time, if we replace the symmetric matrix A and the Euklidian inner product in $\mathbb{R}^n$ by a symmetric operator $\mathcal{A}$: $\underline{\mathbb{U}} \to \underline{\mathbb{U}}$ with respect to another inner product $\langle \cdot, \cdot \rangle$. (Nothing but this symmetry is used in the proofs for the basic CG [5]).
So, let $\mathcal{A} : \underline{\mathbb{U}} \to \underline{\mathbb{U}}$ be symmetric and positive definite with respect to the inner product

$$\langle \cdot, \cdot \rangle : \underline{\mathbb{U}} \times \underline{\mathbb{U}} \to \mathbb{R}^1 .$$

Then the CG method for solving $\mathcal{A}\underline{u} = \tilde{\underline{b}}$ reads as

$$\begin{array}{ll} \text{Start:} & \underline{u} \in \underline{\mathbb{U}} \text{ arbitrary,} \\ & \underline{w} := \mathcal{A}\underline{u} - \tilde{\underline{b}} \\ & \underline{q} := \underline{w}, \quad \gamma := \langle \underline{w}, \underline{w} \rangle \end{array}$$

$$\begin{array}{lll} \text{Iteration:} & 1. & \delta := \langle \mathcal{A}\underline{q}, \underline{q} \rangle, \quad \alpha := -\gamma/\delta \\ & 2. & \hat{\underline{u}} := \underline{u} + \alpha \underline{q} \\ & 3. & \hat{\underline{w}} := \underline{w} + \alpha \mathcal{A}\underline{q} \\ & 4. & \hat{\gamma} := \langle \hat{\underline{w}}, \hat{\underline{w}} \rangle, \quad \beta := \hat{\gamma}/\gamma \\ & 5. & \hat{\underline{q}} := \hat{\underline{w}} + \beta \underline{q} \end{array}$$

with $(\hat{\underline{u}}, \hat{\underline{q}}, \hat{\gamma})$ instead of $(\underline{u}, \underline{q}, \gamma)$ *goto* 1.

As is well–known, the rate of convergence depends on the eigenvalues of $\mathcal{A}$: If

$$\underline{\gamma} \leq \lambda_i(\mathcal{A}) \leq \bar{\gamma}, \tag{2.1.1}$$

then the k–th step of the iteration has

$$\langle \mathcal{A}(\underline{u} - \underline{u}^*), (\underline{u} - \underline{u}^*) \rangle \leq \eta^{2k} \cdot const \tag{2.1.2}$$

with $\eta = \frac{1-\sqrt{\xi}}{1+\sqrt{\xi}}$, $\xi = \underline{\gamma}/\overline{\gamma}$ and $\underline{u}^*$ the exakt solution.

2.2 Preconditioned Conjugate Gradient Method

If we try to solve a linear $(n \times n)$ system

$$K\underline{u}^* = \underline{b},$$

with an ill–conditioned s.p.d. matrix K, we have to introduce a preconditioner C with 'good' eigenvalues of $C^{-1}K$. Then PCGM follows directly from 2.1 by replacing

$$\mathcal{A} = C^{-1}K, \quad \tilde{b} = C^{-1}b$$

and

$$\langle \underline{u}, \underline{v} \rangle = (C\underline{u}, \underline{v})$$

with $(\cdot, \cdot)$ the Euklidian inner product in $\mathbb{U} = \mathbb{R}^n$, because $\mathcal{A}$ is s.p.d. w.r.t. $\langle \cdot, \cdot \rangle$.

So, the rate of convergence depends on the spectral bounds of $\mathcal{A} = C^{-1}K$:

$$\underline{\gamma} \leq \lambda_i(C^{-1}K) \leq \bar{\gamma} \tag{2.2.1}$$

$$(\text{which is } \underline{\gamma}(C\underline{x}, \underline{x}) \leq (K\underline{x}, \underline{x}) \leq \bar{\gamma}(C\underline{x}, \underline{x}) \quad \forall \underline{x} \in \mathbb{R}^n)$$

and (2.1.2) reads as

$$(K(\underline{u} - \underline{u}^*), \underline{u} - \underline{u}^*) \leq \eta^{2k} \cdot const.$$

In the implementation, the step 3. is often replaced by

$$\begin{array}{lll} 3a) & \underline{\hat{r}} & := & \underline{r} + \alpha\, K\underline{q} \\ 3b) & \underline{\hat{w}} & := & C^{-1}\underline{\hat{r}} \end{array}$$

with the residuum $\underline{r} = K\underline{u} - \underline{b}$ of the original linear system.

2.3 Projected PCGM

Now we try to solve the linear system

$$P^T \tilde{K} P \underline{u}^* = P^T \underline{\tilde{b}}$$

with $\underline{u}^* = P\underline{u}^* \in \mathbb{U} \subset \mathbb{R}^n$ and $\mathbb{U}$ a proper subspace of dimension $n_0 < n$.
Let $\mathbb{U} = im\, P$ and $\mathbb{V} = im\, P^T$, where P is a projector $\mathbb{R}^n \rightarrow \mathbb{U}$, then $K = P^T\tilde{K}P$ is a unique mapping $\mathbb{U} \rightarrow \mathbb{V}$, so there exists a unique solution $\underline{u}^* \in \mathbb{U}$ for each $P^T\tilde{b} \in \mathbb{V}$. Let $\tilde{K}$ be a symmetric positive definit $(n \times n)$–matrix. For each $\underline{u} \in \mathbb{U}$, the residiuum

$$\underline{r} = P^T\tilde{K}P\underline{u} - P^T\underline{\tilde{b}} = P^T(\tilde{K}\underline{u} - \underline{\tilde{b}})$$

is a vector in $\mathbb{V}$. This means that a basic CGM without preconditioning $(\mathcal{A} = P^T\tilde{K}P)$ is impossible ($\underline{\hat{u}} := \underline{u} + \alpha\underline{r}$ makes no sense).

So, we have to define a preconditioner $C : \mathbb{U} \rightarrow \mathbb{V}$ and especially $C^{-1} : \mathbb{V} \rightarrow \mathbb{U}$ is required to form $\underline{w} := C^{-1}\underline{r} \in \mathbb{U}$.
In order to obtain a well–defined CGM from the basics in 2.1, we consider the following restriction of the Euklidian inner product in $\mathbb{R}^n$ to $\mathbb{U}$ and $\mathbb{V}$ as a dual pairing:

$$(\underline{u}, \underline{v})_D = (\underline{u}, \underline{v}) \quad \text{for each } \underline{u} \in \mathbb{U} \text{ (first argument)} \\ \text{and } \underline{v} \in \mathbb{V} \text{ (second argument).}$$

Then $K = P^T \tilde{K} P$ and C are symmetric positive definit operators w.r.t. $(\cdot,\cdot)_D$ in the following sense:

$$\begin{aligned}
(\underline{u}_1, K\underline{u}_2)_D &= (\underline{u}_2, K\underline{u}_1)_D \quad \forall \underline{u}_1, \underline{u}_2 \in \mathbb{U} \\
(\underline{u}_1, C\underline{u}_2)_D &= (\underline{u}_2, C\underline{u}_1)_D \quad \forall \underline{u}_1, \underline{u}_2 \in \mathbb{U} \\
(C^{-1}\underline{v}_1, \underline{v}_2)_D &= (C^{-1}\underline{v}_2, \underline{v}_1)_D \quad \forall \underline{v}_1, \underline{v}_2 \in \mathbb{V}.
\end{aligned}$$

Now, we define $\mathcal{A} = C^{-1}K : \mathbb{U} \to \mathbb{U}$ and $\langle \underline{u}_1, \underline{u}_2 \rangle = (\underline{u}_1, C\underline{u}_2)_D \; \forall \underline{u}_1, \underline{u}_2 \in \mathbb{U}$.
Here, $\langle \cdot,\cdot \rangle$ is an inner product $\mathbb{U} \times \mathbb{U} \to \mathbb{R}^1$ and $\mathcal{A}$ is symmetric positive definit w.r.t. $\langle \cdot,\cdot \rangle$:

$$\begin{aligned}
\langle \mathcal{A}\underline{u}_1, \underline{u}_2 \rangle &= (C^{-1}K\underline{u}_1, C\underline{u}_2)_D = \left(\underline{u}_2, C(C^{-1}K\underline{u}_1)\right)_D = (\underline{u}_2, K\underline{u}_1)_D \\
&= (\underline{u}_1, K\underline{u}_2)_D = (\underline{u}_1, CC^{-1}K\underline{u}_2)_D = \langle \underline{u}_1, \mathcal{A}\underline{u}_2 \rangle.
\end{aligned}$$

Hence, all presuppositions of (2.1) are fulfilled and the CGM for solving $\mathcal{A}\underline{u}^* = \underline{b}$ (with $\underline{b} = C^{-1}P^T\tilde{\underline{b}} \in \mathbb{U}$) reads as:

Start: $\underline{u} \in \mathbb{U}$ arbitrary
calculate $\underline{r} = \tilde{K}\underline{u} - \tilde{\underline{b}}$,
note: $P^T\underline{r} \in \mathbb{V}$ is the original residuum
$\underline{q} := \underline{w} := C^{-1}P^T\underline{r} \in \mathbb{U}$
$\gamma := \langle \underline{w}, \underline{w} \rangle = (\underline{w}, C\underline{w})_D$
$= (\underline{w}, P^T\underline{r})_D$
$= (\underline{w}, \underline{r})$ (from $\underline{w} \in \mathbb{U}$)

Iteration:

$$\begin{aligned}
&1. & \delta &:= \langle \mathcal{A}\underline{q}, \underline{q} \rangle = (C^{-1}K\underline{q}, C\underline{q})_D = (\underline{q}, K\underline{q})_D = \\
& & &= (\underline{q}, \tilde{K}\underline{q}) \quad , \quad \alpha := -\gamma/\delta \\
&2. & \hat{\underline{u}} &:= \underline{u} + \alpha\underline{q} \quad \text{(update in } \mathbb{U}) \\
&3a) & \hat{\underline{r}} &:= \underline{r} + \alpha\tilde{K}\underline{q} \\
&3b) & \hat{\underline{w}} &:= C^{-1}P^T\hat{\underline{r}} \\
&4. & \hat{\gamma} &:= \langle \hat{\underline{w}}, \hat{\underline{w}} \rangle = (\hat{\underline{w}}, \hat{\underline{r}}), \quad \beta := \hat{\gamma}/\gamma \\
&5. & \hat{\underline{q}} &:= \hat{\underline{w}} + \beta\underline{q} \quad \text{(update in } \mathbb{U})
\end{aligned}$$

Remark 1: The projection $P^T\underline{r}$ in step 3a) is not explicitly done, if the preconditioner is chosen as

$$C^{-1} = P\tilde{C}^{-1}P^T : \mathbb{V} \to \mathbb{U}. \tag{2.3.1}$$

In this case $P^T\underline{r}$ (and $P^T\hat{\underline{r}}$) never occur in the iteration but $\hat{\underline{w}}$ and $\hat{\gamma}$ are well–defined from the structure of C^{-1}.

Remark 2: The rate of convergence depends on the eigenvalues of the operator $\mathcal{A}$:

$$\underline{\gamma} \le \lambda_i(P\tilde{C}^{-1}P^T\tilde{K}P) \le \bar{\gamma}. \tag{2.3.2}$$

We have for the k–th step

$$\begin{aligned}
&\langle \mathcal{A}(\underline{u} - \underline{u}^*), \underline{u} - \underline{u}^* \rangle = (\underline{u} - \underline{u}^*, K(\underline{u} - \underline{u}^*))_D \\
&= \left(\tilde{K}(\underline{u} - \underline{u}^*), \underline{u} - \underline{u}^*\right) \le \eta^{2k} \cdot const
\end{aligned}$$

with η as in 2.1 . For investigating these eigenvalues for our application the Ficticious Space Lemma [8] has to be used (see Chapter 4).

Remark 3: The special structure of the projectors P and P^T occur only once within the precondtioning step 3b), so we have the usual PCGM with a special projected preconditioner running within the subspace $\underline{\mathbb{U}}$.

3 Implementing the Projection

From Chapter 2 we conclude that the PCGM solution of the linear system

$$P^T \tilde{K} \underline{u} = P^T \underline{\tilde{b}}$$

for $\underline{u} \in \underline{\mathbb{U}}$ is a typical PCGM, if the projection, introduced into the preconditoner

$$C^{-1} = P \tilde{C}^{-1} P^T$$

can be cheaply implemented. To clarify the structure of the matrix P, we have to consider the finite element spaces as defined in the Introduction.

For sake of simplicity, we require for our actual mesh at most one 'hanging node' per edge in the linear case as in fig. 1 (resp. one pair of 'hanging nodes' in the quadratic case as in fig. 2). Each triangle contains at most one edge with a 'hanging node' (i.e. the adaptive mesh generator subdivides each triangle 'red' if more than one of its edges are subdivided).

3.1 The Case of (Bi–, Tri–) Linear Elements

Without loss of generality we can suppose that exactly one 'hanging node' (the last one, node n) has been produced by a 'red' subdivision of one triangle without subdividing the other one sharing an edge (k, k').
So, the node n is 'son' of the 'fathers' k and k' in the hierarchical generation of the actual fine mesh and we have the situation as in fig. 1.

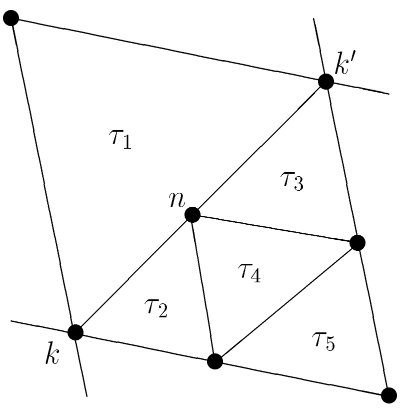

Figure 1

Then, the nonconformal basis $\tilde{\Phi}$ contains 3 discontinuous piecewise linear functions:

$\tilde{\varphi}_n$ (nonzero in τ_2, τ_4 and τ_3; zero in τ_1)
$\tilde{\varphi}_k$ (the support contains τ_1 and τ_2) and
$\tilde{\varphi}_{k'}$ (the support contains τ_1 and τ_3).

A function $u = \tilde{\Phi}\underline{u}$ (with $\underline{u} = (u_1, \ldots, u_n)^T$) is continuous if and only if

$$u_n = \frac{1}{2}(u_k + u_{k'}), \tag{3.1.1}$$

because this implies a linear dependence on u_k and $u_{k'}$ from both sides of this edge (k, k'). The matrix representation of the projector

$$P : \mathbb{R}^n \to \underline{\mathbb{U}} = \left\{ \underline{u} : u_n = \frac{1}{2}(u_k + u_{k'}) \right\}$$

is obviously $P = I - e_n e_n^T + \frac{1}{2} e_n e_k^T + \frac{1}{2} e_n e_{k'}^T$.

A much more simple formula is found with respect to the hierarchical finite element basis $\tilde{\Psi}$. This basis was used for efficient hierarchical preconditioning 2D f.e. systems by YSERENTANT[12].

Let $\tilde{\Psi} = (\tilde{\psi}_1, \ldots, \tilde{\psi}_n)$ the hierarchical basis of the same (discontinuous) f.e. space $\mathbb{V}^{non} = span\tilde{\Phi}$. Then, before subdividing the edge (k, k') we had no 'hanging node', so

$$\mathbb{V}^{con} = span(\tilde{\psi}_1, \ldots, \tilde{\psi}_{n-1})$$

and here the functions $\tilde{\psi}_k$ and $\tilde{\psi}_{k'}$ are continuous with a support containing τ_1 and $\bigcup_{i=2}^{5} \tau_i$. So, only $\tilde{\psi}_n = \tilde{\varphi}_n$ is the discontinuous basis function in the basis $\tilde{\Psi}$ and we have

$$u = \tilde{\Psi}\underline{v} \in \mathbb{V}^{con} \Longleftrightarrow v_n = 0, \tag{3.1.2}$$

a much more simple representation of $\mathbb{V}^{con}$.
For representing $\underline{\mathbb{U}}$ (subspace of $\mathbb{R}^n$ of coefficient vectors with respect to the nodal basis functions $\tilde{\Phi}$), we compare

$$u = \tilde{\Psi}\underline{v} = \tilde{\Phi}\underline{u}.$$

With the well–known transformation matrix Q mapping the nodal basis to the hierarchical one:

$$\tilde{\Psi} = \tilde{\Phi}Q, \tag{3.1.3}$$

we obtain

$$u = \tilde{\Psi}\underline{v} = \tilde{\Phi}Q\underline{v} = \tilde{\Phi}\underline{u} \Longleftrightarrow Q\underline{v} = \underline{u}.$$

This leads to another (hierarchical) representation of the matrix P:

$$\underline{u} \in \underline{\mathbb{U}} \Longleftrightarrow \underline{u} = P\underline{u} \Longleftrightarrow \underline{v} = Q^{-1}\underline{u} \text{ fulfilles } v_n = 0,$$

hence

$$P = Q\hat{P}Q^{-1} \tag{3.1.4}$$

with

$$\begin{aligned} \hat{P} &= diag(1, 1, \ldots, 1, 0) \\ &= I - e_n e_n^T. \end{aligned}$$

The implementation of P (and P^T) within the precondtioning step 3b) of the PCG algorithm in 2.3 is then best combined with the hierarchical preconditioner $\tilde{C}^{-1} = QQ^T$ in $\mathbb{R}^n$: From (3.1.4) and (2.3.1) we have

$$\begin{aligned} C^{-1} = P\tilde{C}^{-1}P^T &= Q\hat{P}Q^{-1}QQ^TQ^{-T}\hat{P}Q^T \\ &= Q\hat{P}Q^T. \end{aligned}$$

So, the projection into the proper subspace is done after transforming the residual $\underline{r} = \tilde{K}\underline{u} - \underline{\tilde{b}}$ into the hierarchical basis.
For better convergence and for 3D–calculations this can be generalized to the BPX–preconditioner in a straight forward manner.

3.2 The Case of Quadratic Elements

Here, we consider 6–node triangles or 8–node quadrilaterals in 2D (resp. 10–node tetrahedrons or 20–node bricks in 3D).
Again, for simple description, we consider only one subdivided edge, that produced 2 'hanging nodes', following fig.2:

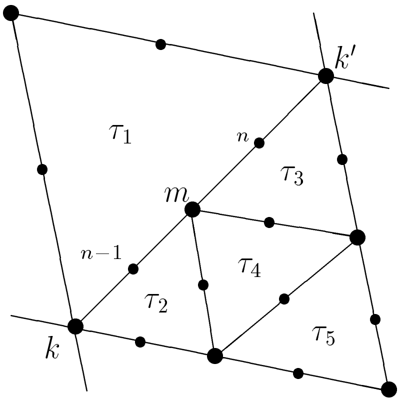

Figure 2

Let the edge (k, m, k') be subdivided into the two smaller edges

$$(m, n-1, k) \text{ and } (m, n, k')$$

(again the two last nodes $n-1$ and n are 'hanging nodes').
Now, the subspace $\mathbb{V}^{con}$ is again represented by restrictions of u_n and u_{n-1} depending on

u_k, u_m and $u_{k'}$ but this is no more a local information on the two subedges. A cheap implementation of P (and P^T) is again possible in considering the hierarchical basis $\tilde{\Psi}$.

For quadratic elements, the hierarchical basis is defined as follows: On the finest mesh, we forget the edge–midnodes and define the hierarchical basis as in the linear case for the functions belonging to vertex nodes. Then the quadratic edge bubbles are added to the basis for completing $\tilde{\Psi}$. Obviously $span\tilde{\Psi} = span\tilde{\Phi}$ and $\tilde{\Psi} = \tilde{\Phi}Q$ remain valid with the same matrix Q as in the linear case.

From the element by element definition of the basis functions as sums of shape functions, we obtain the following 3 discontinuous functions in the hierarchical basis:

$$\begin{aligned}
\tilde{\psi}_m &= \begin{cases} \text{quadratic bubble} & \text{from } \tau_1\text{–side} \\ \text{piecewise linear} & \text{from } \tau_2/\tau_3\text{–side} \end{cases} \\
\tilde{\psi}_n &= \begin{cases} 0 & \text{from } \tau_1\text{–side} \\ \text{quadratic bubble} & \text{in } \tau_2 \end{cases} \\
\tilde{\psi}_{n-1} &= \begin{cases} 0 & \text{from } \tau_1\text{–side} \\ \text{quadratic bubble} & \text{in } \tau_3 \end{cases}
\end{aligned}$$

In the hierarchical basis $\tilde{\Psi}$, the functions $\tilde{\psi}_k$ and $\tilde{\psi}_{k'}$ are continuous (the same as in the linear case).
So, $u = \tilde{\Psi}\underline{v} \in \mathbb{V}^{con}$ is a continuous function, when $v_m\tilde{\psi}_m + v_n\tilde{\psi}_n + v_{n-1}\tilde{\psi}_{n-1}$ is continuous over the edge (k, m, k'), which is equivalent to

$$v_n = v_{n-1} = \frac{1}{4}v_m.$$

($v_n\tilde{\psi}_n + v_{n-1}\tilde{\psi}_{n-1}$ corrects the jump in $v_m\tilde{\psi}_m$ at the edge).
Now, the projection $\hat{P}$ w.r.t. the hierarchical basis is non–symmetric

$$\hat{P} = I - e_n e_n^T - e_{n-1}e_{n-1}^T + \frac{1}{4}e_n e_m^T + \frac{1}{4}e_{n-1}e_m^T$$

and the same calculation as in 3.1 yields

$$C^{-1} = Q\hat{P}\hat{P}^T Q^T$$

as a generalization of the hierarchical precondtioner to 'hanging nodes' for quadratic elements. In the general case, the implementation of $\hat{P}$ and $\hat{P}^T$ is a simple edge–oriented algorithm:

$\underline{v} := \hat{P}^T\underline{v}$: for each edge (m, i, j) do
if node i is 'hanging' then
$v_m := v_m + \frac{1}{4}v_i, v_i := 0$

$\underline{w} := \hat{P}\underline{w}$: for each edge (m, i, j) do
if node i is 'hanging' then
$w_i := \frac{1}{4}w_m$

4 The Estimation of the Eigenvalue Bounds

For complete use of the ideas above, we have to prove that

$$\kappa(P\tilde{C}^{-1}P^T\ P^T\tilde{K}P)$$

is bounded independent on h (or growing as $\sim |\ln h|$ for the simple hierarchical preconditioner $\tilde{C}^{-1} = QQ^T$). Here the Ficticous Space Lemma [8] has to be used:

Ficticious Space Lemma: If we have

1. a symmetric p.d. operator $\tilde{\mathcal{A}} : \tilde{\mathbb{H}} \to \tilde{\mathbb{H}}$
(Hilbert space with inner product $\langle\cdot,\cdot\rangle_\sim$, the 'ficticous space')
2. $\tilde{C}^{-1}$ a 'good' precondtioner for $\tilde{\mathcal{A}}$, i.e.
$\gamma_1\langle\tilde{\mathcal{A}},\tilde{u},\tilde{u}\rangle_\sim \le \langle\tilde{\mathcal{A}}\tilde{C}^{-1}\tilde{\mathcal{A}}\tilde{u},\tilde{u}\rangle_\sim \le \gamma_2\langle\tilde{\mathcal{A}}\tilde{u},\tilde{u}\rangle_\sim \quad \forall\tilde{u}\in\tilde{\mathbb{H}}$.
3. Let $\mathcal{A} : \mathbb{H} \to \mathbb{H}$ s.p.d. w.r.t. $\langle\cdot,\cdot\rangle$ – inner product in $\mathbb{H}$.
4. Let $\mathcal{R} : \tilde{\mathbb{H}} \to \mathbb{H}$ a restriction operator with
$$\langle\mathcal{A}\mathcal{R}\tilde{u},\mathcal{R}\tilde{u}\rangle \le c_R\langle\tilde{\mathcal{A}}\tilde{u},\tilde{u}\rangle_\sim \quad \forall\tilde{u}\in\tilde{\mathbb{H}}$$
5. Let $\mathcal{Q} : \mathbb{H} \to \tilde{\mathbb{H}}$ a prolongation with $\mathcal{R}\mathcal{Q}u = u\ \forall u\in\mathbb{H}$ and
$$\langle\tilde{\mathcal{A}}\mathcal{Q}u,\mathcal{Q}u\rangle_\sim \le c_Q^{-1}\langle\mathcal{A}u,u\rangle\ \forall u\in\mathbb{H}$$

then : $\mathcal{C}^{-1} = \mathcal{R}\tilde{\mathcal{C}}^{-1}\mathcal{R}^*$ is a 'good' preconditioner for $\mathcal{A}$ with

$$\underline{\gamma}\langle\mathcal{A}u,u\rangle \le \langle\mathcal{A}\mathcal{C}^{-1}\mathcal{A}u,u\rangle \le \bar{\gamma}\langle\mathcal{A}u,u\rangle$$

and

$$\underline{\gamma} \ge \gamma_1 c_Q\ , \qquad \bar{\gamma} \le \gamma_2 c_R.$$

We would like to use this Lemma with

$\mathbb{H} = \mathbb{V}^{con}$	($\mathcal{A}$ is defined from the underlying bilinear form $a(\cdot,\cdot)$ with the conformal f. e. basis $span\Phi$ and has K as matrix representation)
$\tilde{\mathbb{H}} = \mathbb{V}^{non}$	($\tilde{\mathcal{A}}$ is defined from $a(\cdot,\cdot)$ w.r.t. the basis $\tilde{\Phi}$, belonging to the stiffness matrix $\tilde{K}$)

Then $\mathcal{R} : \mathbb{V}^{non} \to \mathbb{V}^{con}$ has the previous matrix representation P, and from $\mathbb{V}^{con} \subset \mathbb{V}^{non}$ we

can choose $\mathcal{Q}$ as the idendity.
The F.S.L. could be applied yielding our preconditioner $P\tilde{C}^{-1}P^T$ as matrix representation of $\mathcal{C}^{-1}$, but for the definition of $\tilde{\mathcal{C}}^{-1}$ acting on the nonconformal f.e. space, the spectral bounds γ_1, γ_2 are unclear. So, we consider another auxiliary ficticious space $\tilde{\mathbb{H}} = \mathbb{V}^{green}$.
Let $\mathbb{V}^{green} = span\hat{\Phi}$ the f.e. space on the same triangulation for which $\mathbb{V}^{con}$ and $\mathbb{V}^{non}$ are defined but instead of letting 'hanging nodes', the triangles (τ_1 in the examples) are subdivided 'green' into 2 parts. Then, in the example of 3.1 we have

$$\tilde{\varphi}_i = \hat{\varphi}_i \ \forall i \neq k, k', n,$$

when

$$\begin{aligned} span\tilde{\Phi} &= span(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n) = \mathbb{V}^{non} \\ span\hat{\Phi} &= span(\hat{\varphi}_1, \ldots, \hat{\varphi}_n) = \mathbb{V}^{green} \end{aligned}$$

Now, $\hat{\varphi}_k, \hat{\varphi}_{k'}$ and $\hat{\varphi}_n$ are continuous functions from the usual conformal mesh.
If for a function $u = \hat{\Phi}\underline{u} \in \mathbb{V}^{green}$, we define $\mathcal{R}u = \hat{\Phi}P\underline{u}$ with the same projection matrix as in 3.1 or 3.2, the arising function coincides with the analogous definition from chapters 3.1/3.2 using the nonconformal basis:

$$\mathcal{R}u = \tilde{\Phi}P\underline{u} = \hat{\Phi}P\underline{u} \in \mathbb{V}^{con}.$$

Hence, although the stiffness matrices $\tilde{K}$ and $\hat{K} = (a(\hat{\varphi}_j, \hat{\varphi}_i))_{i,j=1}^n$ do not coincide the projections $P^T\tilde{K}P = P^T\hat{K}P$ do.

Now, we can use the F.S.Lemma with
$\tilde{\mathbb{H}} = \mathbb{V}^{green}$, $\hat{K}$, preconditioner $\tilde{C}^{-1}$ and
$\mathbb{H} = \mathbb{V}^{con}, K = P^T\tilde{K}P = P^T\hat{K}P$ and $\mathcal{R} : \tilde{\mathbb{H}} \to \mathbb{H}$ is represented by P.
For completing this chapter the constant c_R has to be estimated:

$$a(\mathcal{R}u, \mathcal{R}u) \leq c_R a(u,u) \ \forall u \in \mathbb{V}^{green}.$$

Knowing the fact that the dimension of $\mathbb{V}^{con} = \mathcal{R}\mathbb{V}^{green}$ is (n–#hanging nodes) and these 'hanging nodes' can occur in the (locally) finest level only, the angel between the subspaces $\mathbb{V}^{con}$ and $\mathbb{W}$ (when $\mathbb{V}^{green} = \mathbb{V}^{con} + \mathbb{W}$) is 'good':

$$a(u,v) \leq \gamma \left(a(u,u)a(v,v)\right)^{1/2} \quad \forall u \in \mathbb{V}^{con} \quad \forall v \in \mathbb{W}$$

with $0 < \gamma < 1$ (independent on h , see i.e. [2, 6]).
So, c_R follows with $u = \mathcal{R}u + v \ \forall u \in \mathbb{V}^{green}, v \in \mathbb{W}$ and

$$\begin{aligned} a(u,u) &= a(\mathcal{R}u, \mathcal{R}u) + 2a(\mathcal{R}u, v) + a(v,v) \\ &\geq a(\mathcal{R}u, \mathcal{R}u) - 2\gamma \left(a(\mathcal{R}u, \mathcal{R}u) \cdot a(v,v)\right)^{1/2} + a(v,v) \\ &\geq (1-\gamma)\left(a(\mathcal{R}u, \mathcal{R}u) + a(v,v)\right) \end{aligned}$$

yielding $c_R = (1-\gamma)^{-1}$.

Using the Ficticous Space Lemma we obtain for

$$\kappa(C^{-1}K) \leq \bar{\gamma}/\underline{\gamma} \leq \gamma_2/\gamma_1 \cdot c_R.$$

So, the precondtioner $C^{-1} = P\tilde{C}^{-1}P^T$ for $K = P^T\tilde{K}P$ is as good as $\tilde{C}^{-1}$ for $\hat{K}$.

Remark 1: We have
$\gamma_2/\gamma_1 = \mathcal{O}(|\ln h|^2)$ for the hierarchical preconditioner in 2D [12] or
$\gamma_2/\gamma_1 = \mathcal{O}(1)$ for BPX precondtioners [4, 11, 9].

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