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Structured Jordan Canonical Forms for Structured Matrices that are Hermitian, Skew Hermitian or Unitary with Respect to Indefinite Inner Products

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Contents

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Abstract

For inner products defined by a symmetric indefinite matrix $\Sigma_{p,q}$, we study canonical forms for real or complex $\Sigma_{p,q}$-Hermitian matrices, $\Sigma_{p,q}$-skew Hermitian matrices and $\Sigma_{p,q}$-unitary matrices under equivalence transformations which keep the class invariant.

Keywords. structured eigenvalue problems, Lie group, Lie algebra, Jordan algebra
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1 Introduction

In several recent papers [1, 14, 15, 6] the topic of canonical forms for structured matrices and pencils associated with classical Lie groups, Lie algebras and Jordan algebras has been studied. Although the invariants under equivalence transformations have been classified already for quite a while [5, 17], the recent interest comes from the fact that structure preserving transformations can be used very effectively in numerical computations. They allow a reduction in complexity and at the same time often give a better perturbation and error analysis, see for example [2, 3, 4]. The main goal today is to use equivalence transformations that preserve the algebraic structure, but that can also be implemented in a numerically stable way, which usually means that they are unitary transformations. A second goal is to obtain canonical forms that are essentially triangular (within the given structure), and where the nonunitary structured canonical form is just a more condensed version of an analogous unitary structured canonical form.

A complete analysis for the case of Hamiltonian, skew Hamiltonian and symplectic matrices, i.e., matrices that are Hermitian, skew Hermitian and unitary with respect to an indefinite scalar product given by a skew symmetric matrix, has recently been given in [14]. In this paper we now derive analogous results for the matrices that are Hermitian, skew Hermitian and unitary with respect to an inner product defined via the indefinite symmetric matrix $\Sigma_{p,q} := \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, where I_p is the $p \times p$ identity matrix. We consider the following classes of matrices.

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Definition 1 *Let* $\mathbf{R}$ *and* $\mathbf{C}$ *denote the real and complex field, respectively.*

- *A matrix* $\mathcal{C} \in \mathbf{C}^{(p+q)\times(p+q)}$ *is called* $\Sigma_{p,q}$*-Hermitian if* $\mathcal{C}\Sigma_{p,q} = (\mathcal{C}\Sigma_{p,q})^H$*.* $\mathcal{C}$ *is called* $\Sigma_{p,q}$*-symmetric if it is* $\Sigma_{p,q}$*-Hermitian and real.*
- *A matrix* $\mathcal{C} \in \mathbf{C}^{(p+q)\times(p+q)}$ *is called* $\Sigma_{p,q}$*-skew Hermitian if* $\mathcal{C}\Sigma_{p,q} = -(\mathcal{C}\Sigma_{p,q})^H$*.* $\mathcal{C}$ *is called* $\Sigma_{p,q}$*-skew symmetric if it is* $\Sigma_{p,q}$*-skew Hermitian and real.*
- *A matrix* $\mathcal{G} \in \mathbf{C}^{(p+q)\times(p+q)}$ *is called* $\Sigma_{p,q}$*-unitary if* $\mathcal{G}^H\Sigma_{p,q}\mathcal{G} = \Sigma_{p,q}$*. It is called* $\Sigma_{p,q}$*-orthogonal if it is* $\Sigma_{p,q}$*-unitary and real.*

Note that the $\Sigma_{p,q}$-Hermitian matrices form a Lie algebra, while the $\Sigma_{p,q}$-skew Hermitian matrices from a Jordan algebra. Both algebras are invariant under similarity transformations with $\Sigma_{p,q}$-unitary matrices.

Proposition 1

1. *If* $\mathcal{C}$ *is* $\Sigma_{p,q}$*-Hermitian and* $\mathcal{G}$ *is* $\Sigma_{p,q}$*-unitary then* $\mathcal{G}^{-1}\mathcal{C}\mathcal{G}$ *is* $\Sigma_{p,q}$*-Hermitian.*
2. *If* $\mathcal{C}$ *is* $\Sigma_{p,q}$*-skew Hermitian and* $\mathcal{G}$ *is* $\Sigma_{p,q}$*-unitary then* $\mathcal{G}^{-1}\mathcal{C}\mathcal{G}$ *is* $\Sigma_{p,q}$*-skew Hermitian.*
3. *If* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are* $\Sigma_{p,q}$*-unitary then* $\mathcal{G}_1\mathcal{G}_2$ *is also* $\Sigma_{p,q}$*-unitary.*

Similar to the approach for Hamiltonian and symplectic matrices in [14] we derive structured Jordan canonical forms for these classes of matrices. But different from the case of Hamiltonian and symplectic matrices and pencils, for matrices that are $\Sigma_{p,q}$-Hermitian, skew Hermitian or unitary, it is difficult to derive structured triangular Schur like forms with similarity transformations that are both unitary and $\Sigma_{p,q}$-unitary, since this class has only a very small dimension. Currently the best that one can do in this respect are the fishbone like forms of [1]. The approach that we present here is different. To describe the general idea let us consider the case of $\Sigma_{p,q}$-Hermitian matrices. The discussion for the other cases is similar. There are many different approaches that one can take to derive canonical and condensed forms for such matrices. A very simple approach to obtain a canonical form is the idea to express the $\Sigma_{p,q}$-Hermitian matrix $\mathcal{C}$ as an Hermitian pencil $\lambda\Sigma_{p,q} - \Sigma_{p,q}\mathcal{C}$. Using congruence transformations $U^H(\lambda\Sigma_{p,q} - \Sigma_{p,q}\mathcal{C})U$, we obtain a canonical form via classical results, see e.g., [8, 18, 19]. In view of our goals, however, this is not quite what we want, since in general these forms do not give that $U^H\Sigma_{p,q}U = \Sigma_{p,q}$, hence they do not lead directly to the structured form that we want. Clearly, however, the characteristic quantities that we obtain from this canonical form will have to appear in our canonical from, too.

The outline of the paper is as follows:

We will present some basic preliminary results in Section 2 and then present structured canonical forms for $\Sigma_{p,q}$-Hermitian matrices and $\Sigma_{p,q}$-skew Hermitian matrices under $\Sigma_{p,q}$ unitary similarity transformations in Section 3 and Section 4, respectively. By combining the Cayley transformation and the structured canonical forms for $\Sigma_{p,q}$-skew Hermitian matrices we will then derive the structured canonical forms for $\Sigma_{p,q}$-unitary matrices in Section 5. All canonical forms are represented both for real and complex matrices.

2 Preliminaries

In this section we introduce some further notation, derive some preliminary results and state some basic facts that are needed in the following analysis.

Let $\Lambda(A)$ denote the spectrum of a matrix A. We begin with a well-known fact on the relationship between left and right invariant subspaces, see e.g., [7].

Proposition 2 *Let the columns of U span the left invariant subspace of a square matrix A corresponding to $\lambda_1 \in \Lambda(A)$ and let the columns of V span the right invariant subspace corresponding to $\lambda_2 \in \Lambda(A)$. If $\lambda_1 \neq \lambda_2$ then $U^H V = 0$ and if $\lambda_1 = \lambda_2$ then* $\det(U^H V) \neq 0$.

Our construction of structured Jordan froms will be based on the combination of different blocks of the classical, unstructured Jordan form. To do this we need to study in particular canonical forms under congruence for matrices K that satisfy

$$KM = \pm M^H K, \quad K = K^H, \quad \det K \neq 0. \tag{1}$$

Let us recall some facts from the classical theory. Let

$$N_r := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \tag{2}$$

be an $r \times r$ nilpotent Jordan block and let

$$N(r,m) := \mathrm{diag}(\underbrace{N_r, \ldots, N_r}_{m}), \tag{3}$$

then the general structure of the Jordan canonical form of a nilpotent matrix is

$$M = \mathrm{diag}(N(r_1, m_1), \ldots, N(r_s, m_s)). \tag{4}$$

Consider, analogously

$$P_r := \begin{bmatrix} & & -1 \\ & (-1)^2 & \\ \cdot & & \\ (-1)^r & & \end{bmatrix}, \quad \hat{P}_r := \begin{bmatrix} & & 1 \\ & \cdot & \\ \cdot & & \\ 1 & & \end{bmatrix}_{r \times r} \tag{5}$$

and

$$P(r,m) := \mathrm{diag}(\underbrace{P_r, \ldots, P_r}_{m}), \quad \hat{P}(r,m) := \mathrm{diag}(\underbrace{\hat{P}_r, \ldots, \hat{P}_r}_{m}).$$

Then the following relations can be easily verified.

Proposition 3

i) $P_r^H = P_r^{-1} = (-1)^{r-1} P_r$;

ii) $P_r^{-1} N_r^H P_r = -N_r$;

iii) $\hat{P}_r = \hat{P}_r^H = \hat{P}_r^{-1}$;

iv) $\hat{P}_r^{-1} N_r^H \hat{P}_r = N_r$.

For any given nilpotent matrix $N = \mathrm{diag}(N_{r_1}, \ldots, N_{r_s})$ we set

$$P_N := \mathrm{diag}(P_{r_1}, \ldots, P_{r_s}), \quad \hat{P}_N := \mathrm{diag}(\hat{P}_{r_1}, \ldots, \hat{P}_{r_s}) \tag{6}$$

and we denote by $\mathbf{G}(N)$ the set of all matrices that commute with N. To characterize the matrices in $\mathbf{G}(N)$, we need *upper triangular Toeplitz matrices* of the form

$$T := \begin{bmatrix} \tau_0 & \tau_1 & \ldots & \tau_{r-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \tau_1 \\ 0 & & & \tau_0 \end{bmatrix} = \sum_{k=0}^{r-1} \tau_k N_r^k. \tag{7}$$

The diagonal element of such a matrix is denoted by $\Theta(T) := \tau_0$. We have the following well-known Lemma, see Lemma 4.4.11 in [11].

Lemma 2 *Let N_j, N_k be as in (2). A matrix $E \in \mathbf{C}^{j\times k}$ satisfies $N_j E = E N_k$ if and only if E has the form*

$$E = \begin{cases} T & j = k, \\ \begin{bmatrix} 0 & T \end{bmatrix} & j < k, \\ \begin{bmatrix} T \\ 0 \end{bmatrix} & j > k, \end{cases} \tag{8}$$

where T has the form (7).

For more complicated nilpotent matrices in Jordan form we have the following well-known Lemma, see [7, 10], where we denote the set of $j \times k$ rectangular upper triangular Toeplitz matrices E as in (8) by $\mathbf{G}^{j\times k}$.

Lemma 3 *Let N be a nilpotent Jordan matrix of the form $N = \mathrm{diag}(N_{r_1}, \ldots N_{r_s})$. A matrix E commutes with N if and only if E has the block structure $E = [E_{i,j}]_{s\times s}$, where each $E_{i,j} \in \mathbf{C}^{r_i \times r_j}$ is a rectangular upper triangular Toeplitz matrix of the form (8).*

For the particular nilpotent matrix $N_{(r,m)}$ as in (3), it follows that $E \in \mathbf{G}(N_{(r,m)})$ if and only if E has the block structure $E = [E_{i,j}]_{m\times m}$, partitioned conformally with $N_{(r,m)}$, where $E_{i,j} \in \mathbf{G}^{r\times r}$. Collecting the diagonal elements of each of the blocks in one matrix we obtain an $m \times m$ matrix

$$\Theta(E) := \begin{bmatrix} \Theta(E_{1,1}) & \ldots & \Theta(E_{1,m}) \\ \vdots & \ddots & \vdots \\ \Theta(E_{m,1}) & \ldots & \Theta(E_{m,m}) \end{bmatrix},$$

which we call the *main submatrix* of E. Defining

$$\Omega := [e_1, e_{r+1}, \ldots, e_{(m-1)r+1}; e_2, e_{r+2}, \ldots, e_{(m-1)r+2}; \ldots; e_r, e_{2r}, \ldots, e_{mr}],$$

where e_k is the k-th unit vector, we have for each $E \in \mathbf{G}(N_{(r,m)})$, that

$$\omega(E) := \Omega^T E \Omega = \begin{bmatrix} E_0 & E_1 & \dots & E_{r-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & E_1 \\ 0 & & & E_0 \end{bmatrix} \tag{9}$$

with $E_0 = \Theta(E)$. This transformation sets up a one-to-one relationship between $\mathbf{G}(N_{(r,m)})$ and the set of *block upper triangular Toeplitz* matrices. We then have the following result.

Lemma 4 *Let M be as in (4) and P_M as in (6). Let $E \in \mathbf{G}(M)$ and partition E conformally with the block structure of M in (4), i.e., $E = [E_{i,j}]_{s\times s}$ and $E_{k,k} \in \mathbf{G}(N(r_k, m_k))$. Let $\Theta(E_{k,k})$ be the main submatrices of the diagonal blocks $E_{k,k}$, $k = 1, \dots, s$. Then E is nonsingular if and only if $\det(\Theta(E_{k,k})) \neq 0$, for all $k = 1, \dots, s$.*

If E is nonsingular, then there exists a matrix $Y \in \mathbf{G}(M)$, such that

$$(P_M^{-1} Y^H P_M) E Y = \begin{bmatrix} \hat{E}_{1,1} & & & 0 \\ * & \hat{E}_{2,2} & & \\ \vdots & \ddots & \ddots & \\ * & \dots & * & \hat{E}_{s,s} \end{bmatrix} \in \mathbf{G}(M), \tag{10}$$

where

$$\Theta(\hat{E}_{k,k}) = \Theta(E_{k,k}), \quad k = 1, \dots, s, \tag{11}$$

and where for each k, $\Theta(\hat{E}_{k,k})$ is the main submatrix of the diagonal block $\hat{E}_{k,k} \in \mathbf{G}(N(r_k, m_k))$.

Similarly there exists $Y \in \mathbf{G}(M)$, such that $(\hat{P}_M^{-1} Y^H \hat{P}_M) E Y$ and $(\hat{P}_M^{-1} Y^T \hat{P}_M) E Y$, respectively, have the block lower triangular forms as in (10) with the diagonal blocks $\hat{E}_{k,k}$ satisfying (11).

If E is a real matrix, then in all cases Y can be chosen real as well.

Proof. The proof of the first part is given in [14], the other parts follow analogously. □

The following Lemma is an extension of Lemma 11 in [14].

Lemma 5 *Let $E \in \mathbf{G}(N(r, m))$ be nonsingular, where $N(r, m)$ is defined in (3).*

i) If $P(r, m)E$ is Hermitian then there exists a matrix $Y \in \mathbf{G}(N(r, m))$, such that

$$Y^H (P(r, m)E) Y = \operatorname{diag}(\pi_1 P_r, \dots, \pi_m P_r),$$

with $\pi_k \in \{\pm i\}$ if r is even and $\pi_k \in \{\pm 1\}$ if r is odd.

If E is real, then we have two cases. If r is odd, then Y can be chosen real as well and if r is even, then there exists a real Y such that

$$Y^T (P(r, m)E) Y = \operatorname{diag}(\underbrace{\begin{bmatrix} 0 & P_r \\ P_r^T & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & P_r \\ P_r^T & 0 \end{bmatrix}}_{\frac{m}{2}}).$$

ii) If $\hat{P}(r,m)E$ is Hermitian, then there exists a matrix $Y \in \mathbf{G}(N(r,m))$, such that

$$Y^H(\hat{P}(r,m)E)Y = \mathrm{diag}(\pi_1 \hat{P}_r, \ldots, \pi_m \hat{P}_r),$$

with $\pi_k \in \{\pm 1\}$ for all $k = 1, \ldots, m$.

If E is real, then Y can be chosen real.

iii) If $\hat{P}(r,m)E$ is complex symmetric then there exists a matrix $Y \in \mathbf{G}(N(r,m))$ such that

$$Y^T(\hat{P}(r,m)E)Y = \mathrm{diag}(\underbrace{\hat{P}_r, \ldots, \hat{P}_r}_{m}).$$

Proof. i) For convenience we abbreviate $P(r,m)$ by P. With the linear operator ω in (9), we have that $\tilde{E} = \omega(E)$ is a block upper triangular Toeplitz matrix with diagonal block $E_0 = \Theta(E)$. Using Kronecker products ($A \otimes B = [a_{ij}B]$, see [12]), $\tilde{E}$ can be expressed as $\tilde{E} = \sum_{k=0}^{r-1} N_r^k \otimes E_k$, and

$$\tilde{P} = \omega(P) = \begin{bmatrix} 0 & & & -I_m \\ & & (-I_m)^2 & \\ & \cdots & & \\ (-I_m)^r & & & 0 \end{bmatrix} = P_r \otimes I_m.$$

Since PE is Hermitian, so is $\tilde{P}\tilde{E}$ and by symmetry if r is even then $E_0, E_2, \ldots, E_{r-2}$ are skew Hermitian and $E_1, E_3, \ldots, E_{r-1}$ are Hermitian. Similarly if r is odd then $E_0, E_2, \ldots, E_{r-1}$ are Hermitian and $E_1, E_3, \ldots, E_{r-2}$ are skew Hermitian. Suppose that $\tilde{Y}$ is a block upper triangular Toeplitz matrix with the same block structure as $\tilde{E}$. Let $\tilde{Y} = \sum_{k=0}^{r-1} N_r^k \otimes Y_k$. Then using properties of the Kronecker product [12], we obtain

$$\tilde{P}^{-1}\tilde{Y}^H\tilde{P} = \sum_{k=0}^{r-1} (P_r^{-1}N_r^H P_r)^k \otimes Y_k^H = \sum_{k=0}^{r-1} (-1)^k N_r^k \otimes Y_k^H$$

and hence

$$(\tilde{P}^{-1}\tilde{Y}^H\tilde{P})\tilde{E}\tilde{Y} = \sum_{k=0}^{r-1} N_r^k \otimes \{\sum_{p=0}^{k} (-1)^p Y_p^H (\sum_{q=0}^{k-p} E_{k-p-q}Y_q)\}.$$

(Note that $N_r^k = 0$ for $k \geq r$.) We will choose $\tilde{Y}$ such that

$$(\tilde{P}^{-1}\tilde{Y}^H\tilde{P})\tilde{E}\tilde{Y} = I_r \otimes \Pi, \quad \Pi = \mathrm{diag}(\pi_1, \ldots, \pi_m). \tag{12}$$

To this end we will determine matrices $Y_0, \ldots, Y_{r-1}$ that satisfy

$$Y_0^H E_0 Y_0 = \Pi \tag{13}$$

and for $k = 1, \ldots, r-1$

$$Y_0^H E_0 Y_k + (-1)^k Y_k^H E_0 Y_0 = - \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{k-1} \end{bmatrix}^H \begin{bmatrix} E_k & E_{k-1} & \ldots & E_1 \\ -E_{k-1} & -E_{k-2} & \ldots & -E_0 \\ \vdots & \vdots & \cdot & \\ (-1)^{k-1}E_1 & (-1)^{k-1}E_0 & & 0 \end{bmatrix} \begin{bmatrix} Y_0 \\ Y_1 \\ \vdots \\ Y_{k-1} \end{bmatrix}, \tag{14}$$

Since $E_0 = \Theta(E)$, solving (13) is equivalent to determining the canonical form of E_0 under congruence, which is clearly possible. Once Y_0 has been determined, each equation in (14) has the form

$$Y_0^H E_0 Y_k + (-1)^k Y_k^H E_0 Y_0 = -C_k.$$

We now consider different cases: If r and k are both odd or both even then C_k is skew Hermitian and otherwise C_k is Hermitian. By Lemma 4, $\det E \neq 0$ implies $\det E_0 \neq 0$. So in any case Y_k can be chosen successively (but not necessarily uniquely) as $Y_k = -\frac{1}{2}(Y_0^H E_0)^{-1} C_k$ to satisfy (14).

Now we apply the inverse transformation ω^{-1} in (12). Using $Y = \omega^{-1}(\tilde{Y})$, it follows by (9) that $Y \in \mathbf{G}(N(r,m))$ and

$$(P^{-1} Y^H P) E Y = \omega^{-1}(I_r \otimes \Pi) = \mathrm{diag}(\pi_1 I_r, \ldots, \pi_m I_r).$$

Pre-multiplying by P we obtain the assertion.

When E is real, if r is odd then, since $E_0 = \Theta(E)$ is real symmetric, Y_0 can be chosen real in (13). From (14) then all Y_k and hence also $\tilde{Y}$ and therefore Y can be chosen real. If r is even, then E_0 is real skew symmetric but we can still choose a real Y_0 to transform E_0 under congruence to the real skew symmetric canonical form $\mathrm{diag}(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$. For this real Y_0 the remaining Y_k can and therefore also $\tilde{Y}$ and Y again can be chosen real and we obtain a real form.

ii) In this case with $\tilde{E}$ as before, since $\omega(\hat{P}(r,m)) = \hat{P}_r \otimes I_m$ it follows that all E_k are Hermitian. Similarly as before, we can determine a matrix $\tilde{Y} = \sum_{k=0}^{r-1} N_r^k \otimes Y_k$ so that

$$\{\omega(\hat{P}(r,m))^{-1} \tilde{Y}^H \omega(\hat{P}(r,m))\} \tilde{E} \tilde{Y} = \mathrm{diag}(\Pi, \ldots, \Pi),$$

where Π is the canonical form of E_0 under congruence. Taking again the inverse transformation of ω we obtain the result and clearly if E is real we can determine a real Y.

iii) This case is proved analogous as ii). $\square$

The final result in this section presents special condensed forms of the matrices K satisfying (1).

Theorem 6 *Consider a matrix M as in (4).*

i) If K is nonsingular Hermitian and satisfies $KM + M^H K = 0$, then there exists a nonsingular matrix $Y \in \mathbf{G}(M)$ such that

$$Y^H K Y = \mathrm{diag}(\pi_{1,1} P_{r_1}, \ldots, \pi_{1,m_1} P_{r_1}; \ldots; \pi_{s,1} P_{r_s}, \ldots, \pi_{s,m_s} P_{r_s}),$$

where for $j = 1, \ldots, m_k$, $k = 1, \ldots, s$ we have $\pi_{k,j} \in \{\pm i\}$ if r_k is even and $\pi_{k,j} \in \{\pm 1\}$ otherwise.

If K is real then there exists a real nonsingular matrix Y such that

$$Y^T K Y = \mathrm{diag}(K_1, \ldots, K_s),$$

where $K_k = \mathrm{diag}(\pi_{k,1} P_{r_k}, \ldots, \pi_{k,m_k} P_{r_k})$ for odd r_k and

$$K_k = \mathrm{diag}(\underbrace{\begin{bmatrix} 0 & P_{r_k} \\ P_{r_k}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & P_{r_k} \\ P_{r_k}^T & 0 \end{bmatrix}}_{\frac{m_k}{2}})$$

for even r_k.

ii) If K is nonsingular Hermitian and satisfies $KM = M^H K$, then there exists a nonsingular matrix $Y \in \mathbf{G}(M)$ such that

$$Y^H KY = \mathrm{diag}(\pi_{1,1}\hat{P}_{r_1}, \dots, \pi_{1,m_1}\hat{P}_{r_1}; \dots; \pi_{s,1}\hat{P}_{r_s}, \dots, \pi_{s,m_s}\hat{P}_{r_s}),$$

where $\pi_{k,j} \in \{\pm 1\}$ for all $j = 1, \dots, m_k$, $k = 1, \dots, s$.

If K is real then Y can be chosen real.

iii) If K is nonsingular symmetric and satisfies $KM = M^H K$, then there exists a nonsingular matrix $Y \in \mathbf{G}(M)$ such that

$$Y^T KY = \mathrm{diag}(\hat{P}(r_1, m_1), \dots, \hat{P}(r_s, m_s)),$$

where $\hat{P}(r_k, m_k) = \mathrm{diag}(\underbrace{\hat{P}_{r_k}, \dots, \hat{P}_{r_k}}_{m_k})$.

Furthermore all results still hold if M is replaced by a more general matrix $N = \mathrm{diag}(N_{r_1}, \dots, N_{r_s})$.

Proof. We only prove i). The proofs of the other cases are similar. Since $KM + M^H K = 0$, by Proposition 3 we have $KM = P_M M P_M^{-1} K$, or equivalently $(P_M^{-1} K)M = M(P_M^{-1}K)$. This implies that $P_M^{-1}K$ commutes with M, and hence by Lemma 3 there exists a matrix $E \in \mathbf{G}(M)$ such that $K = P_M E$. Since K is nonsingular, so is E. Applying Lemma 4 and using that $K = K^H$, there exists a nonsingular matrix $Y_1 \in \mathbf{G}(M)$ such that

$$Y_1^H KY_1 = P_M \,\mathrm{diag}(E_1, \dots, E_s) = \mathrm{diag}(P(r_1, m_1)E_1, \dots, P(r_s, m_s)E_s),$$

where $E_k \in \mathbf{G}(N(r_k, m_k))$. Moreover for all $k = 1, \dots, s$ the matrices $P(r_k, m_k)E_k$ are nonsingular Hermitian. Finally applying Lemma 5 for each $P(r_k, m_k)E_k$ finishes the proof in the complex case. The real case directly follows from Lemma 4 and 5.

For the last statement we recognize that any matrix N can be transformed to the form of M via an appropriate permutation. □

Remark 1 By Lemma 4 and 5 we see that the parameters π_{ij} that occur in the different cases of Theorem 6 are only related to the matrix K and the nilpotent matrix M. They are invariant under the transformations with matrices in $\mathbf{G}(M)$.

Remark 2 The structured canonical forms can be obtained in an analogous way for all problems associated with nonsingular matrices K satisfying one of the following relations,

1) $KM + M^H K = 0$, $K = K^H$,

2) $KM = M^H K$, $K = K^H$,

3) $KM + M^H K = 0$, $K = -K^H$,

4) $KM = M^H K$, $K = -K^H$,

5) $KM + M^H K = 0$, $K = K^T$,

6) $KM = M^H K$, $K = K^T$,

7) $KM + M^H K = 0$, $K = -K^T$,

8) $KM = M^H K$, $K = -K^T$.

Here M is as in (4).

We have now finished the preliminary considerations and come to the structured Jordan forms.

3 $\Sigma_{p,q}$-Hermitian matrices

In this section we derive structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian matrices. We will always consider two cases, a structured canonical form where the transformation matrices are not necessarily $\Sigma_{p,q}$-unitary and a structured canonical form under $\Sigma_{p,q}$-unitary matrices. We first prove some Lemmas which then will be combined to prove the canonical forms.

Lemma 7 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix and let $\lambda \in \Lambda(\mathcal{C})$ be nonreal. Let $N(\lambda) = \lambda I + N$ with $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ be the Jordan structure associated with λ and let $\hat{P}_N$ be as in (6). Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(\overline{\lambda}) \end{bmatrix}.$$

Furthermore, $\overline{\lambda} \in \Lambda(\mathcal{C})$ has the same algebraic and geometric multiplicity as λ.

If $\mathcal{C}$ is real, then the matrix U can be chosen of the form $[V, \overline{V}]$.

Proof. Since $N(\lambda)$ is the Jordan structure associated with one eigenvalue λ, there exists a full rank matrix U_1, whose columns are composed by the chains of root vectors such that

$$\mathcal{C}U_1 = U_1 N(\lambda). \tag{15}$$

Substituting $\mathcal{C}^H = \Sigma_{p,q}\mathcal{C}\Sigma_{p,q}$ into the conjugate transpose of (15) we obtain

$$U_1^H \Sigma_{p,q} \mathcal{C} = N(\lambda)^H U_1^H \Sigma_{p,q}. \tag{16}$$

Now since λ is nonreal, we have $\overline{\lambda} \neq \lambda$ and it is clear that $\overline{\lambda} \in \Lambda(\mathcal{C})$. Using the relationship between (15) and (16) we obtain that λ and $\overline{\lambda}$ have the same algebraic and geometric multiplicity.

It follows that there exists a full rank matrix U_2 such that

$$\mathcal{C}U_2 = U_2 N(\overline{\lambda}), \tag{17}$$

i.e., the columns of U_2 span the right invariant subspace of $\mathcal{C}$ corresponding to $\overline{\lambda}$. Similarly as before we obtain

$$U_2^H \Sigma_{p,q} \mathcal{C} = N(\overline{\lambda})^H U_2^H \Sigma_{p,q}. \tag{18}$$

Using (15)-(18) we get that the columns of $U_1, \Sigma_{p,q}U_2$ and $U_2, \Sigma_{p,q}U_1$ form bases of the right and left invariant subspaces of $\mathcal{C}$ corresponding to λ and $\overline{\lambda}$, respectively. Since $\lambda \neq \overline{\lambda}$, by Proposition 2,

$$U_1^H \Sigma_{p,q} U_1 = 0, \quad U_2^H \Sigma_{p,q} U_2 = 0, \quad \det(U_1^H \Sigma_{p,q} U_2) \neq 0. \tag{19}$$

Premultiplying by $U_1^H \Sigma_{p,q}$ in (17) and postmultiplying by U_2 in (16) we obtain

$$(U_1^H \Sigma_{p,q} U_2) N(\overline{\lambda}) = N(\lambda)^H (U_1^H \Sigma_{p,q} U_2)$$

and thus $(U_1^H \Sigma_{p,q} U_2) N = N^H (U_1^H \Sigma_{p,q} U_2)$. Since by Proposition 3, $\hat{P}_N^H N^H \hat{P}_N = N$, then $\hat{P}_N (U_1^H \Sigma_{p,q} U_2)$ commutes with N, so does $(U_1^H \Sigma_{p,q} U_2)^{-1} \hat{P}_N$. Therefore we have

$$N(\overline{\lambda})(U_1^H \Sigma_{p,q} U_2)^{-1} \hat{P}_N = (U_1^H \Sigma_{p,q} U_2)^{-1} \hat{P}_N N(\overline{\lambda}). \tag{20}$$

Let $U = [U_1, U_2 (U_1^H \Sigma_{p,q} U_2)^{-1} \hat{P}_N]$. From (19) we have $U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}$ and from (15), (17), (20) we obtain $\mathcal{C} U = U \operatorname{diag}(N(\lambda), N(\overline{\lambda}))$.

If $\mathcal{C}$ is real, then (15) implies that

$$\mathcal{C} \overline{U_1} = \overline{U_1} N(\overline{\lambda}).$$

Replacing U_2 by $\overline{U_1}$ we still have (19) and

$$(U_1^H \Sigma_{p,q} \overline{U_1}) N(\overline{\lambda}) = N(\lambda)^H (U_1^H \Sigma_{p,q} \overline{U_1}),$$

or $(U_1^H \Sigma_{p,q} \overline{U_1}) N = N^H (U_1^H \Sigma_{p,q} \overline{U_1})$. Note that $U_1^H \Sigma_{p,q} \overline{U_1}$ is complex symmetric, and since it is also nonsingular by Theorem 6 iii), there exists a nonsingular matrix $Y \in \mathbf{G}(N)$ such that $Y^T (U_1^H \Sigma_{p,q} \overline{U_1}) Y = \hat{P}_N$. Now set $V = U_1 \overline{Y}$ and $U = [V, \overline{V}]$ then we also have

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \quad \mathcal{C} U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(\overline{\lambda}) \end{bmatrix}.$$

This finishes the proof. $\square$

The main difficulty in the construction of structured Jordan canonical forms arises from the real eigenvalues. For this case we will employ the results of Section 2.

Lemma 8 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix, let $\lambda \in \Lambda(\mathcal{C})$ be real and let $N(\lambda) = \lambda I + N$ with $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ be the Jordan structure associated with λ. Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \operatorname{diag}(\pi_1 \hat{P}_{r_1}, \ldots, \pi_s \hat{P}_{r_s}),$$

with $\pi_k \in \{\pm 1\}$ and $\mathcal{C} U = U N(\lambda)$. Moreover, if $\mathcal{C}$ is real then the matrix U can be chosen real as well.

Proof. Let U_1 be a matrix formed from the chains of root vectors corresponding to λ. Then from (15) and (16), since λ is real, U_1 and $\Sigma_{p,q} U_1$ are the bases of the right and left invariant subspaces of $\mathcal{C}$ corresponding to λ. By Proposition 2 we have $\det U_1^H \Sigma_{p,q} U_1 \neq 0$.

Premultiplying $U_1^H \Sigma_{p,q}$ in (15) and postmultiplying U_1 in (16) we get

$$U_1^H \Sigma_{p,q} \mathcal{C} U_1 = (U_1^H \Sigma_{p,q} U_1) N(\lambda) = N(\lambda)^H (U_1^H \Sigma_{p,q} U_1),$$

which implies that

$$(U_1^H \Sigma_{p,q} U_1) N = N^H (U_1^H \Sigma_{p,q} U_1).$$

Since $U_1^H \Sigma_{p,q} U_1$ is Hermitian and nonsingular, we can apply Theorem 6 ii) and obtain that there exists a nonsingular matrix $Y \in \mathbf{G}(N)$ such that

$$(U_1 Y)^H \Sigma_{p,q} (U_1 Y) = \operatorname{diag}(\pi_1 \hat{P}_{r_1}, \ldots, \pi_s \hat{P}_{r_s})$$

Set $U = U_1 Y$, then, since Y and N commute, Y and $N(\lambda)$ also commute and hence

$$\mathcal{C} U = \mathcal{C} U_1 Y = U_1 N(\lambda) Y = U_1 Y N(\lambda) = U N(\lambda).$$

If $\mathcal{C}$ is real, then since λ is real, also the initial U_1 can be chosen real and by Theorem 6, Y can be chosen real and hence U is real. □

It follows from Remark 1 that the parameters π_k in Lemma 8 are uniquely determined by the Jordan structure associated with the eigenvalue λ. For this reason we call the parameters π_k the *structure inertia indices* of $\mathcal{C}$ corresponding to the real eigenvalue λ, or simply the structure inertia indices of λ. We will denote the complete set of parameters by $\operatorname{Ind}(\lambda) = \{\pi_1, \ldots, \pi_s\}$. Note that for each Jordan block there is a corresponding structure inertia index.

Using Lemma 7 and 8 we now obtain the structured Jordan canonical form for $\Sigma_{p,q}$-Hermitian matrices under general similarity transformations.

Theorem 9 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$, with positive imaginary parts. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1} \mathcal{C} \mathcal{U} = \operatorname{diag}(R_c^+, R_c^-, R_r), \tag{21}$$

where the blocks are

$$\begin{aligned} R_c^+ &= \operatorname{diag}(H_1(\lambda_1), \ldots, H_\mu(\lambda_\mu)), \quad R_c^- = \operatorname{diag}(H_1(\overline{\lambda_1}), \ldots, H_\mu(\overline{\lambda_\mu})), \\ R_r &= \operatorname{diag}(M_1(\alpha_1), \ldots, M_\nu(\alpha_\nu)), \end{aligned}$$

with substructures

$$H_k(\lambda_k) = \lambda_k I + H_k, \quad H_k(\overline{\lambda_k}) = \overline{\lambda_k} I + H_k, \quad H_k = \operatorname{diag}(N_{p_{k,1}}, \ldots, N_{p_{k,s_k}}),$$

for $k = 1, \ldots, \mu$, and

$$M_k(\alpha_k) = \alpha_k I + M_k, \quad M_k = \operatorname{diag}(N_{q_{k,1}}, \ldots N_{q_{k,t_k}}),$$

for $k = 1, \ldots, \nu$.

The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_r \end{bmatrix}, \tag{22}$$

with $W_c = \operatorname{diag}(\hat{P}_{H_1}, \ldots, \hat{P}_{H_\mu})$ and $W_r = \operatorname{diag}(W_1^r, \ldots, W_\nu^r)$, where for $k = 1, \ldots, \mu$ we have $\hat{P}_{H_k} = \operatorname{diag}(\hat{P}_{p_{k,1}}, \ldots, \hat{P}_{p_{k,s_k}})$ and for $k = 1, \ldots, \nu$ and $\operatorname{Ind}(\alpha_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ we have $W_k^r = \operatorname{diag}(\pi_{k,1} \hat{P}_{q_{k,1}}, \ldots, \pi_{k,t_k} \hat{P}_{q_{k,t_k}})$.

Proof. For each nonreal eigenvalue λ_k with the corresponding Jordan structure $H_k(\lambda_k)$, by Lemma 7 we can choose a matrix U_k such that

$$U_k^H \Sigma_{p,q} U_k = \begin{bmatrix} 0 & \hat{P}_{H_k} \\ \hat{P}_{H_k}^H & 0 \end{bmatrix}, \quad \mathcal{C} U_k = U_k \operatorname{diag}(H_k(\lambda_k), H_k(\overline{\lambda_k})). \tag{23}$$

Partition $U_k = [U_{k,1}, U_{k,2}]$, where $U_{k,1}$, $U_{k,2}$ have the same size and set

$$\mathcal{U}_c = [U_{1,1}, \ldots, U_{\mu,1}; U_{1,2}, \ldots, U_{\mu,2}] = [\mathcal{U}_1^c; \mathcal{U}_2^c].$$

Note that the columns of $\mathcal{U}_1^c$, $\Sigma_{p,q}\mathcal{U}_2^c$ and $\mathcal{U}_2^c$, $\Sigma_{p,q}\mathcal{U}_1^c$ form bases of the right and left invariant subspaces corresponding to the two disjoint sets of eigenvalues $\{\lambda_1, \ldots, \lambda_\mu\}$ and $\{\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}\}$, respectively. By Proposition 2 and (23) we have

$$\mathcal{U}_c^H \Sigma_{p,q} \mathcal{U}_c = \begin{bmatrix} 0 & W_c \\ W_c^H & 0 \end{bmatrix}, \quad \mathcal{C}\mathcal{U} = \mathcal{U} \operatorname{diag}(R_c^+, R_c^-),$$

with W_c, R_c^+ and R_c^- as asserted.

For each real eigenvalue α_k with the corresponding Jordan structure $M_k(\alpha_k)$, by Lemma 8 we can choose a matrix V_k such that

$$V_k^H \Sigma_{p,q} V_k = \operatorname{diag}(\pi_{k1} \hat{P}_{q_{k,1}}, \ldots, \pi_{k,t_k} \hat{P}_{q_{k,t_k}}),$$

where $\operatorname{Ind}(\alpha_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ and $\mathcal{C} V_k = V_k M_k(\lambda_k)$. Set $\mathcal{U}_r = [V_1, \ldots, V_\nu]$, then by Proposition 2 we have

$$\mathcal{U}_r^H \Sigma_{p,q} \mathcal{U}_r = W_r, \quad \mathcal{C}\mathcal{U}_r = \mathcal{U}_r R_r,$$

where W_r and R_r are of the asserted form and with $\mathcal{U} = [\mathcal{U}_c, \mathcal{U}_r]$ the result follows from Proposition 2. □

The canonical form in Theorem 9 is just the classical Jordan canonical form, but the transformation matrix is constructed in such a way that it satisfies the relationship (22) associated with $\Sigma_{p,q}$. This is not quite what we want, since we would rather like to have that the transformation matrix is $\Sigma_{p,q}$-unitary. In order to obtain this we have to look at the structure in more detail. For this we need to use transformations with

$$\Upsilon_r = \frac{\sqrt{2}}{2} \begin{bmatrix} I_r & -I_r \\ I_r & I_r \end{bmatrix}, \tag{24}$$

for which we have

$$\Upsilon_r^H \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} \Upsilon_r = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}.$$

We also need in the following the symmetric and skew symmetric part of Jordan blocks

$$N_r^+ = \frac{1}{2}(N_r + N_r^T), \quad N_r^- = \frac{1}{2}(N_r - N_r^T).$$

and

$$N_r^+(\lambda) = \lambda I_r + N_r^+, \quad N_r^-(\lambda) = \lambda I_r + N_r^-.$$

By Lemma 7 and 8 we only need to analyse the structure of the left and right chains of root vectors corresponding to a pair of Jordan blocks $N_r(\lambda)$, $N_r(\overline{\lambda})$ for a nonreal λ and to a Jordan block $N_r(\alpha)$ for a real α. We first consider nonreal eigenvalues.

Lemma 10 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix and let $\lambda \in \Lambda(\mathcal{C})$ be nonreal. If $N_r(\lambda) = \lambda I + N_r$ is a Jordan block of $\mathcal{C}$, then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_r^+(\operatorname{Re}\lambda) & -N_r^-(i\operatorname{Im}\lambda) \\ -N_r^-(i\operatorname{Im}\lambda) & N_r^+(\operatorname{Re}\lambda) \end{bmatrix}. \tag{25}$$

Proof. By Lemma 7 for the Jordan block $N_r(\lambda)$, there exists a full rank matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & \hat{P}_r \\ \hat{P}_r & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \operatorname{diag}(N_r(\lambda), N_r(\overline{\lambda})).$$

Set

$$Z_r = \operatorname{diag}(I_r, \hat{P}_r)\Upsilon_r \tag{26}$$

and let $U = \hat{U} Z_r$. Using Proposition 3 we can easily verify that U satisfies (25). $\square$

The next lemma analyses the Jordan blocks associated with a real eigenvalue.

Lemma 11 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix, let $\alpha \in \Lambda(\mathcal{C})$ be real. Let $N_r(\alpha) = \alpha I + N_r$ be a Jordan block and let $\pi \in \operatorname{Ind}(\alpha)$ be the structure inertia index associated with this Jordan block. Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^+(\alpha) + \frac{1}{2}\pi e_s e_s^H & -N_s^- + \frac{1}{2}\pi e_s e_s^H \\ -N_s^- - \frac{1}{2}\pi e_s e_s^H & N_p^+(\alpha) - \frac{1}{2}\pi e_s e_s^H \end{bmatrix} \tag{27}$$

if $r = 2s$ and

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^+(\alpha) & \frac{\sqrt{2}}{2} e_s & -N_s^- \\ \frac{\sqrt{2}}{2}\pi e_s^H & \alpha & \frac{\sqrt{2}}{2}\pi e_s^H \\ -N_s^- & -\frac{\sqrt{2}}{2} e_s & N_s^+(\alpha) \end{bmatrix} \tag{28}$$

if $r = 2s+1$.

Moreover, if $\mathcal{C}$ is real, then U can be chosen real.

Proof. By Lemma 8, for the Jordan block $N_r(\alpha)$ there exists a matrix $\hat{U}$ such that $\hat{U}^H \Sigma_{p,q} \hat{U} = \pi \hat{P}_r$ and $\mathcal{C}\hat{U} = \hat{U} N_r(\alpha)$.

If $r = 2s$, then we form the partition

$$\pi \hat{P}_r = \begin{bmatrix} 0 & \pi \hat{P}_s \\ (\pi \hat{P}_s)^H & 0 \end{bmatrix}, \qquad N_r(\alpha) = \begin{bmatrix} N_s(\alpha) & e_s e_1^H \\ 0 & N_s(\alpha) \end{bmatrix}.$$

With $\hat{Z}_r = \operatorname{diag}(I_s, \pi \hat{P}_s)\Upsilon_s$ and $U = \hat{U}\hat{Z}_r$, then a simple calculation yields (27).

If $r = 2s+1$, then we form the partition

$$\pi \hat{P}_r = \begin{bmatrix} 0 & 0 & \pi \hat{P}_s \\ 0 & \pi & 0 \\ (\pi \hat{P}_s)^H & 0 & 0 \end{bmatrix}, \qquad N_r(\alpha) = \begin{bmatrix} N_s(\alpha) & e_s & 0 \\ 0 & \alpha & e_1^H \\ 0 & 0 & N_s(\alpha) \end{bmatrix}.$$

With

$$\hat{Z}_r = \begin{bmatrix} I_{s+1} & 0 \\ 0 & \pi \hat{P}_s \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} I_s & 0 & -\frac{\sqrt{2}}{2} I_s \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} I_s & 0 & \frac{\sqrt{2}}{2} I_s \end{bmatrix}$$

and $U = \hat{U}\hat{Z}_r$, then a simple calculation yields (28).

If $\mathcal{C}$ is real, by Lemma 8 $\hat{U}$ can be chosen real and since in both cases $\hat{Z}_r$ is real, U can be chosen real. $\square$

Combining Lemmas 10 and 11 we obtain the structured Jordan form of $\Sigma_{p,q}$-Hermitian matrices under $\Sigma_{p,q}$-unitary similarity transformations.

Theorem 12 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-Hermitian matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$, with positive imaginary parts. Then there exists a $\Sigma_{p,q}$-unitary matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_r^+ & & T_r \\ -T_c^H & & R_c & \\ & -T_r^H & & R_r^- \end{bmatrix}. \tag{29}$$

For the blocks we have the following substructures.

i) The blocks with index c, associated with the nonreal eigenvalues, are $R_c = \operatorname{diag}(R_1^c, \ldots, R_\mu^c)$ and $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$, where for $k = 1, \ldots, \mu$ we have

$$\begin{aligned} R_k^c &= \operatorname{diag}(N_{p_{k,1}}^+(\operatorname{Re}\lambda_k), \ldots, N_{p_{k,s_k}}^+(\operatorname{Re}\lambda_k)), \\ T_k^c &= -\operatorname{diag}(N_{p_{k,1}}^-(i\operatorname{Im}\lambda_k), \ldots, N_{p_{k,s_k}}^-(i\operatorname{Im}\lambda_k)). \end{aligned}$$

ii) The blocks with index r, associated with the real eigenvalues are

$$R_r^+ = \operatorname{diag}(C_1, \ldots, C_\nu), \quad R_r^- = \operatorname{diag}(D_1, \ldots, D_\nu), \quad T_r = \operatorname{diag}(F_1, \ldots, F_\nu).$$

For $k = 1, \ldots, \nu$ these have the substructures

$$C_k = \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-), \quad F_k = \operatorname{diag}(F_k^e, F_k^+, F_k^-),$$

where

$$\begin{aligned} C_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\alpha_k) + \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H, \ldots, N_{q_{k,t_k}}^+(\alpha_k) + \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H), \\ D_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\alpha_k) - \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H, \ldots, N_{q_{k,t_k}}^+(\alpha_k) - \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H), \\ F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^- + \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H, \ldots, -N_{q_{k,t_k}}^- + \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H), \\ C_k^+ &= \operatorname{diag}(\begin{bmatrix} N_{u_{k,1}}^+(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,1}} \\ \frac{\sqrt{2}}{2}e_{u_{k,1}}^H & \alpha_k \end{bmatrix}, \ldots, \begin{bmatrix} N_{u_{k,w_k}}^+(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,w_k}} \\ \frac{\sqrt{2}}{2}e_{u_{k,w_k}}^H & \alpha_k \end{bmatrix}), \\ D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^+(\alpha_k), \ldots, N_{u_{k,w_k}}^+(\alpha_k)), \\ F_k^+ &= \operatorname{diag}(\begin{bmatrix} -N_{u_{k,1}}^- \\ \frac{\sqrt{2}}{2}e_{u_{k,1}} \end{bmatrix}, \ldots, \begin{bmatrix} -N_{u_{k,w_k}}^- \\ \frac{\sqrt{2}}{2}e_{u_{k,w_k}} \end{bmatrix}); \\ C_k^- &= \operatorname{diag}(N_{v_{k,1}}^+(\alpha_k), \ldots, N_{v_{k,z_k}}^+(\alpha_k)), \\ D_k^- &= \operatorname{diag}(\begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e_{v_{k,1}}^H \\ -\frac{\sqrt{2}}{2}e_{v_{k,1}} & N_{v_{k,1}}^+(\alpha_k) \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e_{v_{k,z_k}}^H \\ -\frac{\sqrt{2}}{2}e_{v_{k,z_k}} & N_{v_{k,z_k}}^+(\alpha_k) \end{bmatrix}), \\ F_k^- &= \operatorname{diag}([\frac{\sqrt{2}}{2}e_{v_{k,1}}, -N_{v_{k,1}}^-], \ldots, [\frac{\sqrt{2}}{2}e_{v_{k,z_k}}, -N_{v_{k,z_k}}^-]). \end{aligned}$$

Here each nonreal λ_k $(\overline{\lambda_k})$ *has* s_k *Jordan blocks of sizes* $p_{k,1}, \ldots, p_{k,s_k}$ *and each real eigenvalue* α_k *has*

a) t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *and the corresponding structure inertia indices* $\pi_{k,1}, \ldots, \pi_{k,t_k}$*;*

b) w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$*, corresponding to the structure inertia index* 1*;*

c) z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$*, corresponding to the structure inertia index* -1*.*

Proof. Let λ_k be a nonreal eigenvalue with associated Jordan structure $\lambda_k I + H_k$, where $H_k = \operatorname{diag}(N_{p_{k,1}}, \ldots, N_{p_{k,s_k}})$. By Lemma 7 we can determine a matrix $\hat{\mathcal{U}}$ such that

$$\mathcal{C}\hat{U}_k = \hat{U}_k \operatorname{diag}(N_{p_{k,1}}(\lambda_k), N_{p_{k,1}}(\overline{\lambda_k}), \ldots, N_{p_{k,s_k}}(\lambda_k), N_{p_{k,s_k}}(\overline{\lambda_k}))$$

and

$$\hat{U}_k^H \Sigma_{p,q} \hat{U}_k = \operatorname{diag}(\begin{bmatrix} 0 & \hat{P}_{p_{k,1}} \\ \hat{P}_{p_{k,1}}^H & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \hat{P}_{p_{k,s_k}} \\ \hat{P}_{p_{k,s_k}}^H & 0 \end{bmatrix}).$$

In fact this form is obtained by an appropriate permutation applied simultaneously to the equations in Lemma 7. Partition $\hat{U}_k = [\hat{U}_{k,1}, \ldots, \hat{U}_{k,s_k}]$, where for $j = 1, \ldots, s_k$ the part $\hat{U}_{k,j}$ has $2p_{k,j}$ columns. Applying Lemma 10 to each diagonal block composed from the pair $N_{p_{k,j}}(\lambda_k)$, $N_{p_{k,j}}(\overline{\lambda_k})$ we obtain that for each $U_{k,j} = \hat{U}_{k,j} Z_{p_{k,j}}$, with $Z_{p_{k,j}}$ as in (26), we have

$$U_{k,j}^H \Sigma_{p,q} U_{k,j} = \begin{bmatrix} I_{p_{k,j}} & 0 \\ 0 & -I_{p_{k,j}} \end{bmatrix}, \quad \mathcal{C} U_{k,j} = U_{k,j} \begin{bmatrix} N_{p_{k,j}}^+(\operatorname{Re}\lambda_k) & -N_{p_{k,j}}^-(i \operatorname{Im}\lambda_k) \\ -N_{p_{k,j}}^-(i \operatorname{Im}\lambda_k) & N_{p_{k,j}}^+(\operatorname{Re}\lambda_k) \end{bmatrix}.$$

Partition $U_{k,j} = [V_{k,j}, W_{k,j}]$, with $V_{k,j}$ and $W_{k,j}$ of the same dimensions and set $U_k = [V_{k,1}, \ldots, V_{k,s_k} | W_{k,1}, \ldots, W_{k,s_k}]$, then a simple calculation yields

$$U_k^H \Sigma_{p,q} U_k = \begin{bmatrix} I_{m_k} & 0 \\ 0 & -I_{m_k} \end{bmatrix}, \quad \mathcal{C} U_k = U_k \begin{bmatrix} R_k^c & T_k^c \\ -(T_k^c)^H & R_k^c \end{bmatrix},$$

where $m_k = \sum_{j=1}^{s_k} p_{k,j}$. Let $m = \sum_{k=1}^{\mu} m_k$ and partition $U_k = [V_k, W_k]$, where V_k and W_k have the same dimensions. Let

$$V_c = [V_1, \ldots, V_\mu], \qquad W_c = [W_1, \ldots, W_\mu], \qquad \mathcal{U}_c = [V_c, W_c].$$

Then by the invariant subspace property in Proposition 2 we obtain

$$\mathcal{U}_c^H \Sigma_{p,q} \mathcal{U}_c = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}, \quad \mathcal{C}\mathcal{U}_c = \mathcal{U}_c \begin{bmatrix} R_c & T_c \\ -T_c^H & R_c \end{bmatrix}. \tag{30}$$

Similarly let α_k be a real eigenvalue with associated even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ and associated odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ and $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$

corresponding to the structure inertia indices 1 and -1, respectively. By Lemma 8 and Lemma 11 for each even block there is a matrix $\hat{U}^e_{k,j}$ such that

$$\begin{aligned}
(\hat{U}^e_{k,j})^H \Sigma_{p,q} \hat{U}^e_{k,j} &= \begin{bmatrix} I_{q_{k,j}} & 0 \\ 0 & -I_{q_{k,j}} \end{bmatrix}, \\
\mathcal{C}\hat{U}^e_{k,j} &= \hat{U}^e_{k,j} \begin{bmatrix} N^+_{q_{k,j}}(\alpha_k) + \frac{1}{2}\pi_{k,j} e_{q_{k,j}} e^H_{q_{k,j}} & -N^-_{q_{k,j}} + \frac{1}{2}\pi_{k,j} e_{q_{k,j}} e^H_{q_{k,j}} \\ -N^-_{q_{k,j}} - \frac{1}{2}\pi_{k,j} e_{q_{k,j}} e^H_{q_{k,j}} & N^+_{q_{k,j}}(\alpha_k) - \frac{1}{2}\pi_{k,j} e_{q_{k,j}} e^H_{q_{k,j}} \end{bmatrix}.
\end{aligned}$$

For each odd sized Jordan block we have two cases. If the structure inertia index is 1 then there exists a matrix $\hat{U}^+_{k,j}$ such that

$$(\hat{U}^+_{k,j})^H \Sigma_{p,q} \hat{U}^+_{k,j} = \begin{bmatrix} I_{u_{k,j}+1} & 0 \\ 0 & -I_{u_{k,j}} \end{bmatrix}, \quad \mathcal{C}\hat{U}^+_{k,j} = \hat{U}^+_{k,j} \left[\begin{array}{cc|c} N^+_{u_{k,j}}(\alpha_k) & \frac{\sqrt{2}}{2} e_{u_{k,j}} & -N^-_{u_{k,j}} \\ \frac{\sqrt{2}}{2} e^H_{u_{k,j}} & \alpha_k & \frac{\sqrt{2}}{2} e^H_{u_{k,j}} \\ \hline -N^-_{u_{k,j}} & -\frac{\sqrt{2}}{2} e_{u_{k,j}} & N^+_{u_{k,j}}(\alpha_k) \end{array}\right].$$

If the structure inertia index is -1, then there exists a matrix $\hat{U}^-_{k,j}$ such that

$$(\hat{U}^-_{k,j})^H \Sigma_{p,q} \hat{U}^-_{k,j} = \begin{bmatrix} I_{v_{k,j}} & 0 \\ 0 & -I_{v_{k,j}+1} \end{bmatrix}, \quad \mathcal{C}\hat{U}^-_{k,j} = \hat{U}^-_{k,j} \left[\begin{array}{c|cc} N^+_{v_{k,j}}(\alpha_k) & \frac{\sqrt{2}}{2} e_{v_{k,j}} & -N^-_{v_{k,j}} \\ \hline -\frac{\sqrt{2}}{2} e^H_{v_{k,j}} & \alpha_k & -\frac{\sqrt{2}}{2} e^H_{v_{k,j}} \\ -N^-_{v_{k,j}} & -\frac{\sqrt{2}}{2} e_{v_{k,j}} & N^+_{v_{k,j}}(\alpha_k) \end{array}\right].$$

Partition $\hat{U}^e_{k,j} = [V^e_{k,j}, W^e_{k,j}]$ with $V^e_{k,j}, W^e_{k,j}$ having the same number of columns, partition $\hat{U}^+_{k,j} = [V^+_{k,j}, W^+_{k,j}]$, where $V^+_{k,j}$ has $u_{k,j}+1$ columns and $W^+_{k,j}$ has $u_{k,j}$ columns and partition $\hat{U}^-_{k,j} = [V^-_{k,j}, W^-_{k,j}]$, where $V^-_{k,j}$ has $v_{k,j}$ columns and $W^-_{k,j}$ has $v_{k,j}+1$ columns. Set

$$\begin{aligned}
V_k &= [V^e_{k,1}, \ldots, V^e_{k,t_k} | V^+_{k,1}, \ldots, V^+_{k,w_k} | V^-_{k,1}, \ldots, V^-_{k,z_k}], \\
W_k &= [W^e_{k,1}, \ldots, W^e_{k,t_k} | W^+_{k,1}, \ldots, W^+_{k,w_k} | W^-_{k,1}, \ldots, W^-_{k,z_k}]
\end{aligned}$$

and $U_k = [V_k, W_k]$. Then a simple calculation yields

$$U_k^H \Sigma_{p,q} U_k = \begin{bmatrix} I_{n_{k,1}} & 0 \\ 0 & -I_{n_{k,2}} \end{bmatrix}, \quad \mathcal{C} U_k = U_k \begin{bmatrix} C_k & F_k \\ -F_k^H & D_k \end{bmatrix},$$

where $n_{k,1} = w_k + \sum_{l=1}^{t_k} q_{k,l} + \sum_{l=1}^{w_k} u_{k,l} + \sum_{l=1}^{z_k} v_{k,l}$ and $n_{k,2} = z_k + \sum_{l=1}^{t_k} q_{k,l} + \sum_{l=1}^{w_k} u_{k,l} + \sum_{l=1}^{z_k} v_{k,l}$. Set $n_1 = \sum_{k=1}^{\nu} n_{k,1}$, $n_2 = \sum_{k=1}^{\nu} n_{k,2}$. Then with

$$V_r = [V_1, \ldots, V_\nu], \qquad W_r = [W_1, \ldots, W_\nu], \qquad \mathcal{U}_r = [V_r, W_r]$$

we have

$$\mathcal{U}_r^H \Sigma_{p,q} \mathcal{U}_r = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}, \quad \mathcal{C}\mathcal{U}_r = \mathcal{U}_r \begin{bmatrix} R_r^+ & T_r \\ -T_r^H & R_r^- \end{bmatrix}, \tag{31}$$

Finally set $\mathcal{U} = [V_c, V_r | W_c, W_r]$, then by Proposition 2 and by above construction we have

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} I_{m+n_1} & 0 \\ 0 & -I_{m+n_2} \end{bmatrix}.$$

Since $\mathcal{U}$ is nonsingular it follows that $\mathcal{U}^H \Sigma_{p,q} \mathcal{U}$ is congruent to $\Sigma_{p,q}$ and hence $m + n_1 = p$, $m + n_2 = q$ and $\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \Sigma_{p,q}$. Equation (29) then follows from (30) and (31). $\square$

Remark 3 The difference between the structured canonical form of Theorems 9 and 12 is that in order to get a $\Sigma_{p,q}$-unitary transformation matrix we need to refine further and combine different blocks together. This leads to a loss in structure in the Jordan canonical form, which becomes more complicated, but shows that the classical Jordan canonical form somehow obscures the extra structure in the chains of root vectors.

Remark 4 By the structured Jordan from we immediately obtain the following relationships

$$\begin{aligned} p &= \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} p_{k,j} + \sum_{k=1}^{\nu}(w_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}), \\ q &= \sum_{k=1}^{\mu}\sum_{j=1}^{s_k} p_{k,j} + \sum_{k=1}^{\nu}(z_k + \sum_{j=1}^{t_k} q_{k,j} + \sum_{j=1}^{w_k} u_{k,j} + \sum_{j=1}^{z_k} v_{k,j}), \\ |p-q| &= |\sum_{k=1}^{\nu}(w_k - z_k)|. \end{aligned} \tag{32}$$

These relationship show that the parameters p, q will affect the eigenstructure of $\mathcal{C}$. For example, we get in the case $q = 0$ that $\mathcal{C}$ is unitarily similar to a real diagonal matrix. Another direct consequence is that for a real eigenvalue the largest size of the associated Jordan block is not larger than $2\min\{p,q\}+1$, and for a nonreal eigenvalue the largest size of the associated Jordan block is not larger than $\min\{p,q\}$. Furthermore, it is clear that if $|p-q| \neq 0$, then $\mathcal{C}$ must have real eigenvalues with at least $|p-q|$ odd sized Jordan blocks.

In the case of a real matrix we can obtain real structured canonical forms under real $\Sigma_{p,q}$-orthogonal similarity transformations combining blocks associated with pairs of complex conjugate eigenvalues. Note that for real eigenvalues based on Lemma 8 and 11 we have already real canonical forms. Hence we only need to consider the Jordan structure associated with nonreal eigenvalues. Parallel to the complex case we only need to set up the real forms as in Lemma 7 and 10. Before we do this we need some further notation.

Let

$$\Psi_{2r} = [e_1, e_{r+1}, e_2, e_{r+2}, \ldots, e_r, e_{2r}], \quad \Phi_{2r} = \mathrm{diag}(\underbrace{\Phi_2, \Phi_2, \ldots, \Phi_2}_{r}), \tag{33}$$

where

$$\Phi_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}.$$

Then the following properties hold, see [14].

Lemma 13 *If $A = [a_{i,j}] \in \mathbf{C}^{r\times r}$, then*

$$(\Psi_{2r}\Phi_{2r})^H \begin{bmatrix} A & 0 \\ 0 & \overline{A} \end{bmatrix} (\Psi_{2r}\Phi_{2r}) = B =: [B_{ij}],$$

where for $i, j = 1, \ldots, r$ the blocks are

$$B_{ij} = \begin{bmatrix} \mathrm{Re}\, a_{ij} & \mathrm{Im}\, a_{ij} \\ -\mathrm{Im}\, a_{ij} & \mathrm{Re}\, a_{ij} \end{bmatrix}$$

and if $U \in \mathbf{C}^{n\times r}$, then $[U, \overline{U}]\Psi_{2r}\Phi_{2r}$ is a real matrix.

For a nonreal eigenvalue $\lambda \in \Lambda(\mathcal{C})$ set $\Lambda = \begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix}$ and for a real 2×2 matrix A we set

$$N_r(A) = I_r \otimes A + N_r \otimes I_2 = \begin{bmatrix} A & I_2 & & \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ & & & A \end{bmatrix}.$$

In particular we denote by $N_r(\Lambda)$ the real Jordan block of $\mathcal{C}$ corresponding to the eigenvalues λ and $\overline{\lambda}$. Similarly for $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ we set

$$N(\Lambda) = \operatorname{diag}(N_{r_1}(\Lambda), \ldots, N_{r_s}(\Lambda))$$

and with the 2×2 zero matrix 0_2 we set

$$N_r^+(0_2) = \frac{1}{2}(N_r(0_2) + N_r(0_2)^T), \quad N_r^-(0_2) = \frac{1}{2}(N_r(0_2) - N_r(0_2)^T),$$

and analogously

$$N_r^+(A) = I_r \otimes A + N_r^+(0_2), \quad N_r^-(A) = I_r \otimes A + N_r^-(0_2)$$

for a real 2×2 matrix A.

Lemma 14 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix and let $\lambda \in \Lambda(\mathcal{C})$ be nonreal. If $N(\lambda) = \lambda I + N$, where $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$, is the Jordan structure corresponding to the eigenvalue λ, then there exists a real full rank matrix U such that*

$$\begin{aligned} U^T \Sigma_{p,q} U &= \hat{P}_N \otimes \Sigma_{1,1} = \operatorname{diag}(\hat{P}_{r_1} \otimes \Sigma_{1,1}, \ldots, \hat{P}_{r_s} \otimes \Sigma_{1,1}), \\ \mathcal{C}U &= UN(\Lambda). \end{aligned} \tag{34}$$

Proof. By Lemma 7, if $\mathcal{C}$ is real, then there exists a matrix $\hat{U} = [V, \overline{V}]$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(\overline{\lambda}) \end{bmatrix}.$$

Partition $V = [V_1, \ldots, V_s]$, where for $k = 1, \ldots, s$, V_k has r_k columns. Let $\tilde{V}_k = [V_k, \overline{V_k}]$ and set $\tilde{U} = [\tilde{V}_1, \ldots, \tilde{V}_s]$, then

$$\begin{aligned} \tilde{U}^H \Sigma_{p,q} \tilde{U} &= \operatorname{diag}(\begin{bmatrix} 0 & \hat{P}_{r_1} \\ \hat{P}_{r_1}^H & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & \hat{P}_{r_s} \\ \hat{P}_{r_s}^H & 0 \end{bmatrix}), \\ \mathcal{C}\tilde{U} &= \tilde{U} \operatorname{diag}(\begin{bmatrix} N_{r_1}(\lambda) & 0 \\ 0 & N_{r_1}(\overline{\lambda}) \end{bmatrix}, \ldots, \begin{bmatrix} N_{r_s}(\lambda) & 0 \\ 0 & N_{r_s}(\overline{\lambda}) \end{bmatrix}). \end{aligned}$$

Setting $U = \tilde{U} \operatorname{diag}(\Psi_{2r_1}\Phi_{2r_1}, \ldots, \Psi_{2r_s}\Phi_{2r_s})$ it follows by Lemma 13 that U is real and satisfies (34). $\square$

The nonreal eigenvalues of a complex $\Sigma_{p,q}$-Hermitian matrix are already coupled in conjugate pairs. But in the real case Lemma 14 shows that the root vectors have additional structure.

Lemma 15 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix and let $\lambda \in \Lambda(\mathcal{C})$ be nonreal. Let $\Lambda = \begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix}$ and let $N_r(\Lambda)$ be a real Jordan block of $\mathcal{C}$. Then there exists a real full rank matrix U such that*

$$U^T\Sigma_{p,q}U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U\begin{bmatrix} A_1 & B \\ -B^T & A_2 \end{bmatrix}, \tag{35}$$

where we have the following two cases.

i) If $r = 2s$, then

$$A_1 = N_s^+((\operatorname{Re}\lambda)I_2) + E_r, \quad A_2 = N_s^+((\operatorname{Re}\lambda)I_2) - E_r, \quad B = -N_s^-((\operatorname{Im}\lambda)J_1) + E_r,$$

with $E_r = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{1,1} \end{bmatrix}$.

ii) If $r = 2s+1$, then

$$A_1 = \begin{bmatrix} N_s^+((\operatorname{Re}\lambda)I_2) & \frac{\sqrt{2}}{2}e_{r-2} \\ \frac{\sqrt{2}}{2}e_{r-2}^T & \operatorname{Re}\lambda \end{bmatrix}, \quad A_2 = \begin{bmatrix} N_s^+((\operatorname{Re}\lambda)I_2) & \frac{\sqrt{2}}{2}e_{r-1} \\ \frac{\sqrt{2}}{2}e_{r-1}^T & \operatorname{Re}\lambda \end{bmatrix},$$

$$B = \begin{bmatrix} -N_s^-((\operatorname{Im}\lambda)J_1) & -\frac{\sqrt{2}}{2}e_{r-1} \\ \frac{\sqrt{2}}{2}e_{r-2}^T & -\operatorname{Im}\lambda \end{bmatrix},$$

where $J_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Proof. By Lemma 14 for a real Jordan block there exists a real matrix $\hat{U}$ such that

$$\hat{U}^T\Sigma_{p,q}\hat{U} = \hat{P}_r \otimes \Sigma_{1,1}, \quad \mathcal{C}\hat{U} = \hat{U}N_r(\Lambda).$$

i) If $r = 2s$, then we partition

$$\hat{U}^T\Sigma_{p,q}\hat{U} = \begin{bmatrix} 0 & \hat{P}_s \otimes \Sigma_{1,1} \\ \hat{P}_s \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \quad N_r(\Lambda) = \left[\begin{array}{c|c} N_s(\Lambda) & \begin{matrix} 0 & 0 \\ I_2 & 0 \end{matrix} \\ \hline 0 & N_s(\Lambda) \end{array}\right]$$

and we can easily verify that

$$(\hat{P}_s \otimes \Sigma_{1,1})^{-1} = \hat{P}_s \otimes \Sigma_{1,1} = (\hat{P}_s \otimes \Sigma_{1,1})^T, \quad (\hat{P}_s \otimes \Sigma_{1,1})^T(N_s(\Lambda))(\hat{P}_s \otimes \Sigma_{1,1}) = (N_s(\Lambda))^T.$$

Let $U = \hat{U}\operatorname{diag}(I_r, \hat{P}_s \otimes \Sigma_{1,1})\Upsilon_r$, where Υ_r is defined in (24). Then U is real and we have (35).

ii) If $r = 2s+1$, then we partition

$$\hat{U}^T\Sigma_{p,q}\hat{U} = \begin{bmatrix} 0 & 0 & \hat{P}_s \otimes \Sigma_{1,1} \\ 0 & \Sigma_{1,1} & 0 \\ \hat{P}_s \otimes \Sigma_{1,1} & 0 & 0 \end{bmatrix}, \quad N_r(\Lambda) = \left[\begin{array}{c|c|c} N_s(\Lambda) & \begin{matrix} 0 \\ I_2 \end{matrix} & 0 \\ \hline 0 & \Lambda & I_2 \quad 0 \\ \hline 0 & 0 & N_s(\Lambda) \end{array}\right].$$

Setting $U = \hat{U}\operatorname{diag}(I_{r+1}, \hat{P}_s \otimes \Sigma_{1,1}) \begin{bmatrix} \frac{\sqrt{2}}{2}I_{r-1} & 0 & -\frac{\sqrt{2}}{2}I_{r-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ \frac{\sqrt{2}}{2}I_{r-1} & 0 & \frac{\sqrt{2}}{2}I_{r-1} & 0 \end{bmatrix}$, it follows that U is real and we obtain (35). □

Combining Lemmas 8, 11, 14 and15, we obtain the real versions of Theorems 9 and 12.

Theorem 16 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$, with positive imaginary parts.*

Then there exists a real full rank matrix $\mathcal{U}$ such that

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c, R_r),$$

where

$$R_c = \operatorname{diag}(H_1(\Lambda_1), \ldots, H_\mu(\Lambda_\mu))$$

and for $k = 1, \ldots, \mu$ the subblocks are $H_k(\Lambda_k) = \operatorname{diag}(N_{p_{k,1}}(\Lambda_k), \ldots, N_{p_{k,s_k}}(\Lambda_k))$ with $\Lambda_k = \begin{bmatrix} \operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k \\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k \end{bmatrix}$. The other diagonal block is

$$R_r = \operatorname{diag}(M_1(\alpha_1), \ldots, M_\nu(\alpha_\nu)),$$

where for $k = 1, \ldots, \nu$ the subblocks are $M_k(\alpha_k) = \alpha_k I + M_k$ with $M_k = \operatorname{diag}(N_{q_{k,1}}, \ldots N_{q_{k,t_k}})$. The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^T\Sigma_{p,q}\mathcal{U} = \begin{bmatrix} W_c & 0 \\ 0 & W_r \end{bmatrix},$$

where $W_c = \operatorname{diag}(\hat{P}_{H_1} \otimes \Sigma_{1,1}, \ldots, \hat{P}_{H_\mu} \otimes \Sigma_{1,1})$, $W_r = \operatorname{diag}(W_1^r, \ldots, W_\nu^r)$, and where for $k = 1, \ldots, \mu$ we have $\hat{P}_{H_k} = \operatorname{diag}(\hat{P}_{p_{k,1}}, \ldots, \hat{P}_{p_{k,s_k}})$ and for $\operatorname{Ind}(\alpha_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ and $k = 1, \ldots, \nu$ we have $W_k^r = \operatorname{diag}(\pi_{k,1}\hat{P}_{q_{k,1}}, \ldots, \pi_{k,t_k}\hat{P}_{q_{k,t_k}})$.

Proof. The proof is similar to the proof of Theorem 9. □

The real structured Jordan canonical form under real $\Sigma_{p,q}$-orthogonal matrices is also obtained analogously.

Theorem 17 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-symmetric matrix with pairwise different real eigenvalues $\alpha_1, \ldots, \alpha_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive imaginary parts. Then there exists a real $\Sigma_{p,q}$-orthogonal matrix $\mathcal{U}$, such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c^+ & & T_c & \\ & R_r^+ & & T_r \\ -T_c^T & & R_c^- & \\ & -T_r^T & & R_r^- \end{bmatrix}. \tag{36}$$

i) The blocks with index c, associated with nonreal eigenvalues, are $R_c^+ = \operatorname{diag}(A_1, \ldots, A_\mu)$, $R_c^- = \operatorname{diag}(B_1, \ldots, B_\mu)$ and $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$, where for $k = 1, \ldots, \mu$ we have

$$\begin{aligned} A_k &= \operatorname{diag}(A_k^e, A_k^o), \quad B_k = \operatorname{diag}(B_k^e, B_k^o), \quad T_k^c = \operatorname{diag}(T_k^e, T_k^o), \\ A_k^e &= \operatorname{diag}(N_{p_{k,1}}^+((\operatorname{Re}\lambda_k)I_2) + E_{k,1}, \ldots, N_{p_{k,s_k}}^+((\operatorname{Re}\lambda_k)I_2) + E_{k,s_k}), \end{aligned}$$

$$
\begin{aligned}
B_k^e &= \operatorname{diag}(N^+_{p_{k,1}}((\operatorname{Re}\lambda_k)I_2) - E_{k,1}, \ldots, N^+_{p_{k,s_k}}((\operatorname{Re}\lambda_k)I_2) - E_{k,s_k}),\\
T_k^e &= -\operatorname{diag}(N^-_{p_{k,1}}((\operatorname{Im}\lambda_k)J_1) - E_{k,1}, \ldots, N^-_{p_{k,s_k}}((\operatorname{Im}\lambda_k)J_1) - E_{k,s_k}),\\
A_k^o &= \operatorname{diag}(\begin{bmatrix} N^+_{l_{k,1}}((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,1}-1}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,1}-1} & \operatorname{Re}\lambda_k \end{bmatrix}, \ldots, \begin{bmatrix} N^+_{l_{k,x_k}}((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,x_k}-1}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,x_k}-1} & \operatorname{Re}\lambda_k \end{bmatrix}),\\
B_k^o &= \operatorname{diag}(\begin{bmatrix} N^+_{l_{k,1}}((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,1}}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,1}} & \operatorname{Re}\lambda_k \end{bmatrix}, \ldots, \begin{bmatrix} N^+_{l_{k,x_k}}((\operatorname{Re}\lambda_k)I_2) & \frac{\sqrt{2}}{2}e_{2l_{k,x_k}}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,x_k}} & \operatorname{Re}\lambda_k \end{bmatrix}),\\
T_k^o &= \operatorname{diag}(\begin{bmatrix} -N^-_{l_{k,1}}((\operatorname{Im}\lambda_k)J_1) & -\frac{\sqrt{2}}{2}e_{2l_{k,1}}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,1}-1} & -\operatorname{Im}\lambda_k \end{bmatrix}, \ldots, \begin{bmatrix} -N^-_{l_{k,x_k}}((\operatorname{Im}\lambda_k)J_1) & -\frac{\sqrt{2}}{2}e_{2l_{k,x_k}}\\ \frac{\sqrt{2}}{2}e^T_{2l_{k,x_k}-1} & -\operatorname{Im}\lambda_k \end{bmatrix}),
\end{aligned}
$$

with $E_{k,j} = \frac{1}{2}\begin{bmatrix} 0 & 0\\ 0 & \Sigma_{1,1}\end{bmatrix}$.

ii) The blocks with index r*, associated with real eigenvalues, are*

$$R_r^+ = \operatorname{diag}(C_1, \ldots, C_\nu), \quad R_r^- = \operatorname{diag}(D_1, \ldots, D_\nu), \quad T_r = \operatorname{diag}(F_1, \ldots, F_\nu).$$

These have for $k = 1, \ldots, \nu$ *the substructures*

$$C_k = \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-), \quad F_k = \operatorname{diag}(F_k^e, F_k^+, F_k^-),$$

where

$$
\begin{aligned}
C_k^e &= \operatorname{diag}(N^+_{q_{k,1}}(\alpha_k) + \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}}, \ldots, N^+_{q_{k,t_k}}(\alpha_k) + \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
D_k^e &= \operatorname{diag}(N^+_{q_{k,1}}(\alpha_k) - \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}}, \ldots, N^+_{q_{k,t_k}}(\alpha_k) - \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
F_k^e &= \operatorname{diag}(-N^-_{q_{k,1}} + \frac{1}{2}\pi_{k,1}e_{q_{k,1}}e^T_{q_{k,1}}, \ldots, -N^-_{q_{k,t_k}} + \frac{1}{2}\pi_{k,t_k}e_{q_{k,t_k}}e^T_{q_{k,t_k}}),\\
C_k^+ &= \operatorname{diag}(\begin{bmatrix} N^+_{u_{k,1}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,1}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,1}} & \alpha_k\end{bmatrix}, \ldots, \begin{bmatrix} N^+_{u_{k,w_k}}(\alpha_k) & \frac{\sqrt{2}}{2}e_{u_{k,w_k}}\\ \frac{\sqrt{2}}{2}e^T_{u_{k,w_k}} & \alpha_k\end{bmatrix}),\\
D_k^+ &= \operatorname{diag}(N^+_{u_{k,1}}(\alpha_k), \ldots, N^+_{u_{k,w_k}}(\alpha_k)),\\
F_k^+ &= \operatorname{diag}(\begin{bmatrix} -N^-_{u_{k,1}}\\ \frac{\sqrt{2}}{2}e_{u_{k,1}}\end{bmatrix}, \ldots, \begin{bmatrix} -N^-_{u_{k,w_k}}\\ \frac{\sqrt{2}}{2}e_{u_{k,w_k}}\end{bmatrix});\\
C_k^- &= \operatorname{diag}(N^+_{v_{k,1}}(\alpha_k), \ldots, N^+_{v_{k,z_k}}(\alpha_k)),\\
D_k^- &= \operatorname{diag}(\begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e^T_{v_{k,1}}\\ -\frac{\sqrt{2}}{2}e_{v_{k,1}} & N^+_{v_{k,1}}(\alpha_k)\end{bmatrix}, \ldots, \begin{bmatrix} \alpha_k & -\frac{\sqrt{2}}{2}e^T_{v_{k,z_k}}\\ -\frac{\sqrt{2}}{2}e_{v_{k,z_k}} & N^+_{v_{k,z_k}}(\alpha_k)\end{bmatrix}),\\
F_k^- &= \operatorname{diag}([\frac{\sqrt{2}}{2}e_{v_{k,1}}, -N^-_{v_{k,1}}], \ldots [\frac{\sqrt{2}}{2}e_{v_{k,z_k}}, -N^-_{v_{k,z_k}}]).
\end{aligned}
$$

Each λ_k *(*$\overline{\lambda_k}$*) has* s_k *even sized Jordan blocks of sizes* $2p_{k,1}, \ldots, 2p_{k,s_k}$*, and* x_k *odd sized Jordan blocks of sizes* $2l_{k,1}+1, \ldots, 2l_{k,x_k}+1$.

For each real eigenvalue α_k *there are* *a)* t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *corresponding to the structure inertia indices* $\pi_{k,1}, \ldots, \pi_{k,t_k}$*;*

b) w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ *corresponding to the structure inertia index* 1*;*

c) z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ *corresponding to the structure inertia index* -1*.*

Proof. The proof is analogous to the proof of Theorem 12. $\square$

In this section we have obtained real and complex structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian matrices. In the next section we obtain analogous results for $\Sigma_{p,q}$-skew Hermitian matrices.

4 $\Sigma_{p,q}$-skew Hermitian matrices

In this section we discuss structured Jordan canonical forms for $\Sigma_{p,q}$-skew Hermitian matrices. The construction is similar to that for $\Sigma_{p,q}$-Hermitian matrices discussed in Section 3 and therefore we can omit much of the detail. The essential difference is that the role of the real eigenvalues is now taken by the purely imaginary eigenvalues.

But let us first discuss Jordan structures associated with eigenvalues that are not purely imaginary.

Lemma 18 *Let* $\mathcal{C}$ *be a* $\Sigma_{p,q}$*-skew Hermitian matrix and let* $\lambda \in \Lambda(\mathcal{C})$ *have nonzero real part. Let* $N(\lambda) = \lambda I + N$ *with* $N = \mathrm{diag}(N_{r_1}, \ldots, N_{r_s})$ *be the Jordan structure associated with* λ *and let* P_N *be as in (6). Then there exists a full rank matrix* U *such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \\ P_N^H & 0 \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\overline{\lambda}) \end{bmatrix}.$$

Furthermore, $-\overline{\lambda} \in \Lambda(\mathcal{C})$ *and has the same algebraic and geometric multiplicity as* λ*.*

In the case that $\mathcal{C}$ *is real, we have two cases. If* λ *is real nonzero, then there exists a real full rank matrix* U *such that*

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \\ P_N^T & 0 \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\lambda) \end{bmatrix}$$

and if λ *is nonreal, then there exists a real full rank matrix* U *such that*

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \otimes \Sigma_{1,1} \\ P_N^T \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N(\Lambda) & 0 \\ 0 & N(-\Lambda) \end{bmatrix},$$

with $\Lambda = \begin{bmatrix} \mathrm{Re}\,\lambda & \mathrm{Im}\,\lambda \\ -\mathrm{Im}\,\lambda & \mathrm{Re}\,\lambda \end{bmatrix}$.

Proof. By hypothesis there exists a full rank matrix U_1 such that

$$\mathcal{C}U_1 = U_1 N(\lambda). \tag{37}$$

Since $\mathcal{C}$ is $\Sigma_{p,q}$-skew Hermitian, we have $\mathcal{C}^H = -\Sigma_{p,q}\mathcal{C}\Sigma_{p,q}$. Substituting this into the conjugate transpose of (37) we obtain

$$U_1^H \Sigma_{p,q}\mathcal{C} = -N(\lambda)^H U_1^H \Sigma_{p,q}. \tag{38}$$

Now since $\operatorname{Re}\lambda \neq 0$, we have $-\overline{\lambda} \neq \lambda$ and as in the proof of Lemma 7 for the $\Sigma_{p,q}$-Hermitian case we obtain $-\overline{\lambda} \in \Lambda(\mathcal{C})$, and that $-\overline{\lambda}$ and λ have the same algebraic and geometric multiplicities.

Let U_2 be a full rank matrix such that

$$\mathcal{C}U_2 = U_2 N(-\overline{\lambda}). \tag{39}$$

Then

$$U_2^H \Sigma_{p,q}\mathcal{C} = -N(-\overline{\lambda})^H U_2^H \Sigma_{p,q} \tag{40}$$

and we have

$$U_1^H \Sigma_{p,q} U_1 = 0, \quad U_2^H \Sigma_{p,q} U_2 = 0, \quad \det(U_1^H \Sigma_{p,q} U_2) \neq 0. \tag{41}$$

Premultiplying (39) with $U_1^H \Sigma_{p,q}$ and postmultiplying (38) with U_2, we obtain

$$(U_1^H \Sigma_{p,q} U_2) N(-\overline{\lambda}) = -N(\lambda)^H (U_1^H \Sigma_{p,q} U_2).$$

Since $P_N N^H P_N^H = -N$, we have $P_N(-N(\lambda)^H)P_N^H = N(-\overline{\lambda})$ and therefore $P_N^H(U_1^H \Sigma_{p,q} U_2)$ and $N(-\overline{\lambda})$ commute, or equivalently,

$$N(-\overline{\lambda})(P_N^H(U_1^H \Sigma_{p,q} U_2))^{-1} = (P_N^H(U_1^H \Sigma_{p,q} U_2))^{-1} N(-\overline{\lambda}). \tag{42}$$

Let $U = [U_1, U_2(U_1^H \Sigma_{p,q} U_2)^{-1} P_N]$, then from (41) we have $U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \\ P_N^H & 0 \end{bmatrix}$ and by (37), (39), (42) we obtain $\mathcal{C}U = U \operatorname{diag}(N(\lambda), N(-\overline{\lambda}))$.

If $\mathcal{C}$ is real and λ is real then U_1, U_2 can be chosen real and hence U is real. If λ is nonreal then from (37) and (39), we obtain

$$\mathcal{C}\overline{U_1} = \overline{U_1} N(\overline{\lambda}), \quad \mathcal{C}\overline{U_2} = \overline{U_2} N(-\lambda),$$

which implies that $\overline{\lambda}, -\lambda \in \Lambda(\mathcal{C})$ have the same algebraic and geometric multiplicities as λ. Since $\operatorname{Re}\lambda, \operatorname{Im}\lambda \neq 0$, the four eigenvalues λ, $\overline{\lambda}$, $-\lambda$ and $-\overline{\lambda}$ are pairwise distinct. Let $\hat{U}_2 = U_2(U_1^H \Sigma_{p,q} U_2)^{-1} P_N$ and set $\hat{U} = [U_1, \overline{U_1}, \overline{\hat{U}_2}, \hat{U}_2]$. Then by the invariant subspace property of Proposition 2 we obtain

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} & & & P_N \\ & & P_N & \\ & P_N^H & & \\ P_N^H & & & \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} = \begin{bmatrix} N(\lambda) & & & \\ & N(\overline{\lambda}) & & \\ & & N(-\lambda) & \\ & & & N(-\overline{\lambda}) \end{bmatrix}.$$

Let $U = \hat{U} \operatorname{diag}(\Psi\Phi, \Psi\Phi)$ with Ψ, Φ as in (33), then by Lemma 13, U is real and we can easily verify that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & P_N \otimes \Sigma_{1,1} \\ P_N^T \otimes \Sigma_{1,1} & \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N(\Lambda) & 0 \\ 0 & N(-\Lambda) \end{bmatrix}.$$

□

While in the $\Sigma_{p,q}$-Hermitian case difficulties arise from the real eigenvalues, here the purely imaginary eigenvalues are causing difficulties. The characterization of the Jordan structure for the purely imaginary eigenvalues is given in the following Lemma analogous to Lemma 11.

Lemma 19 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix and let $\sigma \in \Lambda(\mathcal{C})$ be purely imaginary. Let $N(\sigma) = \sigma I + N$ with $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ be the Jordan structure associated with σ. Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \operatorname{diag}(\pi_1 P_{r_1}, \ldots, \pi_s P_{r_s}), \qquad \mathcal{C}U = UN(\sigma),$$

with $\pi_k \in \{\pm i\}$ for even r_k and $\pi_k \in \{\pm 1\}$ for odd r_k.

If $\mathcal{C}$ is real then there are again two cases.

i) If σ is zero then there exists a real matrix U such that

$$\begin{aligned} U^T \Sigma_{p,q} U &= \operatorname{diag}(\pi_1 P_{2u_1+1}, \ldots, \pi_a P_{2u_a+1}, \begin{bmatrix} 0 & P_{2v_1} \\ P_{2v_1}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & P_{2v_b} \\ P_{2v_b}^T & 0 \end{bmatrix}), \\ \mathcal{C}U &= UN = U \operatorname{diag}(N_{2u_1+1}, \ldots, N_{2u_a+1}, N_{2v_1}, N_{2v_1}, \ldots, N_{2v_b}, N_{2v_b}). \end{aligned}$$

This implies that for zero eigenvalues the number of even sized Jordan blocks must be even and the corresponding structure inertia indices must occur in $i, -i$ pairs.

ii) If σ is nonzero then there exists a real matrix V such that

$$V^T \Sigma_{p,q} V = \operatorname{diag}(P_{r_1} \otimes \Xi_1, \ldots, P_{r_s} \otimes \Xi_s), \quad \mathcal{C}V = VN((\operatorname{Im}\sigma)J_1),$$

where $\Xi_k = \pi_k I_2$ if r_k is odd and $\Xi_k = (\operatorname{Im}\pi_k)J_1$ if r_k is even.

Proof. Let the columns of U_1 be the chains of root vectors corresponding to σ. Then we have (37) and (38), by replacing λ with σ. Since σ is purely imaginary, U_1 and $\Sigma_{p,q}U_1$ are the bases of the right and left invariant subspaces of $\mathcal{C}$ corresponding to σ. By Proposition 2 we have $\det(U_1^H \Sigma_{p,q} U_1) \neq 0$. Premultiplying (37) by $U_1^H \Sigma_{p,q}$ and postmultiplying (38) by U_1 we obtain

$$U_1^H \Sigma_{p,q} \mathcal{C} U_1 = (U_1^H \Sigma_{p,q} U_1) N(\sigma) = -N(\sigma)^H (U_1^H \Sigma_{p,q} U_1),$$

which implies that

$$(U_1^H \Sigma_{p,q} U_1) N = -N^H (U_1^H \Sigma_{p,q} U_1).$$

Note that $U_1^H \Sigma_{p,q} U_1$ is Hermitian and nonsingular. Thus, we can apply Theorem 6 i) and hence there exists a nonsingular matrix $Y \in \mathbf{G}(N)$ such that

$$(U_1 Y)^H \Sigma_{p,q} (U_1 Y) = \operatorname{diag}(\pi_1 P_{r_1}, \ldots, \pi_s P_{r_s})$$

Set $U = U_1 Y$, then

$$\mathcal{C}U = \mathcal{C}U_1 Y = U_1 N(\sigma) Y = U_1 Y N(\sigma) = UN(\sigma)$$

and U is as required.

Now consider the case that $\mathcal{C}$ is real.

i) If $\sigma = 0$, then clearly U_1 can be chosen real and as before we can apply Theorem 6 i) which yields the assertion.

ii) If σ is nonzero then we also have

$$\overline{U}^H \Sigma_{p,q} \overline{U} = \operatorname{diag}(\overline{\pi_1} P_{r_1}, \ldots, \overline{\pi_s} P_{r_s}), \quad \mathcal{C}\overline{U} = \overline{U} N(\overline{\sigma}).$$

Set $V = [U, \overline{U}]\Psi\Phi$. Then, since $\sigma \neq \overline{\sigma}$, we have

$$\Phi_2^H \operatorname{diag}(\pi_k, \overline{\pi_k})\Phi_2 = \begin{cases} \pi_k I_2 & \text{if } r_k \text{ is odd} \\ (\operatorname{Im} \pi_k) J_1 & \text{if } r_k \text{ is even} \end{cases}$$

and by Lemma 13 we obtain that V is real and as desired. □

We can combine Lemmas 18 and 19 to obtain the structured Jordan canonical form for $\Sigma_{p,q}$-skew Hermitian matrices.

Theorem 20 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix with pairwise different purely imaginary eigenvalues $\sigma_1, \ldots, \sigma_\nu$ and pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive real parts. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c^+, R_c^-, R_g), \tag{43}$$

i) The diagonal blocks with index c, associated with eigenvalues not on the imaginary axis, are

$$R_c^+ = \operatorname{diag}(H_1(\lambda_1), \ldots, H_\mu(\lambda_\mu)), \qquad R_c^- = \operatorname{diag}(H_1(-\overline{\lambda_1}), \ldots, H_\mu(-\overline{\lambda_\mu})),$$

where for $k = 1, \ldots, \mu$ we have

$$H_k(\lambda_k) = \lambda_k I + H_k, \qquad H_k(-\overline{\lambda_k}) = -\overline{\lambda_k} I + H_k \qquad H_k = \operatorname{diag}(N_{p_{k,1}}, \ldots, N_{p_{k,s_k}}).$$

ii) The block R_g, associated with purely imaginary eigenvalues, has the form

$$R_g = \operatorname{diag}(M_1(\sigma_1), \ldots, M_\nu(\sigma_\nu)),$$

where $M_k(\sigma_k) = \sigma_k I + M_k$ and for $k = 1, \ldots, \nu$ we have $M_k = \operatorname{diag}(N_{q_{k,1}}, \ldots N_{q_{k,t_k}})$.

The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_g \end{bmatrix}, \tag{44}$$

where

$$W_c = \operatorname{diag}(P_{H_1}, \ldots, P_{H_\mu}), \qquad W_g = \operatorname{diag}(W_1^g, \ldots, W_\nu^g),$$

and for $k = 1, \ldots, \mu$ we have $P_{H_k} = \operatorname{diag}(P_{p_{k,1}}, \ldots, P_{p_{k,s_k}})$, and with $\operatorname{Ind}(\sigma_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ for $k = 1, \ldots, \nu$ we have $W_k^g = \operatorname{diag}(\pi_{k,1} P_{q_{k,1}}, \ldots, \pi_{k,t_k} P_{q_{k,t_k}})$.

Proof. The proof is analogous to that for Theorem 9, using Lemmas 18 and 19 instead of Lemma 7 and 8, respectively. □

For real matrices under nonstructured similarity transformation we obtain the following canonical form.

Theorem 21 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-skew symmetric matrix with pairwise different nonzero purely imaginary eigenvalues $\sigma_1, \ldots, \sigma_\nu$ with positive imaginary parts, pairwise different eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive real and imaginary parts and pairwise different real positive eigenvalues $\alpha_1, \ldots, \alpha_\eta$. (Note that the spectrum contains with σ_k also $-\sigma_k$, with α_j also $-\alpha_j$ and with λ_j also $-\lambda_j, \overline{\lambda_j}, -\overline{\lambda_j}$ and also 0 may be a further eigenvalue.) Then there exists a real nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \operatorname{diag}(R_c^+, R_c^-, R_g).$$

i) The blocks with index c, associated with eigenvalues not on the imaginary axis, are $R_c^+ = \operatorname{diag}(\hat{R}_c^+, \tilde{R}_c^+)$*, with* $\hat{R}_c^+ = \operatorname{diag}(K_1(\alpha_1), \ldots, K_\eta(\alpha_\eta))$ *where for* $k = 1, \ldots, \eta$ *we have* $K_k(\alpha_k) = \alpha_k I + K_k$ *and* $K_k = \operatorname{diag}(N_{f_{k,1}}, \ldots, N_{f_{k,l_k}})$*. Analogously* $\tilde{R}_c^+ = \operatorname{diag}(H_1(\Lambda_1), \ldots, H_\mu(\Lambda_\mu))$*, where for* $k = 1, \ldots, \mu$ *we have* $\Lambda_k = \begin{bmatrix} \operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k \\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k \end{bmatrix}$ *and* $H_k(\Lambda_k) = \operatorname{diag}(N_{p_{k,1}}(\Lambda_k), \ldots, N_{p_{k,s_k}}(\Lambda_k))$*.*

The block $R_c^- = \operatorname{diag}(\hat{R}_c^-, \tilde{R}_c^-)$*, has the same substructure as* R_c^+ *just replacing* α_j *with* $-\alpha_j$ *and* Λ_j *by* $-\Lambda_j$*.*

ii) The block R_g*, associated with purely imaginary eigenvalues, has the structure*

$$R_g = \operatorname{diag}(M_1((\operatorname{Im}\sigma_1)J_1), \ldots, M_\nu((\operatorname{Im}\sigma_\nu)J_1), M_0),$$

where for $k = 1, \ldots, \nu$ *we have*

$$M_k((\operatorname{Im}\sigma_k)J_1) = \operatorname{diag}(N_{q_{k,1}}((\operatorname{Im}\sigma_k)J_1), \ldots, N_{q_{k,t_k}}((\operatorname{Im}\sigma_k)J_1))$$

and where

$$M_0 = \operatorname{diag}(N_{2g_1+1}, \ldots, N_{2g_a+1}, N_{2h_1}, N_{2h_1}, \ldots, N_{2h_b}, N_{2h_b})$$

is the structure associated with the eigenvalue 0*.*

The matrix $\mathcal{U}$ *has the form*

$$\mathcal{U}^T \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^T & 0 & 0 \\ 0 & 0 & W_g \end{bmatrix},$$

where $W_c = \operatorname{diag}(\hat{W}_c, \tilde{W}_c)$ *with*

$$\hat{W}_c = \operatorname{diag}(P_{K_1}, \ldots, P_{K_\eta}), \quad \tilde{W}_c = \operatorname{diag}(P_{H_1} \otimes \Sigma_{1,1}, \ldots, P_{H_\mu} \otimes \Sigma_{1,1}),$$

and where

$$P_{K_k} = \operatorname{diag}(P_{f_{k,1}}, \ldots, P_{f_{k,l_k}}), \qquad P_{H_k} = \operatorname{diag}(P_{p_{k,1}}, \ldots, P_{p_{k,s_k}}).$$

The block W_g *has the form* $W_g = \operatorname{diag}(W_1^g, \ldots, W_\nu^g, W_0)$*, where for* $k = 1, \ldots, \nu$ *and* $\operatorname{Ind}(\sigma_k) = \{\pi_{k,1}, \ldots, \pi_{k,t_k}\}$ *we have* $W_k^g = \operatorname{diag}(P_{q_{k,1}} \otimes \Xi_{k,1}, \ldots, P_{q_{k,t_k}} \otimes \Xi_{k,t_k})$*, with* $\Xi_{k,j} = \pi_{k,j} I_2$ *if* $q_{k,j}$ *is odd and* $\Xi_{k,j} = (\operatorname{Im}\pi_{k,j})J_1$ *if* $q_{k,j}$ *is even.*

Finally for $\operatorname{Ind}(0) = \{\pi_1^0, \ldots, \pi_a^0, i, -i, \ldots, i, -i\}$ *we have*

$$W_0 = \operatorname{diag}(\pi_1^0 P_{2g_1+1}, \ldots, \pi_a^0 P_{2g_a+1}, \begin{bmatrix} 0 & P_{2h_1} \\ P_{2h_1}^T & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & P_{2h_b} \\ P_{2h_b}^T & 0 \end{bmatrix}).$$

Proof. The proof follows from Lemmas 18 and 19. □

After determining the Jordan structure under non $\Sigma_{p,q}$-unitary similarity transformations we now derive the corresponding structured canonical form under $\Sigma_{p,q}$-unitary transformations. For this we need the following two lemmas.

Lemma 22 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix and let $\lambda \in \Lambda(\mathcal{C})$ and $\operatorname{Re}\lambda \neq 0$. If $N_r(\lambda) = \lambda I + N_r$ is a Jordan block of $\mathcal{C}$, then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_r^-(i\operatorname{Im}\lambda) & -N_r^+(\operatorname{Re}\lambda) \\ -N_r^+(\operatorname{Re}\lambda) & N_r^-(i\operatorname{Im}\lambda) \end{bmatrix}. \tag{45}$$

If $\mathcal{C}$ is real, then there are two cases.

i) If λ is real nonzero then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_r^- & -N_r^+(\lambda) \\ -N_r^+(\lambda) & N_r^- \end{bmatrix}.$$

ii) If λ is nonreal then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_{2r} & 0 \\ 0 & -I_{2r} \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_r^-((\operatorname{Im}\lambda)J_1) & -N_r^+((\operatorname{Re}\lambda)I_2) \\ -N_r^+((\operatorname{Re}\lambda)I_2) & N_r^-((\operatorname{Im}\lambda)J_1) \end{bmatrix}.$$

Proof. By Lemma 18 for the Jordan block $N_r(\lambda)$ there exists a full rank matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & P_r \\ P_r^H & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \operatorname{diag}(N_r(\lambda), N_r(-\overline{\lambda})).$$

Set $Z_r = \operatorname{diag}(I_r, P_r^{-1})\Upsilon_r$ and $U = \hat{U} Z_r$. Then we can easily verify that (45) holds.

Now let $\mathcal{C}$ be real. Then in case i), if λ is real nonzero, taking $\hat{U}$ real we have the real form. In case ii) if λ is nonreal, by Lemma 18 there exists a real matrix $\hat{U}$ such that

$$\hat{U}^T \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & P_r \otimes \Sigma_{1,1} \\ P_r^T \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \operatorname{diag}(N_r(\Lambda), N_r(-\Lambda)),$$

with $\Lambda = \begin{bmatrix} \operatorname{Re}\lambda & \operatorname{Im}\lambda \\ -\operatorname{Im}\lambda & \operatorname{Re}\lambda \end{bmatrix}$. Setting

$$Z_r = \operatorname{diag}(I_{2r}, (P_r \otimes \Sigma_{1,1})^{-1})\Upsilon_{2r}$$

and $U = \hat{U} Z_r$, the assertion follows. □

Lemma 23 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix and let $\sigma \in \Lambda(\mathcal{C})$ be purely imaginary. If $N_r(\sigma) = \sigma I + N_r$ is a Jordan block of $\mathcal{C}$, then there exists a full rank matrix U such that we have the following cases:*

i) If $r = 2s$, then

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^-(\sigma) + \frac{1}{2} i\beta e_s e_s^H & -N_s^+ + \frac{1}{2} i\beta e_s e_s^H \\ -N_s^+ - \frac{1}{2} i\beta e_s e_s^H & N_s^-(\sigma) - \frac{1}{2} i\beta e_s e_s^H \end{bmatrix}, \tag{46}$$

where $\beta = (-1)^s i\pi$ and $\pi \in \{\pm i\}$ is the structure index corresponding to the Jordan block $N_r(\sigma)$.

ii) If $r = 2s + 1$, then

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^-(\sigma) & \frac{\sqrt{2}}{2} e_s & -N_s^+ \\ -\frac{\sqrt{2}}{2}\beta e_s^H & \sigma & -\frac{\sqrt{2}}{2}\beta e_s^H \\ -N_s^+ & -\frac{\sqrt{2}}{2} e_s & N_s^-(\sigma) \end{bmatrix}, \tag{47}$$

where $\beta = (-1)^{s+1}\pi$ and $\pi \in \{\pm 1\}$ is the structure index corresponding to the Jordan block $N_r(\sigma)$.

If $\mathcal{C}$ is real then we have to distinguish whether $\sigma = 0$ or not.

a) If $\sigma = 0$, then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^- & -N_s^+ \\ -N_s^+ & N_s^- \end{bmatrix}$$

if $r = 2s$ and

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_s \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^- & \frac{\sqrt{2}}{2} e_s & -N_s^+ \\ -\frac{\sqrt{2}}{2}\beta e_s^T & 0 & -\frac{\sqrt{2}}{2}\beta e_s^T \\ -N_s^+ & -\frac{\sqrt{2}}{2} e_s & N_s^- \end{bmatrix},$$

if $r = 2s + 1$. Here $\beta = (-1)^{s+1}\pi$ and $\pi \in \{\pm 1\}$ is the structure index corresponding to the Jordan block N_r.

b) If σ is nonzero then there exists a real full rank matrix U such that if $r = 2s$ then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{C}U = U \begin{bmatrix} N_s^-((\operatorname{Im}\sigma)J_1) + E_r & -N_s^+(0_2) + E_r \\ -N_s^+(0_2) - E_r & N_s^-((\operatorname{Im}\sigma)J_1) - E_r \end{bmatrix},$$

where $E_r = \frac{1}{2}\beta \begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix}$ and β is the same as in the complex case. If $r = 2s + 1$, then

$$\begin{aligned} U^T \Sigma_{p,q} U &= \begin{bmatrix} I_{r-1} & & \\ & \beta I_2 & \\ & & -I_{r-1} \end{bmatrix}, \\ \mathcal{C}U &= U \left[\begin{array}{c|c|c} N_s^-((\operatorname{Im}\sigma)J_1) & \begin{matrix} 0 \\ \frac{\sqrt{2}}{2} I_2 \end{matrix} & -N_s^+(0_2) \\ \hline 0 \quad -\frac{\sqrt{2}}{2}\beta I_2 & (\operatorname{Im}\sigma)J_1 & 0 \quad -\frac{\sqrt{2}}{2}\beta I_2 \\ \hline -N_s^+(0_2) & \begin{matrix} 0 \\ -\frac{\sqrt{2}}{2} I_2 \end{matrix} & N_s^-((\operatorname{Im}\sigma)J_1) \end{array}\right], \end{aligned}$$

where β is again the same as in the complex case.

Proof. By Lemma 19, for the Jordan block $N_r(\sigma)$, there exists a matrix $\hat{U}$ such that $\hat{U}^H \Sigma_{p,q} \hat{U} = \pi P_r$ and $\mathcal{C}\hat{U} = \hat{U} N_r(\sigma)$.

i) If $r = 2s$, then we partition

$$\pi P_r = \begin{bmatrix} 0 & \pi P_s \\ (\pi P_s)^H & 0 \end{bmatrix}, \quad N_r(\sigma) = \begin{bmatrix} N_s(\sigma) & e_s e_1^H \\ 0 & N_s(\sigma) \end{bmatrix}.$$

Here we have used that $P_s^H = (-1)^{s-1}P_s$ and $\pi \in \{\pm i\}$. Setting $\hat{Z}_r = \operatorname{diag}(I_s, (\pi P_s)^{-1})\Upsilon_s$ and $U = \hat{U}\hat{Z}_r$, a simple calculation yields (46) with this U.

ii) If $r = 2s+1$, then we partition

$$\pi P_r = \begin{bmatrix} 0 & 0 & \pi P_s \\ 0 & (-1)^{s+1}\pi & 0 \\ (\pi P_s)^H & 0 & 0 \end{bmatrix}, \quad N_r(\sigma) = \begin{bmatrix} N_s(\sigma) & e_s & 0 \\ 0 & \sigma & e_1^H \\ 0 & 0 & N_s(\sigma) \end{bmatrix},$$

where $\pi \in \{\pm 1\}$. Setting

$$\hat{Z}_r = \begin{bmatrix} I_{s+1} & 0 \\ 0 & (\pi P_s)^{-1} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2}I_s & 0 & -\frac{\sqrt{2}}{2}I_s \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2}I_s & 0 & \frac{\sqrt{2}}{2}I_s \end{bmatrix}$$

and $U = \hat{U}\hat{Z}_r$, again a simple calculation yields (47).

With Lemma 19 the real case follows analogously. □

Using these lemmas we can now derive the structured Jordan canonical form for $\Sigma_{p,q}$-skew Hermitian matrices under similarity transformations with $\Sigma_{p,q}$-unitary matrices.

Theorem 24 *Let $\mathcal{C}$ be a $\Sigma_{p,q}$-skew Hermitian matrix with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive real parts and pairwise distinct $\sigma_1, \ldots, \sigma_\nu$ with real part zero. Then there exists a $\Sigma_{p,q}$-unitary matrix $\mathcal{U}$, such that*

$$\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_g^+ & & T_g \\ T_c^H & & R_c & \\ & T_g^H & & R_g^- \end{bmatrix}. \tag{48}$$

For the different blocks we have the following substructures.

i) The blocks with index c, associated with eigenvalues not on the imaginary axis, are $R_c = \operatorname{diag}(R_1^c, \ldots, R_\mu^c)$ and $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$ where for $k = 1, \ldots, \mu$

$$\begin{aligned} R_k^c &= \operatorname{diag}(N_{p_{k,1}}^-(i \operatorname{Im} \lambda_k), \ldots, N_{p_{k,s_k}}^-(i \operatorname{Im} \lambda_k)), \\ T_k^c &= -\operatorname{diag}(N_{p_{k,1}}^+(\operatorname{Re} \lambda_k), \ldots, N_{p_{k,s_k}}^+(\operatorname{Re} \lambda_k)). \end{aligned}$$

ii) The blocks with index g, associated with purely imaginary eigenvalues, are

$$R_g^+ = \operatorname{diag}(C_1, \ldots, C_\nu), \quad R_g^- = \operatorname{diag}(D_1, \ldots, D_\nu), \quad T_g = \operatorname{diag}(F_1, \ldots, F_\nu),$$

where for $k = 1, \ldots, \nu$ the substructures are

$$C_k = \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-), \quad F_k = \operatorname{diag}(F_k^e, F_k^+, F_k^-),$$

with further partitioning

$$\begin{aligned} C_k^e &= \operatorname{diag}(N_{q_{k,1}}^-(\sigma_k) + \frac{1}{2}i\beta_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H, \ldots, N_{q_{k,t_k}}^-(\sigma_k) + \frac{1}{2}i\beta_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H), \\ D_k^e &= \operatorname{diag}(N_{q_{k,1}}^-(\sigma_k) - \frac{1}{2}i\beta_{k,1}e_{q_{k,1}}e_{q_{k,1}}^H, \ldots, N_{q_{k,t_k}}^-(\sigma_k) - \frac{1}{2}i\beta_{k,t_k}e_{q_{k,t_k}}e_{q_{k,t_k}}^H), \end{aligned}$$

$$
\begin{aligned}
F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^+ + \frac{1}{2} i\beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H, \ldots, -N_{q_{k,t_k}}^+ + \frac{1}{2} i\beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H),\\
C_k^+ &= \operatorname{diag}(\begin{bmatrix} N_{u_{k,1}}^-(\sigma_k) & \frac{\sqrt{2}}{2} e_{u_{k,1}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,1}}^H & \sigma_k \end{bmatrix}, \ldots, \begin{bmatrix} N_{u_{k,w_k}}^-(\sigma_k) & \frac{\sqrt{2}}{2} e_{u_{k,w_k}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H & \sigma_k \end{bmatrix}),\\
D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^-(\sigma_k), \ldots, N_{u_{k,w_k}}^-(\sigma_k)),\\
F_k^+ &= -\operatorname{diag}(\begin{bmatrix} N_{u_{k,1}}^+ \\ \frac{\sqrt{2}}{2} e_{u_{k,1}}^H \end{bmatrix}, \ldots, \begin{bmatrix} N_{u_{k,w_k}}^+ \\ \frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H \end{bmatrix}),\\
C_k^- &= \operatorname{diag}(N_{v_{k,1}}^-(\sigma_k), \ldots, N_{v_{k,z_k}}^-(\sigma_k)),\\
D_k^- &= \operatorname{diag}(\begin{bmatrix} \sigma_k & \frac{\sqrt{2}}{2} e_{v_{k,1}}^H \\ -\frac{\sqrt{2}}{2} e_{v_{k,1}} & N_{v_{k,1}}^-(\sigma_k) \end{bmatrix}, \ldots, \begin{bmatrix} \sigma_k & \frac{\sqrt{2}}{2} e_{v_{k,z_k}}^H \\ -\frac{\sqrt{2}}{2} e_{v_{k,z_k}} & N_{v_{k,z_k}}^-(\sigma_k) \end{bmatrix}),\\
F_k^- &= \operatorname{diag}([\frac{\sqrt{2}}{2} e_{v_{k,1}}, -N_{v_{k,1}}^+], \ldots [\frac{\sqrt{2}}{2} e_{v_{k,z_k}}, -N_{v_{k,z_k}}^+]).
\end{aligned}
$$

Each λ_k ($-\overline{\lambda_k}$) has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$. Each purely imaginary eigenvalue σ_k has

a) *t_k even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ with the corresponding structure inertia indices $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$;*

b) *w_k odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$;*

c) *z_k odd sized Jordan blocks of sizes $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ corresponding to the structure indices $(-1)^{v_{k,1}}, \ldots, (-1)^{v_{k,z_k}}$.*

Proof. The proof is analogous to that of Theorem 12, using Lemmas 22 and 23 instead of Lemma 10 and 11, respectively. □

As the final result in this section we present the real version of Theorem 24.

Theorem 25 *Let $\mathcal{C}$ be a real $\Sigma_{p,q}$-skew symmetric matrix with pairwise distinct real positive eigenvalues $\alpha_1, \ldots, \alpha_\eta$, pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_\mu$ with positive real and imaginary parts and pairwise distinct purely imaginary eigenvalues $\sigma_1, \ldots, \sigma_\nu$ with positive imaginary parts. (Note that we then also have eigenvalues $-\alpha_1, \ldots, -\alpha_\eta$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}$, $-\lambda_1, \ldots, -\lambda_\mu$, $-\overline{\lambda_1}, \ldots, -\overline{\lambda_\mu}$ and $-\sigma_1, \ldots, -\sigma_\eta$ and also 0 may be another eigenvalue.)*

Then there exists a real $\Sigma_{p,q}$-orthogonal matrix $\mathcal{U}$, such that

$$
\mathcal{U}^{-1}\mathcal{C}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_g^+ & & T_g \\ T_c^T & & R_c & \\ & T_g^T & & R_g^- \end{bmatrix}, \qquad (49)
$$

where the different blocks have the following substructures:

i) *The blocks with index c, associated with the eigenvalues with nonzero real part, are*

$$
\begin{aligned}
R_c &= \operatorname{diag}(\hat{R}_c, \tilde{R}_c), \quad T_c = \operatorname{diag}(\hat{T}_c, \tilde{T}_c),\\
\hat{R}_c &= \operatorname{diag}(\hat{R}_1^c, \ldots, \hat{R}_\eta^c), \quad \tilde{R}_c = \operatorname{diag}(\tilde{R}_1^c, \ldots, \tilde{R}_\mu^c),\\
\hat{T}_c &= \operatorname{diag}(\hat{T}_1^c, \ldots, \hat{T}_\eta^c), \quad \tilde{T}_c = \operatorname{diag}(\tilde{T}_1^c, \ldots, \tilde{T}_\mu^c),
\end{aligned}
$$

where for $k = 1, \ldots, \eta$ *the substructures are*

$$\hat{R}_k^c = \operatorname{diag}(N_{f_{k,1}}^-, \ldots, N_{f_{k,l_k}}^-), \quad \hat{T}_k^c = -\operatorname{diag}(N_{f_{k,1}}^+(\alpha_k), \ldots, N_{f_{k,l_k}}^+(\alpha_k)),$$

and for $k = 1, \ldots, \mu$,

$$\begin{aligned} \tilde{R}_k^c &= \operatorname{diag}(N_{p_{k,1}}^-((\operatorname{Im}\lambda_k)J_1), \ldots, N_{p_{k,s_k}}^-((\operatorname{Im}\lambda_k)J_1)), \\ \tilde{T}_k^c &= -\operatorname{diag}(N_{p_{k,1}}^+((\operatorname{Re}\lambda_k)I_2), \ldots, N_{p_{k,s_k}}^+((\operatorname{Re}\lambda_k)I_2)). \end{aligned}$$

ii) The blocks with index g*, associated with the purely imaginary eigenvalues, are*

$$R_g^+ = \operatorname{diag}(C_1, \ldots, C_\nu, C_0), \quad R_g^- = \operatorname{diag}(D_1, \ldots, D_\nu, D_0), \quad T_g = \operatorname{diag}(F_1, \ldots, F_\nu, F_0),$$

with the partitioning

$$C_k = \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-), \quad F_k = \operatorname{diag}(F_k^e, F_k^+, F_k^-),$$

and for $k = 1, \ldots, \nu$ *the blocks have the further substructure*

$$\begin{aligned}
C_k^e &= \operatorname{diag}(N_{q_{k,1}}^-((\operatorname{Im}\sigma_k)J_1) + E_{k,1} \ldots, N_{q_{k,t_k}}^-((\operatorname{Im}\sigma_k)J_1) + E_{k,t_k}), \\
D_k^e &= \operatorname{diag}(N_{q_{k,1}}^-((\operatorname{Im}\sigma_k)J_1) - E_{k,1}, \ldots, N_{q_{k,t_k}}^-((\operatorname{Im}\sigma_k)J_1) - E_{k,t_k}), \\
F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^+(0_2) + E_{k,1}, \ldots, -N_{q_{k,t_k}}^+(0_2) + E_{k,t_k}), \\
C_k^+ &= \operatorname{diag}(\left[\begin{array}{c|c} N_{u_{k,1}}^-((\operatorname{Im}\sigma_k)J_1) & \begin{matrix} 0 \\ \frac{\sqrt{2}}{2}I_2 \end{matrix} \\ \hline \begin{matrix} 0 & -\frac{\sqrt{2}}{2}I_2 \end{matrix} & (\operatorname{Im}\sigma_k)J_1 \end{array}\right], \ldots, \left[\begin{array}{c|c} N_{u_{k,w_k}}^-((\operatorname{Im}\sigma_k)J_1) & \begin{matrix} 0 \\ \frac{\sqrt{2}}{2}I_2 \end{matrix} \\ \hline \begin{matrix} 0 & -\frac{\sqrt{2}}{2}I_2 \end{matrix} & (\operatorname{Im}\sigma_k)J_1 \end{array}\right]), \\
D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^-((\operatorname{Im}\sigma_k)J_1), \ldots, N_{u_{k,w_k}}^-((\operatorname{Im}\sigma_k)J_1)), \\
F_k^+ &= -\operatorname{diag}(\left[\begin{array}{c} N_{u_{k,1}}^+(0_2) \\ \hline \begin{matrix} 0 & \frac{\sqrt{2}}{2}I_2 \end{matrix} \end{array}\right], \ldots, \left[\begin{array}{c} N_{u_{k,w_k}}^+(0_2) \\ \hline \begin{matrix} 0 & \frac{\sqrt{2}}{2}I_2 \end{matrix} \end{array}\right]); \\
C_k^- &= \operatorname{diag}(N_{v_{k,1}}^-((\operatorname{Im}\sigma_k)J_1), \ldots, N_{v_{k,z_k}}^-((\operatorname{Im}\sigma_k)J_1)), \\
D_k^- &= \operatorname{diag}(\left[\begin{array}{c|c} (\operatorname{Im}\sigma_k)J_1 & \begin{matrix} 0 & \frac{\sqrt{2}}{2}I_2 \end{matrix} \\ \hline \begin{matrix} 0 \\ -\frac{\sqrt{2}}{2}I_2 \end{matrix} & N_{v_{k,1}}^-((\operatorname{Im}\sigma_k)J_1) \end{array}\right], \ldots, \left[\begin{array}{c|c} (\operatorname{Im}\sigma_k)J_1 & \begin{matrix} 0 & \frac{\sqrt{2}}{2}I_2 \end{matrix} \\ \hline \begin{matrix} 0 \\ -\frac{\sqrt{2}}{2}I_2 \end{matrix} & N_{v_{k,z_k}}^-((\operatorname{Im}\sigma_k)J_1) \end{array}\right]), \\
F_k^- &= \operatorname{diag}(\left[\begin{array}{c|c} \begin{matrix} 0 \\ \frac{\sqrt{2}}{2}I_2 \end{matrix} & -N_{v_{k,1}}^+(0_2) \end{array}\right], \ldots, \left[\begin{array}{c|c} \begin{matrix} 0 \\ \frac{\sqrt{2}}{2}I_2 \end{matrix} & -N_{v_{k,z_k}}^+(0_2) \end{array}\right]).
\end{aligned}$$

Here for $j = 1, \ldots, t_k$, $E_{k,j} = \frac{1}{2}\beta_{k,j}\begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix}$.

Finally, the blocks with index 0*, associated to the eigenvalue* 0*, are*

$$C_0 = \operatorname{diag}(C_0^e, C_0^+, C_0^-), \quad D_0 = \operatorname{diag}(D_0^e, D_0^+, D_0^-), \quad F_0 = \operatorname{diag}(F_0^e, F_0^+, F_0^-),$$

with substructures

$$C_0^e = D_0^e = \operatorname{diag}(N_{2x_1}^-, \ldots, N_{2x_c}^-), \quad F_0^e = -\operatorname{diag}(N_{2x_1}^+, \ldots, N_{2x_c}^+);$$

$$\begin{aligned}
C_0^+ &= \operatorname{diag}(\begin{bmatrix} N_{g_1}^- & \frac{\sqrt{2}}{2}e_{g_1} \\ -\frac{\sqrt{2}}{2}e_{g_1}^T & 0 \end{bmatrix}, \dots, \begin{bmatrix} N_{g_a}^- & \frac{\sqrt{2}}{2}e_{g_a} \\ -\frac{\sqrt{2}}{2}e_{g_a}^T & 0 \end{bmatrix}), \\
D_0^+ &= \operatorname{diag}(N_{g_1}^-, \dots, N_{g_a}^-), \quad F_0^+ = -\operatorname{diag}(\begin{bmatrix} N_{g_1}^+ \\ \frac{\sqrt{2}}{2}e_{g_1}^T \end{bmatrix}, \dots, \begin{bmatrix} N_{g_a}^+ \\ \frac{\sqrt{2}}{2}e_{g_a}^T \end{bmatrix}); \\
C_0^- &= \operatorname{diag}(N_{h_1}^-, \dots, N_{h_b}^-), \quad F_0^- = \operatorname{diag}([\frac{\sqrt{2}}{2}e_{h_1}, -N_{h_1}^+], \dots, [\frac{\sqrt{2}}{2}e_{h_b}, -N_{h_b}^+]), \\
D_0^- &= \operatorname{diag}(\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}e_{h_1}^T \\ -\frac{\sqrt{2}}{2}e_{h_1} & N_{h_1}^- \end{bmatrix}, \dots, \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}e_{h_b}^T \\ -\frac{\sqrt{2}}{2}e_{h_b} & N_{h_b}^- \end{bmatrix}).
\end{aligned}$$

Each nonzero real eigenvalue α_k ($-\alpha_k$) has l_k Jordan blocks of sizes $f_{k,1}, \dots, f_{k,l_k}$ and each nonreal eigenvalue λ_k ($-\lambda_k, \overline{\lambda_k}, -\overline{\lambda_k}$) that is not on the imaginary axis has s_k Jordan blocks with sizes $p_{k,1}, \dots, p_{k,s_k}$.

For each nonzero purely imaginary eigenvalue σ_k ($-\sigma_k$) we have

a) t_k even sized Jordan blocks of sizes $2q_{k,1}, \dots, 2q_{k,t_k}$ with the corresponding structure inertia indices $i(-1)^{q_{k,1}+1}\beta_{k,1}, \dots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$ for σ_k and $i(-1)^{q_{k,1}}\beta_{k,1}, \dots, i(-1)^{q_{k,t_k}}\beta_{k,t_k}$ for $-\sigma_k$;

b) w_k odd sized Jordan blocks of sizes $2u_{k,1}+1, \dots, 2u_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{u_{k,1}+1}, \dots, (-1)^{u_{k,w_k}+1}$;

c) z_k odd sized Jordan blocks of sizes $2v_{k,1}+1, \dots, 2v_{k,z_k}+1$ corresponding to the structure indices $(-1)^{v_{k,1}}, \dots, (-1)^{v_{k,z_k}}$.

The zero eigenvalue has $2c$ even sized Jordan blocks with sizes of $2x_1, 2x_1, \dots, 2x_c, 2x_c$ with corresponding structure inertia indices $i, -i, \dots, i, -i$, and $a + b$ odd sized Jordan blocks, where a of them have sizes $2g_1+1, \dots, 2g_a+1$ with the corresponding structure inertia indices $(-1)^{g_1+1}, \dots, (-1)^{g_a+1}$ and b of them have sizes $2h_1+1, \dots, 2h_b+1$ with the corresponding structure inertia indices $(-1)^{h_1}, \dots, (-1)^{h_b}$.

Proof. The proof is analogous to the proof of Theorem 17 using Lemmas 22 and 23. □

We have seen that the results for $\Sigma_{p,q}$-Hermitian and skew Hermitian matrices are quite similar, which was to be expected, since both classes have an algebra structure. In the next section we now study the canonical forms for matrices in the associated Lie group of $\Sigma_{p,q}$-unitary matrices.

5 $\Sigma_{p,q}$-unitary matrices

In the previous two sections we have studied structured Jordan canonical forms for $\Sigma_{p,q}$-Hermitian and skew Hermitian matrices. Both these classes have an algebra structure, the $\Sigma_{p,q}$-Hermitian matrices form a Jordan algebra and the $\Sigma_{p,q}$-skew Hermitian matrices a Lie algebra. The Lie group associated with these two algebras is the class of $\Sigma_{p,q}$-unitary matrices. In order to derive structured canonical forms for this group analogous to the results for the algebras, we can make use of the Cayley transformation.

Lemma 26 *If $\mathcal{A}$ is $\Sigma_{p,q}$-unitary and $1 \notin \Lambda(\mathcal{A})$ then the Cayley transformation of $\mathcal{B}$*

$$\mathcal{B} = \rho(\mathcal{A}) = (\mathcal{A} + I)(\mathcal{A} - I)^{-1} \tag{50}$$

is $\Sigma_{p,q}$-skew Hermitian. Conversely, if $\mathcal{A}$ is $\Sigma_{p,q}$-skew Hermitian then $\mathcal{B}$ as in (50) is $\Sigma_{p,q}$-unitary.

Proof. We only prove the result for the case that $\mathcal{A}$ is $\Sigma_{p,q}$-unitary. The other direction follows form the fact that $\rho(\rho(\mathcal{A})) = \mathcal{A}$.

Since $\mathcal{A}$ is $\Sigma_{p,q}$-unitary, $\Sigma_{p,q}\mathcal{A} = \mathcal{A}^{-H}\Sigma_{p,q}$. By this relation

$$\begin{aligned}
\Sigma_{p,q}\mathcal{B} &= \Sigma_{p,q}(\mathcal{A}+I)(\mathcal{A}-I)^{-1} = (\mathcal{A}^{-H}+I)\Sigma_{p,q}(\mathcal{A}-I)^{-1} \\
&= (\mathcal{A}^{-H}+I)(\mathcal{A}^{-H}-I)^{-1}\Sigma_{p,q} = (I+\mathcal{A}^{H})(\mathcal{A}^{-H})(\mathcal{A}^{H})(I-\mathcal{A}^{H})^{-1}\Sigma_{p,q} \\
&= (\mathcal{A}+I)^{H}(I-\mathcal{A})^{-H}\Sigma_{p,q} = -\mathcal{B}^{H}\Sigma_{p,q} = -(\Sigma_{p,q}\mathcal{B})^{H}.
\end{aligned}$$

Therefore $\mathcal{B}$ is $\Sigma_{p,q}$-skew Hermitian. □

Using the Cayley transformation ρ the canonical forms of $\Sigma_{p,q}$-unitary matrices (if 1 is not an eigenvalue) can be easily obtained from the canonical form of the corresponding $\Sigma_{p,q}$-skew Hermitian matrix discussed in Section 4. However, if we Cayley transform the canonical form it is usually not a canonical form again and we need further reductions to obtain again the canonical form. But, obviously it suffices to further reduce each Jordan block separately. Before discussing these reductions, we first split the Jordan structure of a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$ into two parts, the part related to the eigenvalue 1 and the rest.

Lemma 27 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix that has 1 as an eigenvalue. Then, there exists a nonsingular matrix $\mathcal{Y}$, such that*

$$\mathcal{Y}^{H}\mathcal{Y} = \mathrm{diag}(\Sigma_{p_1,q_1}, \Sigma_{p_2,q_2}), \quad \mathcal{Y}^{-1}\mathcal{G}\mathcal{Y} = \mathrm{diag}(\mathcal{G}_1, \mathcal{G}_2),$$

where $p_1+p_2=p, q_1+q_2=q$, $\mathcal{G}_1$ is Σ_{p_1,q_1}-unitary with $1 \notin \Lambda(\mathcal{G}_1)$ and $\mathcal{G}_2$ is Σ_{p_2,q_2}-unitary and has 1 as only eigenvalue.

Furthermore, if $\mathcal{G}$ is real, then $\mathcal{Y}$ can be chosen real, so that also $\mathcal{G}_1, \mathcal{G}_2$ are real.

Proof. Let $\hat{\mathcal{Y}}$ be a nonsingular matrix such that

$$\mathcal{G}\hat{\mathcal{Y}} = \hat{\mathcal{Y}}\,\mathrm{diag}(\hat{\mathcal{G}}_1, \hat{\mathcal{G}}_2) = \hat{\mathcal{Y}}\hat{\mathcal{G}},$$

with $1 \notin \Lambda(\hat{\mathcal{G}}_1)$ and $\Lambda(\hat{\mathcal{G}}_2) = \{1\}$. Then we have $\hat{\mathcal{Y}}^{H}\mathcal{G}^{H} = \hat{\mathcal{G}}^{H}\hat{\mathcal{Y}}^{H}$ and, using the $\Sigma_{p,q}$-unitarity of $\mathcal{G}$ we have the discrete Lyapunov (or Stein) equation

$$\hat{\mathcal{G}}^{H}(\hat{\mathcal{Y}}^{H}\Sigma_{p,q}\hat{\mathcal{Y}})\hat{\mathcal{G}} = \hat{\mathcal{Y}}^{H}\Sigma_{p,q}\hat{\mathcal{Y}}. \tag{51}$$

By the diagonal block form of $\hat{\mathcal{G}}$ and the eigenvalue splitting, the solution of (51) has also block diagonal form, i.e., $\hat{\mathcal{Y}}^{H}\Sigma_{p,q}\hat{\mathcal{Y}} = \mathrm{diag}(T_1, T_2)$. Note that $\hat{\mathcal{Y}}^{H}\Sigma_{p,q}\hat{\mathcal{Y}}$ as well as T_1, T_2 are nonsingular Hermitian. Therefore, there exist nonsingular matrices Z_1, Z_2 such that

$$Z_1^{H}T_1Z_1 = \Sigma_{p_1,q_1}, \qquad Z_2^{H}T_2Z_2 = \Sigma_{p_2,q_2}.$$

To finish the proof, we set $\mathcal{Y} = \hat{\mathcal{Y}}\,\mathrm{diag}(Z_1, Z_2)$, $\mathcal{G}_1 = Z_1^{-1}\hat{\mathcal{G}}_1Z_1$ and $\mathcal{G}_2 = Z_2^{-1}\hat{\mathcal{G}}_2Z_2$.

The real case is clear, since 1 is a real eigenvalue. □

It is well known, that Cayley transformation directly leads to a rational relationship between the eigenvalues, i.e., if $\gamma \neq 1$ is an eigenvalue of a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$, then $\lambda = \rho(\gamma) = \frac{\gamma+1}{\gamma-1}$ is an eigenvalue of the Cayley transformation $\rho(\mathcal{G})$ and we have the following well-known facts.

Proposition 4 *Let $\mathcal{G}$ be $\Sigma_{p,q}$-unitary with $1 \notin \Lambda(\mathcal{G})$. Set $\mathcal{C} = \rho(\mathcal{G})$ and let $\gamma \in \Lambda(\mathcal{G})$ and $\lambda = \rho(\gamma) \in \Lambda(\mathcal{C})$. Then*

i) $\lambda \neq 1, -1$.

ii) γ *and* λ *have the same algebraic and geometric multiplicities.*

iii) $|\gamma| = 1$ *if and only if* λ *is purely imaginary.*

iv) If $\lambda \in \Lambda(\mathcal{C})$ is not purely imaginary, then $-\overline{\lambda} = \rho(\overline{\gamma}^{-1})$ and, furthermore, $\lambda, -\overline{\lambda} \in \Lambda(\mathcal{C})$ if and only if $\gamma, \overline{\gamma}^{-1} \in \Lambda(\mathcal{G})$.

In order to further reduce Cayley transformed Jordan blocks we need the following result.

Lemma 28 *Let $N_r(\lambda)$ be a Jordan block with $\lambda \neq 1$ and let $\gamma = \rho(\lambda)$. Then there exists a nonsingular upper triangular matrix X_r such that*

$$X_r^{-1}\rho(N_r(\lambda))X_r = N_r(\gamma),$$

and $e_r^T X_r e_r \neq 0$.

Proof. See, e.g., [14]. □

We are now prepared to present block by block the transformations of the results in Section 4.

Lemma 29 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $N(\gamma) = \gamma I + N$ with $N = \operatorname{diag}(N_{r_1}, \ldots, N_{r_s})$ be the Jordan structure of $\mathcal{G}$ corresponding to $\gamma \in \Lambda(\mathcal{G})$ with $|\gamma| \neq 1$. Then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^H & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N(\gamma) & 0 \\ 0 & N(\overline{\gamma})^{-1} \end{bmatrix} \tag{52}$$

and $\overline{\gamma}^{-1} \in \Lambda(\mathcal{G})$ has the same algebraic and geometric multiplicities as γ.

If $\mathcal{G}$ is real then we have two cases:

i) If γ is real then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \\ \hat{P}_N^T & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N(\gamma) & 0 \\ 0 & (N(\gamma))^{-1} \end{bmatrix}.$$

ii) If γ is nonreal then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_N \otimes \Sigma_{1,1} \\ \hat{P}_N^T \otimes \Sigma_{1,1} & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N(\Gamma) & 0 \\ 0 & (N(\Gamma))^{-1} \end{bmatrix},$$

with $\Gamma = \begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma \\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma \end{bmatrix}.$

Proof. We may assume without loss of generality that $1 \notin \Lambda(\mathcal{G})$. Otherwise by Lemma 27 we can consider the smaller size matrix $\mathcal{G}_1$. If ρ is the Cayley transformation, then by Lemma 26, $\mathcal{C} = \rho(\mathcal{G})$ is $\Sigma_{p,q}$-skew Hermitian. Furthermore, $\lambda = \rho(\gamma) \in \Lambda(\mathcal{C})$ and by Proposition 4 ii), iv), λ is not purely imaginary and the associated Jordan structure associated with λ is $\lambda I + N$. Applying Lemma 18 there exists a matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & P_N \\ P_N^H & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U} \begin{bmatrix} N(\lambda) & 0 \\ 0 & N(-\overline{\lambda}) \end{bmatrix}.$$

With $\tilde{U} = \hat{U}\,\mathrm{diag}(I, P_N^{-1})$ and, since $P_N N P_N^H = -N^H$, we have

$$\tilde{U}^H \Sigma_{p,q} \tilde{U} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathcal{C}\tilde{U} = \tilde{U} \begin{bmatrix} N(\lambda) & 0 \\ 0 & -(N(\lambda))^H \end{bmatrix}.$$

Using the Cayley transformation then we have

$$\mathcal{G}\tilde{U} = \tilde{U} = \begin{bmatrix} \rho(N(\lambda)) & 0 \\ 0 & \rho(-(N(\lambda))^H) \end{bmatrix}.$$

Note that

$$\begin{aligned} \rho(-N(\lambda))^H) &= (-N(\lambda)^H + I)(-N(\lambda)^H - I)^{-1} \\ &= \{(N(\lambda) - I)(N(\lambda) + I)^{-1})\}^H \\ &= \{\rho(N(\lambda))\}^{-H}. \end{aligned}$$

Applying Lemma 28, there exists a nonsingular matrix $X = \mathrm{diag}(X_{r_1}, \ldots, X_{r_s})$ such that $X^{-1}\rho(N(\lambda))X = N(\gamma)$. Obviously $X^H\{\rho(N(\lambda))\}^{-H}X^{-H} = N(\gamma)^{-H}$.
Setting $V = \tilde{U}\,\mathrm{diag}(X, X^{-H})$ we obtain

$$V^H \Sigma_{p,q} V = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathcal{G}V = V \begin{bmatrix} N(\gamma) & 0 \\ 0 & N(\gamma)^{-H} \end{bmatrix} \tag{53}$$

and taking $U = V\,\mathrm{diag}(I, \hat{P}_N)$ finishes the proof in the complex case..

Since the Cayley transformation of a real matrix is also real, we can apply Lemma 18 to get the result for the real case. □

This result shows for the eigenvalues of a $\Sigma_{p,q}$-unitary matrix that are not of modulus 1, the structured canonical form cannot be of the form of a usual Jordan matrix, only half of these eigenvalues have the classical Jordan structure, while for the other half of the eigenvalues we have to involve the inverses of Jordan blocks.

For eigenvalues with $|\gamma| = 1$ the canonical structure is even more complicated. If we restrict the chains of root vectors to have the proper structures coming form a $\Sigma_{p,q}$-skew Hermitian matrices as in Lemma 19 then no Jordan block will appear in the canonical form. We can do further reductions for which we will need the following simple result.

Lemma 30 *Given a vector $t = [t_1, \ldots, t_r]^T$ and $t_r \neq 0$ then there exists a nonsingular upper triangular Toeplitz matrix T such that $T^{-1}t = e_r$.*

Proof. See [14]. □

We now study the reduction of Cayley transformed blocks arising form unimodular eigenvalues.

Lemma 31 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $\gamma \in \Lambda(\mathcal{G})$ with $|\gamma| = 1$ and $\gamma \neq 1$. Let $N_r(\gamma)$ be a single Jordan block, then there exists a full rank matrix U such that*

$$U^H \Sigma_{p,q} U = \hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(\gamma) & i\beta e_s e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & N_s(\overline{\gamma})^{-1} \end{bmatrix}, \tag{54}$$

if $r = 2s$ and

$$\begin{aligned} U^H \Sigma_{p,q} U &= \beta \hat{P}_r, \\ \mathcal{G}U &= U \begin{bmatrix} N_s(\gamma) & \gamma e_s & \frac{\gamma}{1-\gamma} e_s e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & \gamma & -e_1^H N_s(\overline{\gamma})^{-1} \\ 0 & 0 & N_s(\overline{\gamma})^{-1} \end{bmatrix}, \end{aligned} \tag{55}$$

if $r = 2s + 1$.

Here $\beta = (-1)^s i\pi$ with $\pi \in \{\pm i\}$ if $r = 2s$ and $\beta = (-1)^{s+1}\pi$, $\pi \in \{\pm 1\}$ if $r = 2s+1$ where π is the structure inertia index of the corresponding eigenvalue $\lambda = \rho(\gamma)$.

If $\mathcal{G}$ is real then we have two cases:

i) If $\gamma \neq -1$, then with $\Gamma = \begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma \\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma \end{bmatrix}$, $\hat{P}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $S(\gamma) = -\frac{1}{2} \begin{bmatrix} 1 & \frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} \\ \frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} & -1 \end{bmatrix}$, there exists a real full rank matrix U such that if $r = 2s$, then

$$U^T \Sigma_{p,q} U = \hat{P}_r \otimes \Sigma_{1,1}, \quad \mathcal{G}U = U \begin{bmatrix} N_s(\Gamma) & -\beta \begin{bmatrix} 0 & 0 \\ \hat{P}_2 & 0 \end{bmatrix} N_s(\Gamma)^{-1} \\ 0 & N_s(\Gamma)^{-1} \end{bmatrix} \tag{56}$$

and if $r = 2s + 1$, then

$$\begin{aligned} U^T \Sigma_{p,q} U &= \beta \begin{bmatrix} 0 & 0 & \hat{P}_s \otimes \Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_s \otimes \Sigma_{1,1} & 0 & 0 \end{bmatrix}, \\ \mathcal{G}U &= U \left[\begin{array}{c|c|c} N_s(\Gamma) & \begin{matrix} 0 \\ \Gamma \end{matrix} & \begin{bmatrix} 0 & 0 \\ S(\gamma) & 0 \end{bmatrix} N_s(\Gamma)^{-1} \\ \hline 0 & \Gamma & [-\Sigma_{1,1}, 0] N_s(\Gamma)^{-1} \\ \hline 0 & 0 & N_s(\Gamma)^{-1} \end{array}\right]. \end{aligned} \tag{57}$$

ii) If $\gamma = -1$, then there exists a real full rank matrix U such that

$$U^T \Sigma_{p,q} U = \begin{bmatrix} 0 & \hat{P}_r \\ \hat{P}_r^T & 0 \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N_r(-1) & 0 \\ 0 & N_r(-1)^{-1} \end{bmatrix}, \tag{58}$$

if r is even and

$$U^T \Sigma_{p,q} U = \beta \hat{P}_r, \quad \mathcal{G}U = U \begin{bmatrix} N_s(-1) & -e_s & -\frac{1}{2} e_s e_1^T N_s(-1)^{-1} \\ 0 & -1 & -e_1^T N_s(-1)^{-1} \\ 0 & 0 & N_s(-1)^{-1} \end{bmatrix}, \tag{59}$$

if $r = 2s+1$. Here $\beta = (-1)^{s+1}\pi$ and π is the structure inertia index of 0 *corresponding to $\rho(\mathcal{G})$.*

Proof. We may again assume without loss of generality that $1 \notin \Lambda(\mathcal{G})$ and set $\mathcal{C} = \rho(\mathcal{G})$. By Proposition 4 the corresponding $\lambda = \rho(\gamma)$ now is purely imaginary, and $\mathcal{C}$ has the Jordan block $\lambda I + N_r$. Applying Lemma 19 there exists a matrix $\hat{U}$ such that

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \pi P_r, \quad \mathcal{C}\hat{U} = \hat{U} N_r(\lambda). \tag{60}$$

If $r = 2s$ then $\pi \in \{\pm i\}$ and we partition

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & \pi P_s \\ (\pi P_s)^H & 0 \end{bmatrix}, \quad N_r(\lambda) = \begin{bmatrix} N_s(\lambda) & e_s e_1^H \\ 0 & N_s(\lambda) \end{bmatrix}.$$

Applying the Cayley transformation we obtain

$$\mathcal{G}\hat{U} = \hat{U}\rho(N_r(\lambda)).$$

Using the notation $\hat{N}_s(\gamma) = \rho(N_s(\lambda))$ and the property that $(N_s(\lambda) - I)^{-1} = \frac{1}{2}(\hat{N}_s(\gamma) - I)$ we obtain

$$\rho(N_r(\lambda)) = \begin{bmatrix} \hat{N}_s(\gamma) & \frac{1}{2}(I - \hat{N}_s(\gamma))e_s e_1^H(\hat{N}_s(\gamma) - I) \\ 0 & \hat{N}_s(\gamma) \end{bmatrix}.$$

Setting $\tilde{U} = \hat{U}\,\mathrm{diag}(I_s, (\pi P_s)^{-1})$, then

$$\tilde{U}^H \tilde{U} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad \mathcal{G}\tilde{U} = \tilde{U} \begin{bmatrix} \hat{N}_s(\gamma) & \frac{i\beta}{2}(I - \hat{N}_s(\gamma))e_s e_s^H \hat{N}_s(\gamma)^{-H}(I - \hat{N}_s(\gamma))^H \\ 0 & \hat{N}_s(\gamma)^{-H} \end{bmatrix}.$$

By Lemma 27, there exists a nonsingular upper triangular matrix X such that $X^{-1}\hat{N}_s(\gamma)X = N_s(\gamma)$. Since the last component of $t := X^{-1}(\frac{\sqrt{2}}{2}e_s)$ is nonzero, by Lemma 29 there exists a nonsingular upper triangular Toeplitz matrix T such that $T^{-1}t = e_s$. Setting $Y = X(I - N_s(\gamma))T$ and $U = \tilde{U}\,\mathrm{diag}(Y, Y^{-H}\hat{P}_s)$, we obtain (54), since $(I - N_s(\gamma))T$ commutes with $N_s(\gamma)$ and $\hat{P}_s^{-1} N_s(\gamma)^{-H} \hat{P}_s = N_s(\overline{\gamma})^{-1}$.

If r is odd, following (60) we partition

$$\hat{U}^H \Sigma_{p,q} \hat{U} = \begin{bmatrix} 0 & 0 & \pi P_s \\ 0 & \beta & 0 \\ (\pi P_s)^H & 0 & 0 \end{bmatrix}, \quad N_r(\lambda) = \begin{bmatrix} N_s(\lambda) & e_s & 0 \\ 0 & \lambda & e_1^H \\ 0 & 0 & N_s(\lambda) \end{bmatrix}$$

and as before we obtain

$$\mathcal{G}\hat{U} = \hat{U} \begin{bmatrix} \hat{N}_s(\gamma) & \frac{\gamma-1}{2}(I - \hat{N}_s(\gamma))e_s & \frac{1-\gamma}{4}(I - \hat{N}_s(\gamma))e_s e_1^H(\hat{N}_s(\gamma) - I) \\ 0 & \gamma & \frac{1-\gamma}{2}e_1^H(\hat{N}_s(\gamma) - I) \\ 0 & 0 & \hat{N}_s(\gamma) \end{bmatrix}.$$

With $\tilde{U} = \hat{U}\,\mathrm{diag}(I_{s+1}, (\pi P_s)^{-1})$ we then have

$$\tilde{U}^H \Sigma_{p,q} \tilde{U} = \begin{bmatrix} 0 & 0 & I_s \\ 0 & \beta & 0 \\ I_s & 0 & 0 \end{bmatrix},$$

$$\mathcal{C}\tilde{U} = \tilde{U} \begin{bmatrix} N_s(\lambda) & \frac{\gamma-1}{2}(I - \hat{N}_s(\gamma))e_s & \frac{\gamma-1}{4}\beta(I - \hat{N}_s(\gamma))e_s e_s^H \hat{N}_s(\gamma)^{-H}(I - \hat{N}_s(\gamma)^H) \\ 0 & \gamma & \beta\frac{\gamma-1}{2}e_s^H \hat{N}_s(\gamma)^{-H}(I - \hat{N}_s(\gamma)^H) \\ 0 & 0 & \hat{N}_s(\gamma)^{-H} \end{bmatrix}.$$

Setting $Y = \frac{1-\overline{\gamma}}{2} X(I - N_s(\gamma))T$ and $U = \tilde{U}\operatorname{diag}(Y, 1, Y^{-H}\beta\hat{P}_s)$, we have (55).

If $\mathcal{G}$ is real and $\gamma \neq -1$ then similarly we can use the transformation $\Psi\Phi$ to update the real forms (56) and (57) from (54) and (55), respectively. If $\gamma = -1$ then the corresponding eigenvalue of $\mathcal{C} = \rho(\mathcal{G})$ is 0. By Lemma 19 for an even size Jordan block of $\mathcal{C}$ there exists a real matrix $\hat{U}$ such that

$$\hat{U}^T\Sigma_{p,q}\hat{U} = \begin{bmatrix} 0 & P_r \\ P_r^T & 0 \end{bmatrix}, \quad \mathcal{C}\hat{U} = \hat{U}\begin{bmatrix} N_r & 0 \\ 0 & N_r \end{bmatrix}.$$

Proceeding as in Lemma 29 we obtain (58). For odd size Jordan blocks $\hat{U}$ in (60) can be chosen real and from (55) we obtain (59). □

So far we have restricted ourselves to the Jordan structure associated with eigenvalues not equal to 1. For the eigenvalue 1 we give a separate analysis.

Lemma 32 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $N_r(1)$ be a Jordan block of $\mathcal{G}$. Then there exists a full rank matrix U such that*

$$U^H\Sigma_{p,q}U = \hat{P}_r, \quad \mathcal{G}U = U\begin{bmatrix} N_s(1) & -i\beta e_s e_1^H N_s(1)^{-1} \\ 0 & N_s(1)^{-1} \end{bmatrix},$$

if $r = 2s$ and if $r = 2s+1$, then

$$U^H\Sigma_{p,q}U = \beta\hat{P}_r, \quad \mathcal{G}U = U\begin{bmatrix} N_s(1) & e_s & -\frac{1}{2}e_s e_1^H N_s(1)^{-1} \\ 0 & 1 & -e_1^H N_s(1)^{-1} \\ 0 & 0 & N_s(1)^{-1} \end{bmatrix}. \tag{61}$$

Here $\beta = (-1)^s i\pi$ with $\pi \in \{\pm i\}$ if $r = 2s$ and $\beta = (-1)^{s+1}\pi$ with $\pi \in \{\pm 1\}$ if $r = 2s+1$.

If $\mathcal{G}$ is real, then there exists a real matrix U such that

$$U^T\Sigma_{p,q}U = \hat{P}_{2r}, \quad \mathcal{G}U = U\begin{bmatrix} N_r(1) & 0 \\ 0 & N_r(1)^{-1} \end{bmatrix},$$

if r is even and if $r = 2s+1$ we have again (61).

Proof. By Lemma 27 we may assume without loss of generality that $\Lambda(\mathcal{G}) = \{1\}$. Otherwise we work on the small size matrix $\mathcal{G}_2$. We cannot use the Cayley transformation ρ but a different rational transformation $\hat{\rho}(z) = (1-z)(1+z)^{-1}$. If $\mathcal{A}$ is $\Sigma_{p,q}$-unitary then $\mathcal{B} = \hat{\rho}(\mathcal{A})$ is $\Sigma_{p,q}$-skew Hermitian and conversely. With this new transformation we obtain the proof analogous to the proof of Lemma 31. □

Using these results, we have the following structured canonical form.

Theorem 33 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix $\mathcal{G}$, let $\lambda_1, \ldots, \lambda_\mu$ be the pairwise different eigenvalues of modulus less than one and let $\sigma_1, \ldots, \sigma_\nu$ be the pairwise different eigenvalues of modulus one. Then there exists a nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \operatorname{diag}(R_c, R_c^i, R_u).$$

i) The diagonal blocks R_c, R_c^i, associated with eigenvalues not on the unit circle, are

$$R_c = \operatorname{diag}(H_1(\lambda_1), \ldots, H_\mu(\lambda_\mu)), \quad R_c^i = \operatorname{diag}(H_1(\overline{\lambda_1})^{-1}, \ldots, H_\mu(\overline{\lambda_\mu})^{-1}),$$

where for $k = 1, \ldots, \mu$ we have $H_k(\lambda_k) = \lambda_k I + H_k$, $H_k(\overline{\lambda_k}) = \overline{\lambda_k} I + H_k$ and $H_k = \operatorname{diag}(N_{p_{k,1}}, \ldots, N_{p_{k,s_k}})$.

ii) The diagonal block R_u associated with the unimodular eigenvalues are $R_u = \mathrm{diag}(M_1, \ldots, M_\nu)$, where for $k = 1, \ldots, \nu$, we have $M_k = \mathrm{diag}(A_{k,1}, \ldots, A_{k,t_k}; B_{k,1}, \ldots, B_{k,w_k})$.

Here for $j = 1, \ldots, t_k$ we have

$$A_{k,j} = \begin{bmatrix} N_{q_{k,1}}(\sigma_k) & i\delta_k \beta^e_{k,j} e_{q_{k,j}} e_1^H N_{q_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & N_{q_{k,j}}(\overline{\sigma_k})^{-1} \end{bmatrix},$$

with $\delta_k = 1$ if $\sigma_k \neq 1$ and $\delta_k = -1$ if $\sigma_k = 1$ and furthermore $\beta^e_{k,j} = (-1)^{p_{k,j}} i\pi^e_{k,j}$ with $\pi^e_{k,j} \in \{\pm i\}$.

Moreover, for $j = 1, \ldots, w_k$, we have

$$B_{k,j} = \begin{bmatrix} N_{r_{k,j}}(\sigma_k) & \sigma_k e_{r_{k,j}} & s(\sigma_k) e_{r_{k,j}} e_1^H N_{r_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & \sigma_k & -e_1^H N_{r_{k,j}}(\overline{\sigma_k})^{-1} \\ 0 & 0 & N_{r_{k,j}}(\overline{\sigma_k})^{-1} \end{bmatrix},$$

with $s(\sigma_k) = \frac{\sigma_k}{1-\sigma_k}$ if $\sigma_k \neq 1$ and $s(1) = -\frac{1}{2}$.

The matrix $\mathcal{U}$ has the form

$$\mathcal{U}^H \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^H & 0 & 0 \\ 0 & 0 & W_u \end{bmatrix},$$

with $W_c = \mathrm{diag}(\hat{P}_{H_1}, \ldots, \hat{P}_{H_\mu})$ and $W_u = \mathrm{diag}(W_1^u, \ldots, W_\nu^u)$, where for $k = 1, \ldots, \mu$ we have $\hat{P}_{H_k} = \mathrm{diag}(\hat{P}_{p_{k,1}}, \ldots, \hat{P}_{p_{k,s_k}})$ and for $k = 1, \ldots, \nu$ we have

$$W_k^u = \mathrm{diag}(\hat{P}_{2q_{k,1}}, \ldots, \hat{P}_{2q_{k,t_k}}; \beta^o_{k,1}\hat{P}_{2r_{k,1}+1}, \ldots, \beta^o_{k,w_k}\hat{P}_{2r_{k,w_k}+1}).$$

Here for $j = 1, \ldots, w_k$ we have $\beta^o_{k,j} = (-1)^{r_{k,j}+1}\pi^o_{k,j}$ with $\pi^o_{kj} \in \{\pm 1\}$.

Each eigenvalue λ_k ($\overline{\lambda_k}^{-1}$) has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$ and each unimodular eigenvalue σ_k has

a) t_k even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ corresponding to the structure inertia indices $(-1)^{q_{k,1}+1} i\beta^e_{k,1}, \ldots, (-1)^{q_{k,t_k}+1} i\beta^e_{k,t_k}$ and

b) w_k odd sized Jordan blocks of sizes $2r_{k,1}+1, \ldots, 2r_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{r_{k,1}+1}\beta^o_{k,1}, \ldots, (-1)^{r_{k,w_k}+1}\beta^o_{k,w_k}$.

Proof. The proof follows from Lemmas 29, 31 and 32. □

Note that the structure inertia indices actually arise through the Cayley transformation in the associated $\Sigma_{p,q}$-skew Hermitian matrices, but they inherently describe also the associated structure for the unimodular eigenvalues of $\mathcal{G}$.

In the real case the structure is again more complicated.

Theorem 34 *Let $\mathcal{G}$ be a real $\Sigma_{p,q}$-orthogonal matrix, let $\alpha_1, \ldots, \alpha_\eta$ be pairwise different real eigenvalues of modulus less than one, let $\lambda_1, \ldots, \lambda_\mu$ be pairwise different nonreal eigenvalues with positive imaginary parts of modulus less than one, and let $\gamma_1, \ldots, \gamma_\nu$ be pairwise different nonreal eigenvalues of modulus 1, also with positive imaginary parts. (Note that then also $\alpha_1^{-1}, \ldots, \alpha_\eta^{-1}$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}$; $\lambda_1^{-1}, \ldots, \lambda_\mu^{-1}$; $\overline{\lambda_1}^{-1}, \ldots, \overline{\lambda_\mu}^{-1}$, $\overline{\gamma_1}, \ldots, \overline{\gamma_\nu}$ and possibly also $-1, 1$ are eigenvalues.) Then there exists a real nonsingular matrix $\mathcal{U}$ such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \mathrm{diag}(R_c, R_c^i, R_u).$$

i) The blocks with index c, associated with eigenvalues not on the unit circle, are $R_c = \operatorname{diag}(\hat{R}_c, \tilde{R}_c)$ *and* $R_c^i = \operatorname{diag}(\hat{R}_c^i, \tilde{R}_c^i)$*, with*

$$
\begin{aligned}
\hat{R}_c &= \operatorname{diag}(K_1(\alpha_1), \ldots, K_\eta(\alpha_\eta)), \quad \hat{R}_c^i = \operatorname{diag}(K_1(\alpha_1)^{-1}, \ldots, K_\eta(\alpha_\eta)^{-1}), \\
\tilde{R}_c &= \operatorname{diag}(H_1(\Lambda_1), \ldots, H_\mu(\Lambda_\mu)), \quad \tilde{R}_c^i = \operatorname{diag}(H_1(\Lambda_1)^{-1}, \ldots, H_\mu(\Lambda_\mu)^{-1}),
\end{aligned}
$$

where for $k = 1, \ldots, \eta$ *we have* $K_k(\alpha_k) = \alpha_k I + K_k$ *and* $K_k = \operatorname{diag}(N_{f_{k,1}}, \ldots, N_{f_{k,l_k}})$ *and for* $k = 1, \ldots, \mu$ *we have* $H_k(\Lambda_k) = \operatorname{diag}(N_{p_{k,1}}(\Lambda_k), \ldots, N_{p_{k,s_k}}(\Lambda_k))$*, with* $\Lambda_k = \begin{bmatrix} \operatorname{Re}\lambda_k & \operatorname{Im}\lambda_k \\ -\operatorname{Im}\lambda_k & \operatorname{Re}\lambda_k \end{bmatrix}$.

ii) The block R_u*, associated with the unimodular eigenvalues, is* $R_u = \operatorname{diag}(M_1, \ldots, M_\nu, M_-, M_+)$ *with*

$$
\begin{aligned}
M_k &= \operatorname{diag}(A_{k,1}, \ldots, A_{k,t_k}; B_{k,1}, \ldots, B_{k,w_k}), \\
M_- &= \operatorname{diag}(A_1^-, \ldots, A_{t_-}^-, B_1^-, \ldots, B_{w_-}^-), \\
M_+ &= \operatorname{diag}(A_1^+, \ldots, A_{t_+}^+, B_1^+, \ldots, B_{w_+}^+).
\end{aligned}
$$

Here we have the following substructures:

a) For $j = 1, \ldots, t_k$

$$
A_{k,j} = \left[\begin{array}{c|c}
N_{q_{k,1}}(\Gamma_k) & -\beta_{k,j}^e \begin{bmatrix} 0 & 0 \\ \hat{P}_2 & 0 \end{bmatrix} N_{q_{k,j}}(\Gamma_k)^{-1} \\
\hline
0 & N_{q_{k,j}}(\Gamma_k)^{-1}
\end{array}\right],
$$

with $\beta_{k,j}^e = (-1)^{p_{k,j}} i \pi_{k,j}^e$ *and* $\pi_{k,j}^e \in \{\pm i\}$.

b) For $j = 1, \ldots, w_k$

$$
B_{k,j} = \left[\begin{array}{c|c||c}
N_{r_{k,j}}(\Gamma_k) & \begin{array}{c} 0 \\ \Gamma_k \end{array} & \begin{bmatrix} 0 & 0 \\ S(\Gamma_k) & 0 \end{bmatrix} N_{r_{k,j}}(\Gamma_k)^{-1} \\
\hline
0 & \Gamma_k & [-\Sigma_{1,1}, 0] N_{r_{k,j}}(\Gamma_k)^{-1} \\
\hline
0 & 0 & N_{r_{k,j}}(\Gamma_k)^{-1}
\end{array}\right],
$$

with

$$
\Gamma_k = \begin{bmatrix} \operatorname{Re}\gamma_k & \operatorname{Im}\gamma_k \\ -\operatorname{Im}\gamma_k & \operatorname{Re}\gamma_k \end{bmatrix}, \quad S(\Gamma_k) = -\frac{1}{2} \begin{bmatrix} 1 & \frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k} \\ \frac{\operatorname{Im}\gamma_k}{1-\operatorname{Re}\gamma_k} & -1 \end{bmatrix}.
$$

c) For $k = 1, \ldots, t_-$

$$
A_k^- = \begin{bmatrix} N_{q_k^-}(-1) & 0 \\ 0 & N_{q_k^-}(-1)^{-1} \end{bmatrix},
$$

d) For $k = 1, \ldots, w_-$

$$
B_k^- = \begin{bmatrix} N_{r_k^-}(-1) & -e_{r_k^-} & -\frac{1}{2} e_{r_k^-} e_1^T N_{r_k^-}(-1)^{-1} \\ 0 & -1 & -e_1^T N_{r_k^-}(-1)^{-1} \\ 0 & 0 & N_{r_k^-}(-1)^{-1} \end{bmatrix}.
$$

e) For $k = 1, \ldots, t_+$

$$A_k^+ = \begin{bmatrix} N_{q_k^+}(1) & 0 \\ 0 & N_{q_k^+}(1)^{-1} \end{bmatrix}.$$

f) For $k = 1, \ldots, w_+$

$$B_k^+ = \begin{bmatrix} N_{r_k^+}(1) & e_{r_k^+} & -\frac{1}{2} e_{r_k^+} e_1^T N_{r_k^+}(1)^{-1} \\ 0 & 1 & -e_1^T N_{r_k^+}(1)^{-1} \\ 0 & 0 & N_{r_k^+}(1)^{-1} \end{bmatrix}.$$

The matrix $\mathcal{U}$ *has the form*

$$\mathcal{U}^T \Sigma_{p,q} \mathcal{U} = \begin{bmatrix} 0 & W_c & 0 \\ W_c^T & 0 & 0 \\ 0 & 0 & W_u \end{bmatrix},$$

where

$$\begin{aligned} W_c &= \operatorname{diag}(\hat{W}_c, \tilde{W}_c), \quad W_u = \operatorname{diag}(W_1^u, \ldots, W_\nu^u, W_-^u, W_+^u), \\ \hat{W}_c &= \operatorname{diag}(\hat{P}_{K_1}, \ldots, \hat{P}_{k_\eta}), \quad \tilde{W}_c = \operatorname{diag}(\hat{P}_{H_1} \otimes \Sigma_{1,1}, \ldots, \hat{P}_{H_\mu} \otimes \Sigma_{1,1}) \end{aligned}$$

and as substructures we have for $k = 1, \ldots, \eta$ *that* $\hat{P}_{K_k} = \operatorname{diag}(\hat{P}_{f_{k,1}}, \ldots, \hat{P}_{f_{k,l_k}})$ *and for* $k = 1, \ldots, \mu$ *that* $\hat{P}_{H_k} = \operatorname{diag}(\hat{P}_{p_{k,1}}, \ldots, \hat{P}_{p_{k,s_k}})$.

The substructure for the blocks with index u *is as follows:*

1) For $j = 1, \ldots, w_k$ *we have*

$$\begin{aligned} W_k^u &= \operatorname{diag}\Big(\hat{P}_{2q_{k,1}} \otimes \Sigma_{1,1}, \ldots, \hat{P}_{2q_{k,t_k}} \otimes \Sigma_{1,1}; \beta_{k,1}^o \begin{bmatrix} 0 & 0 & \hat{P}_{r_{k,1}} \otimes \Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_{r_{k,1}}^T \otimes \Sigma_{1,1} & 0 & 0 \end{bmatrix}, \\ &\qquad \ldots, \beta_{k,w_k}^o \begin{bmatrix} 0 & 0 & \hat{P}_{r_{k,w_k}} \otimes \Sigma_{1,1} \\ 0 & I_2 & 0 \\ \hat{P}_{r_{k,w_k}}^T \otimes \Sigma_{1,1} & 0 & 0 \end{bmatrix}\Big), \end{aligned}$$

with $\beta_{k,j}^o = (-1)^{r_{k,j}+1} \pi_{k,j}^o$ *and* $\pi_{k,j}^o \in \{\pm 1\}$.

2) For $k = 1, \ldots, w_-$ *we have*

$$W_-^u = \operatorname{diag}(\hat{P}_{2q_1^-}, \ldots, \hat{P}_{2q_{t_-}^-}; \beta_1^- \hat{P}_{2r_1^-+1}, \ldots, \beta_{w_-}^- \hat{P}_{2r_{w_-}^-+1}),$$

with $\beta_k^- = (-1)^{r_k^-+1} \pi_k^-$ *and* $\pi_k^- \in \{\pm 1\}$.

3) For $k = 1, \ldots, w_+$ *we have*

$$W_+^u = \operatorname{diag}(\hat{P}_{2q_1^+}, \ldots, \hat{P}_{2q_{t_+}^+}; \beta_1^+ \hat{P}_{2r_1^++1}, \ldots, \beta_{w_+}^+ \hat{P}_{2r_{w_+}^++1}),$$

with $\beta_k^+ = (-1)^{r_k^++1} \pi_k^+$, *and* $\pi_k^+ \in \{\pm 1\}$.

Each real eigenvalue α_k (α_k^{-1}) *has* l_k *Jordan blocks of sizes* $f_{k,1},\ldots,f_{k,l_k}$ *and each eigenvalue* λ_k $(\overline{\lambda_k},\ \lambda_k^{-1},\ \overline{\lambda_k}^{-1})$ *has* s_k *Jordan blocks of sizes* $p_{k,1},\ldots,p_{k,s_k}$.

Each nonreal unimodular eigenvalue γ_k $(\overline{\gamma_k})$ *has* t_k *even sized Jordan blocks of sizes* $2q_{k,1},\ldots,2q_{k,t_k}$ *corresponding to the structure inertia indices* $(-1)^{q_{k,1}+1}i\beta^e_{k,1},\ldots,(-1)^{q_{k,t_k}+1}i\beta^e_{k,t_k}$ *and* w_k *odd sized Jordan blocks of sizes* $2r_{k,1}+1,\ldots,2r_{k,w_k}+1$ *corresponding to the structure inertia indices* $(-1)^{r_{k,1}+1}\beta^o_{k,1},\ldots,(-1)^{r_{k,w_k}+1}\beta^o_{k,w_k}$.

The eigenvalue -1 *has* $2t_-$ *even sized Jordan blocks of sizes* $q_1^-,q_1^-,\ldots,q_{t_-}^-,q_{t_-}^-$ *corresponding to the structure inertia indices* $i,-i,\ldots,i,-i$, *and* w_- *odd sized Jordan blocks of sizes* $2r_1^-+1,\ldots,2r_{w_-}^-+1$ *corresponding to the indices* $(-1)^{r_1^-+1}\beta_1^-,\ldots,(-1)^{r_{w_-}^-+1}\beta_{w_-}^-$.

The eigenvalue 1 *has* $2t_+$ *even sized Jordan blocks of sizes* $q_1^+,q_1^+\ldots,q_{t_+}^+,q_{t_+}^+$ *corresponding to the structure inertia indices* $i,-i,\ldots,i,-i$ *and* w_+ *odd size Jordan blocks of sizes* $2r_1^++1,\ldots,2r_{w_+}^++1$ *corresponding to the indices* $(-1)^{r_1^++1}\beta_1^+,\ldots,(-1)^{r_{w_+}^++1}\beta_{w_+}^+$.

Proof. The proof is analogous to the proof of Theorem 33. $\square$

Finally we discuss the canonical forms under $\Sigma_{p,q}$-unitary similarity transformations. To simplify the notation which is even more technical, we introduce for a nonzero scalar γ the blocks

$$N_r^+(\gamma)=\frac{1}{2}(N_r(\gamma)+N_r(\gamma)^{-H}),\quad N_r^-(\gamma)=\frac{1}{2}(N_r(\gamma)-N_r(\gamma)^{-H})$$

and similarly for a 2×2 real nonsingular matrix Γ we set

$$N_r^+(\Gamma)=\frac{1}{2}(N_r(\Gamma)+N_r(\Gamma)^{-T}),\quad N_r^-(\Gamma)=\frac{1}{2}(N_r(\Gamma)-N_r(\Gamma)^{-T}).$$

Lemma 35 *Let* $\mathcal{G}$ *be a* $\Sigma_{p,q}$*-unitary matrix and let* $N_r(\gamma)=\gamma I+N_r$ *be a Jordan block of* $\mathcal{G}$. *If* $|\gamma|\neq 1$, *then there exists a full rank matrix* U *such that*

$$U^H\Sigma_{p,q}U=\begin{bmatrix} I_r & 0\\ 0 & -I_r\end{bmatrix},\quad \mathcal{G}U=U\begin{bmatrix} N_r^+(\gamma) & -N_r^-(\gamma)\\ -N_r^-(\gamma) & N_r^+(\gamma)\end{bmatrix}. \tag{62}$$

If $\mathcal{G}$ *is real then we have two cases:*

i) If γ *is real then there exists a real full rank matrix* U *such that*

$$U^T\Sigma_{p,q}U=\begin{bmatrix} I_r & 0\\ 0 & -I_r\end{bmatrix},\quad \mathcal{G}U=U\begin{bmatrix} N_r^+(\gamma) & -N_r^-(\gamma)\\ -N_r^-(\gamma) & N_r^+(\gamma)\end{bmatrix}. \tag{63}$$

ii) If γ *is nonreal then there exists a real full rank matrix* U *such that*

$$U^T\Sigma_{p,q}U=\begin{bmatrix} I_{2r} & 0\\ 0 & -I_{2r}\end{bmatrix},\quad \mathcal{G}U=U\begin{bmatrix} N_r^+(\Gamma) & -N_r^-(\Gamma)\\ -N_r^-(\Gamma) & N_r^+(\Gamma)\end{bmatrix}, \tag{64}$$

with $\Gamma=\begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma\\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma\end{bmatrix}$.

Proof. By Lemma 29 for $N_r(\gamma)$ there exists a matrix $\hat{U}$ such that

$$\hat{U}^H\Sigma_{p,q}\hat{U}=\begin{bmatrix} 0 & \hat{P}_r\\ \hat{P}_r^H & 0\end{bmatrix},\quad \mathcal{G}\hat{U}=\hat{U}\begin{bmatrix} N_r(\gamma) & 0\\ 0 & N_r(\overline{\gamma})^{-1}\end{bmatrix}.$$

With $U = \hat{U}\,\mathrm{diag}(I_r, \hat{P}_r)\Upsilon_r$ we then have (62).

If $\mathcal{G}$ is real and γ is real, then $\hat{U}$ can be chosen real and hence also U is real and we have (63). If γ is nonreal, then by the real form in Lemma 29 we obtain (64). $\square$

Lemma 36 *Let $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix and let $N_r(\gamma)$ be a Jordan block associated with an eigenvalue γ of modulus one. Then there exists a full rank matrix U with the following properties:*

i) If $r = 2s$, then with $\beta = (-1)^s i\pi$, where $\pi \in \{\pm i\}$ we have

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 \\ 0 & -I_s \end{bmatrix},$$

$$\mathcal{G}U = U \begin{bmatrix} N_s^+(\gamma) + \frac{i\delta\beta}{2} e_s e_s^H N_s(\gamma)^{-H} & -N_s^-(\gamma) + \frac{i\delta\beta}{2} e_s e_s^H N_s(\gamma)^{-H} \\ -N_s^-(\gamma) - \frac{i\delta\beta}{2} e_s e_s^H N_s(\gamma)^{-H} & N_s^+(\gamma) - \frac{i\delta\beta}{2} e_s e_s^H N_s(\gamma)^{-H} \end{bmatrix},$$

where $\delta = 1$ if $\gamma \neq 1$ and $\delta = -1$ if $\gamma = 1$.

ii) If $r = 2s+1$, then with $\beta = (-1)^{s+1}\pi$, where $\pi \in \{\pm 1\}$ we have

$$U^H \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_s \end{bmatrix},$$

$$\mathcal{G}U = U \begin{bmatrix} N_s^+(\gamma) + \frac{\beta s(\gamma)}{2} e_s e_s^H N_s(\gamma)^{-H} & \frac{\sqrt{2}}{2}\gamma e_s & -N_s^-(\gamma) + \frac{\beta s(\gamma)}{2} e_s e_s^H N_s(\gamma)^{-H} \\ -\frac{\sqrt{2}}{2}\beta e_s^H N_s(\gamma)^{-H} & \gamma & -\frac{\sqrt{2}}{2}\beta e_s^H N_s(\gamma)^{-H} \\ -N_s^-(\gamma) - \frac{\beta s(\gamma)}{2} e_s e_s^H N_s(\gamma)^{-H} & -\frac{\sqrt{2}}{2}\gamma e_s & N_s^+(\gamma) - \frac{\beta s(\gamma)}{2} e_s e_s^H N_s(\gamma)^{-H} \end{bmatrix}.$$

where $s(\gamma) = \frac{\gamma}{1-\gamma}$ if $\gamma \neq 1$ and $s(1) = -\frac{1}{2}$.

If $\mathcal{G}$ is real then there exists a real matrix U with the following properties:

a) If $\gamma \neq \pm 1$ and $r = 2s$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N_s^+(\Gamma) + E_r & -N_s^-(\Gamma) + E_r \\ -N_s^-(\Gamma) - E_r & N_s^+(\Gamma) - E_r \end{bmatrix},$$

where $\Gamma = \begin{bmatrix} \operatorname{Re}\gamma & \operatorname{Im}\gamma \\ -\operatorname{Im}\gamma & \operatorname{Re}\gamma \end{bmatrix}$ *and* $E_r = \frac{\beta}{2}\begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix} N_s(\Gamma)^{-T}$.

b) If $\gamma \neq \pm 1$ and $r = 2s+1$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_{2s} & 0 & 0 \\ 0 & \beta I_2 & 0 \\ 0 & 0 & -I_{2s} \end{bmatrix},$$

$$\mathcal{G}U = U \left[\begin{array}{c|c|c} N_s^+(\Gamma) + \hat{E}_r & \begin{matrix} 0 \\ \frac{\sqrt{2}}{2}\Gamma \end{matrix} & -N_s^-(\Gamma) + \hat{E}_r \\ \hline -\frac{\sqrt{2}}{2}\beta[0, I_2]N_s(\Gamma)^{-T} & \Gamma & -\frac{\sqrt{2}}{2}\beta[0, I_2]N_s(\Gamma)^{-T} \\ \hline -N_s^-(\Gamma) - \hat{E}_r & \begin{matrix} 0 \\ -\frac{\sqrt{2}}{2}\Gamma \end{matrix} & N_s^+(\Gamma) - \hat{E}_r \end{array}\right],$$

where $\hat{E}_r = \frac{\beta}{2}\begin{bmatrix} 0 & 0 \\ 0 & \hat{S}(\gamma) \end{bmatrix} N_s(\Gamma)^{-T}$ *and* $\hat{S}(\gamma) = -\frac{1}{2}\begin{bmatrix} 1 & -\frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} \\ \frac{\operatorname{Im}\gamma}{1-\operatorname{Re}\gamma} & 1 \end{bmatrix}$.

c) If $\gamma = -1$ and $r = 2s$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N_r^+(-1) & -N_r^-(-1) \\ -N_r^-(-1) & N_r^-(-1) \end{bmatrix}.$$

d) If $\gamma = -1$ and $r = 2s+1$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_s \end{bmatrix},$$

$$\mathcal{G}U = U \begin{bmatrix} N_s^+(-1) - \frac{\beta}{4} e_s e_s^T N_s(-1)^{-T} & -\frac{\sqrt{2}}{2} e_s & -N_s^-(-1) - \frac{\beta}{4} e_s e_s^T N_s(-1)^{-T} \\ -\frac{\sqrt{2}}{2} \beta e_s^T N_s(-1)^{-T} & -1 & -\frac{\sqrt{2}}{2} \beta e_s^T N_s(-1)^{-T} \\ -N_s^-(-1) + \frac{\beta}{4} e_s e_s^T N_s(-1)^{-T} & \frac{\sqrt{2}}{2} e_s & N_s^+(-1) + \frac{\beta}{4} e_s e_s^T N_s(-1)^{-T} \end{bmatrix}.$$

e) If $\gamma = 1$ and $r = 2s$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix}, \quad \mathcal{G}U = U \begin{bmatrix} N_r^+(1) & -N_s^-(1) \\ -N_s^-(1) & N_r^+(1) \end{bmatrix}.$$

f) If $\gamma = 1$ and $r = 2s+1$, then

$$U^T \Sigma_{p,q} U = \begin{bmatrix} I_s & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -I_s \end{bmatrix},$$

$$\mathcal{G}U = U \begin{bmatrix} N_s^+(1) - \frac{\beta}{4} e_s e_s^T N_s(1)^{-T} & \frac{\sqrt{2}}{2} e_s & -N_s^-(1) - \frac{\beta}{4} e_s e_s^T N_s(1)^{-T} \\ -\frac{\sqrt{2}}{2} \beta e_s^T N_s(1)^{-T} & 1 & -\frac{\sqrt{2}}{2} \beta e_s^T N_s(1)^{-T} \\ -N_s^-(1) + \frac{\beta}{4} e_s e_s^T N_s(1)^{-T} & -\frac{\sqrt{2}}{2} e_s & N_s^+(1) + \frac{\beta}{4} e_s e_s^T N_s(1)^{-T} \end{bmatrix}.$$

Proof. The proof is analogous to the proof of Lemma 35, using the results in Lemmas 31 and 32. □

This finally brings us to the structured canonical forms under $\Sigma_{p,q}$-unitary or in the real case $\Sigma_{p,q}$-orthogonal transformations.

Theorem 37 *Ley $\mathcal{G}$ be a $\Sigma_{p,q}$-unitary matrix with pairwise distinct eigenvalues $\lambda_1, \ldots, \lambda_\mu$ of modulus less than one and pairwise distinct eigenvalues $\gamma_1, \ldots, \gamma_\nu$ of modulus one. Note that then also $\overline{\lambda_1}^{-1}, \ldots, \overline{\lambda_\mu}^{-1}$ are eigenvalues. Then there exists a $\Sigma_{p,q}$-unitary matrix $\mathcal{U}$, such that*

$$\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_u^+ & & T_u \\ T_c & & R_c & \\ & Y_u & & R_u^- \end{bmatrix}.$$

i) The blocks with index c, associated with the eigenvalues that do not have modulus one, have the form $R_c = \operatorname{diag}(R_1^c, \ldots, R_\mu^c)$ and $T_c = \operatorname{diag}(T_1^c, \ldots, T_\mu^c)$, where for $k = 1, \ldots, \mu$

$$R_k^c = \operatorname{diag}(N_{p_{k,1}}^+(\lambda_k), \ldots, N_{p_{k,s_k}}^+(\lambda_k)), \quad T_k^c = -\operatorname{diag}(N_{p_{k,1}}^-(\lambda_k), \ldots, N_{p_{k,s_k}}^-(\lambda_k)).$$

ii) The blocks with index u, associated with the unimodular eigenvalues, are

$$\begin{aligned} R_u^+ &= \operatorname{diag}(C_1,\ldots,C_\nu), \quad R_u^- = \operatorname{diag}(D_1,\ldots,D_\nu),\\ T_u &= \operatorname{diag}(F_1,\ldots,F_\nu), \quad Y_u = \operatorname{diag}(G_1,\ldots,G_\nu). \end{aligned}$$

Here for $k = 1,\ldots,\nu$ the blocks are

$$\begin{aligned} C_k &= \operatorname{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \operatorname{diag}(D_k^e, D_k^+, D_k^-),\\ F_k &= \operatorname{diag}(F_k^e, F_k^+, F_k^-), \quad G_k = \operatorname{diag}(G_k^e, G_k^+, G_k^-), \end{aligned}$$

and with $\delta_k = 1$ for $\gamma_k \neq 1$ and $\delta_k = -1$ if $\gamma_k = 1$ the substructures are

$$\begin{aligned}
C_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H},\\
&\qquad \ldots, N_{q_{k,t_k}}^+(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}),\\
D_k^e &= \operatorname{diag}(N_{q_{k,1}}^+(\gamma_k) - \frac{1}{2} i\delta_k\beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H},\\
&\qquad \ldots, N_{q_{k,t_k}}^+(\gamma_k) - \frac{1}{2} i\delta_k\beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H}),\\
F_k^e &= \operatorname{diag}(-N_{q_{k,1}}^-(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H},\\
&\qquad \ldots, -N_{q_{k,t_k}}^-(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H});\\
G_k^e &= -\operatorname{diag}(N_{q_{k,1}}^-(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,1} e_{q_{k,1}} e_{q_{k,1}}^H N_{q_{k,1}}(\gamma_k)^{-H},\\
&\qquad \ldots, N_{q_{k,t_k}}^-(\gamma_k) + \frac{1}{2} i\delta_k\beta_{k,t_k} e_{q_{k,t_k}} e_{q_{k,t_k}}^H N_{q_{k,t_k}}(\gamma_k)^{-H});\\
C_k^+ &= \operatorname{diag}(\begin{bmatrix} N_{u_{k,1}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,1}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} & \gamma_k \end{bmatrix},\\
&\qquad \ldots, \begin{bmatrix} N_{u_{k,w_k}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} & \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,w_k}} \\ -\frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} & \gamma_k \end{bmatrix}),\\
D_k^+ &= \operatorname{diag}(N_{u_{k,1}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H},\\
&\qquad \ldots, N_{u_{k,w_k}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H}),\\
F_k^+ &= -\operatorname{diag}(\begin{bmatrix} N_{u_{k,1}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} \\ \frac{\sqrt{2}}{2} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H} \end{bmatrix},\\
&\qquad \ldots, \begin{bmatrix} N_{u_{k,w_k}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} \\ \frac{\sqrt{2}}{2} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H} \end{bmatrix});\\
G_k^+ &= -\operatorname{diag}([N_{u_{k,1}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,1}} e_{u_{k,1}}^H N_{u_{k,1}}(\gamma_k)^{-H}, \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,1}}^H],\\
&\qquad \ldots, [N_{u_{k,1}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{u_{k,w_k}} e_{u_{k,w_k}}^H N_{u_{k,w_k}}(\gamma_k)^{-H}, \frac{\sqrt{2}}{2}\gamma_k e_{u_{k,w_k}}^H]);
\end{aligned}$$

$$
\begin{aligned}
C_k^- &= \mathrm{diag}(N_{v_{k,1}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H}, \\
&\quad \ldots, N_{v_{k,z_k}}^+(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H}), \\
D_k^- &= \mathrm{diag}(\begin{bmatrix} \gamma_k & \frac{\sqrt{2}}{2} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \\ -\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,1}} & N_{v_{k,1}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \end{bmatrix}, \\
&\quad \ldots, \begin{bmatrix} \gamma_k & \frac{\sqrt{2}}{2} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H} \\ -\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,z_k}} & N_{v_{k,z_k}}^+(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H} \end{bmatrix}), \\
F_k^- &= \mathrm{diag}([\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,1}}, -N_{v_{k,1}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H}], \\
&\quad \ldots, [\frac{\sqrt{2}}{2}\gamma_k e_{v_{k,z_k}}, -N_{v_{k,z_k}}^-(\gamma_k) - \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H}]), \\
G_k^- &= \mathrm{diag}(\begin{bmatrix} \frac{\sqrt{2}}{2} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \\ -N_{v_{k,1}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,1}} e_{v_{k,1}}^H N_{v_{k,1}}(\gamma_k)^{-H} \end{bmatrix}, \\
&\quad \ldots, \begin{bmatrix} \frac{\sqrt{2}}{2} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H} \\ -N_{v_{k,z_k}}^-(\gamma_k) + \frac{s(\gamma_k)}{2} e_{v_{k,z_k}} e_{v_{k,z_k}}^H N_{v_{k,z_k}}(\gamma_k)^{-H} \end{bmatrix}).
\end{aligned}
$$

In these formulas we have used $s(\gamma_k) = \frac{\gamma_k}{1-\gamma_k}$ *if* $\gamma_k \neq 1$ *and* $s(1) = -\frac{1}{2}$.

Each λ_k ($\overline{\lambda_k}^{-1}$) *has* s_k *Jordan blocks of sizes* $p_{k,1}, \ldots, p_{k,s_k}$. *For each unimodular eigenvalue* γ_k *we have*

- *a)* t_k *even sized Jordan blocks of sizes* $2q_{k,1}, \ldots, 2q_{k,t_k}$ *with the corresponding structure inertia indices* $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$;
- *b)* w_k *odd sized Jordan blocks of sizes* $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ *corresponding to the indices* $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$;
- *c)* z_k *odd sized Jordan blocks of sizes* $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ *corresponding to the indices* $(-1)^{v_{k,j}}, \ldots, (-1)^{v_{k,z_k}}$.

Proof. The proof follows from Lemmas 35 and 36. □

As our last result we present the real version of the structured canonical form of $\Sigma_{p,q}$-orthogonal matrices under $\Sigma_{p,q}$-orthogonal similarity transformations.

Theorem 38 *Let* $\mathcal{G}$ *be a real* $\Sigma_{p,q}$*-orthogonal matrix with pairwise distinct real eigenvalues* $\alpha_1, \ldots, \alpha_\eta$ *of modulus less than one, pairwise distinct nonreal eigenvalues* $\lambda_1, \ldots, \lambda_\mu$ *of modulus less than one with positive imaginary parts, and pairwise different nonreal eigenvalues* $\gamma_1, \ldots, \gamma_\nu$ *of modulus one also with positive imaginary parts. (Note that we then also have the eigenvalues* $\alpha_1^{-1}, \ldots, \alpha_\eta^{-1}$, $\overline{\lambda_1}, \ldots, \overline{\lambda_\mu}$, $\lambda_1^{-1}, \ldots, \lambda_\mu^{-1}$, $\overline{\lambda_1}^{-1}, \ldots, \overline{\lambda_\mu}^{-1}$ *and* $\overline{\gamma_1}, \ldots, \overline{\gamma_\nu}$ *as well as possibly* $-1, 1$.*)*

Then there exists a real $\Sigma_{p,q}$*-orthogonal matrix* $\mathcal{U}$ *such that*

$$
\mathcal{U}^{-1}\mathcal{G}\mathcal{U} = \begin{bmatrix} R_c & & T_c & \\ & R_u^+ & & T_u \\ T_c & & R_c & \\ & Y_u & & R_u^- \end{bmatrix}.
$$

i) The blocks with index c, associated with eigenvalues not on the unit circle, are split further as $R_c = \mathrm{diag}(\hat{R}_c, \tilde{R}_c)$ *and* $T_c = \mathrm{diag}(\hat{T}_c, \tilde{T}_c)$ *with*

$$\begin{aligned} \hat{R}_c &= \mathrm{diag}(\hat{R}_1^c, \ldots, \hat{R}_\eta^c), \quad \tilde{R}_c = \mathrm{diag}(\tilde{R}_1^c, \ldots, \tilde{R}_\mu^c), \\ \hat{T}_c &= \mathrm{diag}(\hat{T}_1^c, \ldots, \hat{T}_\eta^c), \quad \tilde{T}_c = \mathrm{diag}(\tilde{T}_1^c, \ldots, \tilde{T}_\mu^c) \end{aligned}$$

and for $k = 1, \ldots, \eta$ *we have*

$$\hat{R}_k^c = \mathrm{diag}(N_{f_{k,1}}^+(\alpha_k), \ldots, N_{f_{k,l_k}}^+(\alpha_k)), \quad \hat{T}_k^c = -\,\mathrm{diag}(N_{f_{k,1}}^-(\alpha_k), \ldots, N_{f_{k,l_k}}^-(\alpha_k)),$$

while for $k = 1, \ldots, \mu$

$$\tilde{R}_k^c = \mathrm{diag}(N_{p_{k,1}}^+(\Lambda_k), \ldots, N_{p_{k,s_k}}^+(\Lambda_k)), \quad \tilde{T}_k^c = -\,\mathrm{diag}(N_{p_{k,1}}^-(\Lambda_k), \ldots, N_{p_{k,s_k}}^-(\Lambda_k)).$$

ii) The blocks with index u, associated with the unimodular eigenvalues, are split further in real and nonreal eigenvalues, as

$$\begin{aligned} R_u^+ = \mathrm{diag}(C_1, \ldots, C_\nu, C_-, C_+), \quad R_u^- = \mathrm{diag}(D_1, \ldots, D_\nu, D_-, D_+), \\ T_u = \mathrm{diag}(F_1, \ldots, F_\nu, F_-, F_+), \quad Y_u = \mathrm{diag}(G_1, \ldots, G_\nu, G_-, G_+) \end{aligned}$$

and have for $k = 1, \ldots, \nu$ *the partitioning*

$$\begin{aligned} C_k &= \mathrm{diag}(C_k^e, C_k^+, C_k^-), \quad D_k = \mathrm{diag}(D_k^e, D_k^+, D_k^-), \\ F_k &= \mathrm{diag}(F_k^e, F_k^+, F_k^-), \quad G_k = \mathrm{diag}(G_k^e, G_k^+, G_k^-). \end{aligned}$$

In these blocks we have with

$$\begin{aligned} E_{k,j} &= \frac{1}{2}\beta_{k,j} \begin{bmatrix} 0 & 0 \\ 0 & J_1 \end{bmatrix} N_{q_{k,j}}(\Gamma_k)^{-T}, \quad \tilde{E}_{k,j} = \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} N_{v_{k,j}}(\Gamma_k)^{-T}, \\ \hat{E}_{k,j} &= \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & E_1 \end{bmatrix} N_{u_{k,j}}(\Gamma_k)^{-T}, \quad E_1 = -\frac{1}{2} \begin{bmatrix} 1 & -\frac{\mathrm{Im}\,\gamma_k}{1-\mathrm{Re}\,\gamma_k} \\ \frac{\mathrm{Im}\,\gamma_k}{1-\mathrm{Re}\,\gamma_k} & 1 \end{bmatrix}, \end{aligned}$$

the following substructures.

$$\begin{aligned} C_k^e &= \mathrm{diag}(N_{q_{k,1}}^+(\Gamma_k) + E_{k,1} \ldots, N_{q_{k,t_k}}^+(\Gamma_k) + E_{k,t_k}), \\ D_k^e &= \mathrm{diag}(N_{q_{k,1}}^+(\Gamma_k) - E_{k,1}, \ldots, N_{q_{k,t_k}}^+(\Gamma_k) - E_{k,t_k}), \\ F_k^e &= \mathrm{diag}(-N_{q_{k,1}}^-(\Gamma_k) + E_{k,1}, \ldots, -N_{q_{k,t_k}}^-(\Gamma_k) + E_{k,t_k}) \\ G_k^e &= -\,\mathrm{diag}(N_{q_{k,1}}^-(\Gamma_k) + E_{k,1}, \ldots, N_{q_{k,t_k}}^-(\Gamma_k) + E_{k,t_k}) \\ C_k^+ &= \mathrm{diag}(\left[\begin{array}{c|c} N_{u_{k,1}}^+(\Gamma_k) + \hat{E}_{k,1} & \begin{array}{c} 0 \\ \frac{\sqrt{2}}{2}\Gamma_k \end{array} \\ \hline [0, -\frac{\sqrt{2}}{2} I_2] N_{u_{k,1}}(\Gamma_k)^{-T} & \Gamma_k \end{array}\right], \\ &\qquad \ldots, \mathrm{diag}(\left[\begin{array}{c|c} N_{u_{k,w_k}}^+(\Gamma_k) + \hat{E}_{k,w_k} & \begin{array}{c} 0 \\ \frac{\sqrt{2}}{2}\Gamma_k \end{array} \\ \hline [0, -\frac{\sqrt{2}}{2} I_2] N_{u_{k,w_k}}(\Gamma_k)^{-T} & \Gamma_k \end{array}\right]), \\ D_k^+ &= \mathrm{diag}(N_{u_{k,1}}^+(\Gamma_k) - \hat{E}_{k,1}, \ldots, N_{u_{k,w_k}}^+(\Gamma_k) - \hat{E}_{k,w_k}), \end{aligned}$$

$$
\begin{aligned}
F_k^+ &= -\operatorname{diag}\left(\left[\frac{N_{u_{k,1}}^-(\Gamma_k)-\hat{E}_{k,1}}{[0,\frac{\sqrt{2}}{2}I_2]N_{u_{k,1}}(\Gamma_k)^{-T}}\right],\ldots,\left[\frac{N_{u_{k,w_k}}^-(\Gamma_k)-\hat{E}_{k,w_k}}{[0,\frac{\sqrt{2}}{2}I_2]N_{u_{k,w_k}}(\Gamma_k)^{-T}}\right]\right),\\
G_k^+ &= -\operatorname{diag}\left([N_{u_{k,1}}^-(\Gamma_k)+\hat{E}_{k,1}| \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}],\ldots,[N_{u_{k,w_k}}^-(\Gamma_k)+\hat{E}_{k,w_k}| \begin{matrix}0\\ \frac{\sqrt{2}}{2}\Gamma_k\end{matrix}]\right);\\
C_k^- &= \operatorname{diag}(N_{v_{k,1}}^+(\Gamma_k)-\tilde{E}_{k,1},\ldots,N_{v_{k,z_k}}^+(\Gamma_k)-\tilde{E}_{k,z_k}),\\
D_k^- &= \operatorname{diag}\left(\left[\begin{array}{c|c}\Gamma_k & [0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,1}}(\Gamma_k)^{-T}\\ \hline \begin{matrix}0\\ -\frac{\sqrt{2}}{2}\Gamma_k\end{matrix} & N_{v_{k,1}}^+(\Gamma_k)+\tilde{E}_{k,z_k}\end{array}\right],\right.\\
&\qquad \left.\ldots,\left[\begin{array}{c|c}\Gamma_k & [0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,z_k}}(\Gamma_k)^{-T}\\ \hline \begin{matrix}0\\ -\frac{\sqrt{2}}{2}\Gamma_k\end{matrix} & N_{v_{k,z_k}}^+(\Gamma_k)+\tilde{E}_{k,z_k}\end{array}\right]\right),\\
F_k^- &= \operatorname{diag}\left([\begin{matrix}0\\ \frac{\sqrt{2}}{2}I_2\end{matrix} \,|-N_{v_{k,1}}^-(\gamma_k)-\tilde{E}_{k,1}],\ldots,[\begin{matrix}0\\ \frac{\sqrt{2}}{2}I_2\end{matrix}\,|-N_{v_{k,z_k}}^-(\Gamma_k)-\tilde{E}_{k,z_k}]\right),\\
G_k^- &= \operatorname{diag}\left(\begin{bmatrix}[0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,1}}(\Gamma_k)^{-T}\\ -N_{v_{k,1}}^-+\tilde{E}_{k,1}\end{bmatrix},\ldots,\begin{bmatrix}[0,\frac{\sqrt{2}}{2}I_2]N_{v_{k,z_k}}(\Gamma_k)^{-T}\\ -N_{v_{k,z_k}}^-+\tilde{E}_{k,z_k}\end{bmatrix}\right).
\end{aligned}
$$

The blocks associated with eigenvalues $1,-1$ *are partitioned further as*

$$
\begin{aligned}
C_\pm &= \operatorname{diag}(C_\pm^e,C_\pm^+,C_\pm^-),\quad D_\pm=\operatorname{diag}(D_\pm^e,D_\pm^+,D_\pm^-),\\
F_\pm &= \operatorname{diag}(F_\pm^e,F_\pm^+,F_\pm^-),\quad G_\pm=\operatorname{diag}(G_\pm^e,G_\pm^+,G_\pm^-);
\end{aligned}
$$

and have with $E_k^\pm=\frac14 e_{g_k^\pm}e_{g_k^\pm}^TN_{g_k^\pm}(\pm1)^{-T}$ *and* $\hat{E}_k^\pm=\frac14 e_{h_k^\pm}e_{h_k^\pm}^TN_{h_k^\pm}(\pm1)^{-T}$ *the substructures*

$$
\begin{aligned}
C_\pm^e &= D_\pm^e=\operatorname{diag}(N_{2x_1^\pm}^+(\pm1),\ldots,N_{2x_{c_\pm}^\pm}^+(\pm1)),\\
F_\pm^e &= G_\pm^e=-\operatorname{diag}(N_{2x_1^\pm}^-(\pm1),\ldots,N_{2x_{c_\pm}^\pm}^-(\pm1));\\
C_\pm^+ &= \operatorname{diag}\left(\begin{bmatrix}N_{g_1^\pm}^+(\pm1)-E_1^\pm & \pm\frac{\sqrt{2}}{2}e_{g_1^\pm}\\ -\frac{\sqrt{2}}{2}e_{g_1^\pm}^TN_{g_1^\pm}(\pm1)^{-T} & \pm1\end{bmatrix},\ldots,\begin{bmatrix}N_{g_{a_\pm}^\pm}^+(\pm1)-E_{a_\pm}^\pm & \pm\frac{\sqrt{2}}{2}e_{g_{a_\pm}^\pm}\\ -\frac{\sqrt{2}}{2}e_{g_{a_\pm}^\pm}^TN_{g_{a_\pm}^\pm}(\pm1)^{-T} & \pm1\end{bmatrix}\right),\\
D_\pm^+ &= \operatorname{diag}(N_{g_1^\pm}^+(\pm1)+E_1^\pm,\ldots,N_{g_{a_\pm}^\pm}^+(\pm1)+E_{a_\pm}^\pm),\\
F_\pm^+ &= -\operatorname{diag}\left(\begin{bmatrix}N_{g_1^\pm}^-(\pm1)+E_1^\pm\\ \frac{\sqrt{2}}{2}e_{g_1^\pm}^TN_{g_1^\pm}(\pm1)^{-T}\end{bmatrix},\ldots,\begin{bmatrix}N_{g_{a_\pm}^\pm}^-(\pm1)+E_{a_\pm}^\pm\\ \frac{\sqrt{2}}{2}e_{g_{a_\pm}^\pm}^TN_{g_{a_\pm}^\pm}(\pm1)^{-T}\end{bmatrix}\right),\\
G_\pm^+ &= -\operatorname{diag}([N_{g_1^\pm}^-(\pm1)-E_1^\pm|\pm\frac{\sqrt{2}}{2}e_{g_1^\pm}],\ldots,[N_{g_{a_\pm}^\pm}^-(\pm1)-E_{a_\pm}^\pm|\pm\frac{\sqrt{2}}{2}e_{g_{a_\pm}^\pm}]),\\
C_\pm^- &= \operatorname{diag}(N_{h_1^\pm}^+(\pm1)+\hat{E}_1^\pm,\ldots,N_{h_{b_\pm}^\pm}^+(\pm1)+\hat{E}_{b_\pm}^\pm),\\
D_\pm^- &= \operatorname{diag}\left(\begin{bmatrix}\pm1 & \frac{\sqrt{2}}{2}e_{h_1^\pm}^TN_{h_1^\pm}(\pm1)^{-T}\\ \mp\frac{\sqrt{2}}{2}e_{h_1^\pm} & N_{h_1^\pm}^+(\pm1)-\hat{E}_1^\pm\end{bmatrix},\ldots,\begin{bmatrix}\pm1 & \frac{\sqrt{2}}{2}e_{h_{b_\pm}^\pm}^TN_{h_{b_\pm}^\pm}(\pm1)^{-T}\\ \mp\frac{\sqrt{2}}{2}e_{h_{b_\pm}^\pm} & N_{h_{b_\pm}^\pm}^+(\pm1)-\hat{E}_{b_\pm}^\pm\end{bmatrix}\right),
\end{aligned}
$$

$$
\begin{aligned}
F_{\pm}^{-} &= \mathrm{diag}([\pm\frac{\sqrt{2}}{2}e_{h_1^{\pm}}, -N_{h_1^{\pm}}^{-}(\pm 1)+\hat{E}_1^{\pm}], \ldots, [\pm\frac{\sqrt{2}}{2}e_{h_{b_{\pm}}^{\pm}}, -N_{h_{b_{\pm}}^{\pm}}^{-}(\pm 1)+\hat{E}_{b_{\pm}}^{\pm}]),\\
G_{\pm}^{-} &= \mathrm{diag}(\left[\begin{array}{c} \frac{\sqrt{2}}{2}e_{h_1^{\pm}}^{T}N_{h_1^{\pm}}(\pm 1)^{-T} \\ -N_{h_1^{\pm}}^{-}(\pm 1)-\hat{E}_1^{\pm} \end{array}\right], \ldots, \left[\begin{array}{c} \frac{\sqrt{2}}{2}e_{h_{b_{\pm}}^{\pm}}^{T}N_{h_{b_{\pm}}^{\pm}}(\pm 1)^{-T} \\ -N_{h_{b_{\pm}}^{\pm}}^{-}(\pm 1)-\hat{E}_{b_{\pm}}^{\pm} \end{array}\right]).
\end{aligned}
$$

Each real eigenvalue α_k (α_k^{-1}) has l_k Jordan blocks of sizes $f_{k,1}, \ldots, f_{k,l_k}$ and each λ_k (λ_k^{-1}, $\overline{\lambda_k}$, $\overline{\lambda_k}^{-1}$) has s_k Jordan blocks of sizes $p_{k,1}, \ldots, p_{k,s_k}$.

Each nonreal unimodular eigenvalue γ_k ($\overline{\gamma_k}$) has

a) t_k even sized Jordan blocks of sizes $2q_{k,1}, \ldots, 2q_{k,t_k}$ with the corresponding structure inertia indices $i(-1)^{q_{k,1}+1}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}+1}\beta_{k,t_k}$ associated with γ_k and $i(-1)^{q_{k,1}}\beta_{k,1}, \ldots, i(-1)^{q_{k,t_k}}\beta_{k,t_k}$ associated with $\overline{\gamma_k}$;

b) w_k odd sized Jordan blocks of sizes $2u_{k,1}+1, \ldots, 2u_{k,w_k}+1$ corresponding to the structure inertia indices $(-1)^{u_{k,1}+1}, \ldots, (-1)^{u_{k,w_k}+1}$;

c) z_k odd sized Jordan blocks of sizes $2v_{k,1}+1, \ldots, 2v_{k,z_k}+1$ corresponding to the structure inertia indices $(-1)^{v_{k,1}}, \ldots, (-1)^{v_{k,z_k}}$.

The eigenvalue 1 has $2c_+$ even sized Jordan blocks of sizes $2x_1^+, 2x_1^+, \ldots, 2x_{c_+}^+, 2x_{c_+}^+$ corresponding to the structure inertia indices $i, -i, \ldots, i, -i$, and $a_+ + b_+$ odd sized Jordan blocks, a_+ of them of sizes $2g_1^+ + 1, \ldots, 2g_{a_+}^+ + 1$ with the corresponding structure inertia indices $(-1)^{g_1^+ +1}, \ldots, (-1)^{g_{a_+}^+ +1}$ and b_+ of them of sizes $2h_1^+ + 1, \ldots, 2h_{b_+}^+ + 1$ with the corresponding structure inertia indices $(-1)^{h_1^+}, \ldots, (-1)^{h_{b_+}^+}$.

Similarly, the eigenvalue -1 has $2c_-$ even sized Jordan blocks of sizes $2x_1^-, 2x_1^-, \ldots, 2x_{c_-}^-, 2x_{c_-}^-$ corresponding to the structure inertia indices $i, -i, \ldots, i, -i$, and $a_- + b_-$ odd sized Jordan blocks, a_- of them of sizes $2g_1^- + 1, \ldots, 2g_{a_-}^- + 1$ with the corresponding structure inertia indices $(-1)^{g_1^- +1}, \ldots, (-1)^{g_{a_-}^- +1}$ and b_- of them of sizes $2h_1^- + 1, \ldots, 2h_{b_-}^- + 1$ with the corresponding structure inertia indices $(-1)^{h_1^-}, \ldots, (-1)^{h_{b_-}^-}$.

Proof. The proof follows directly from Lemmas 35 and 36. □

6 Conclusion

We have presented real and complex structured Jordan canonical forms under real $\Sigma_{p,q}$-orthogonal and $\Sigma_{p,q}$-unitary matrices, respectively. Combining these results with the structured canonical forms for Hamiltonian, skew Hamiltonian and symplectic matrices in [14] a complete list of the possible structured canonical forms is available.

Actually by Remark 2 the structured Jordan canonical forms for groups of structure matrices such as complex $\Sigma_{p,q}$-symmetric, skew symmetric and orthogonal matrices, complex J symmetric, skew symmetric and orthogonal matrices, can be derived in a similar way, were $J = \left[\begin{array}{cc} 0 & I_n \\ -I_n & 0 \end{array}\right]$. We can also generalize these results to the cases that $\Sigma_{p,q}$ and J are replaced by general nonsingular Hermitian and skew Hermitian matrices, respectively. Due to the large amount of material that we have already presented we have refrained from presenting these results.

It is also possible to generalize all these results to the matrix pencil case with structures as it has been done for Hamiltonian pencils, and symplectic pencils in [14] and for skew Hamiltonian/Hamiltonian pencils in [15, 16]. This generalization can be done as follows: Suppose that for a matrix pencil $\mathcal{A}-\lambda\mathcal{B}$ with say $\mathcal{A}=\mathcal{A}^H$, $\mathcal{B}=\mathcal{B}^H$ the matrix $\mathcal{B}$ is invertible, then the matrix $\hat{A}=\mathcal{B}^{-1}\mathcal{A}$ satisfies $\mathcal{B}\hat{A}=\hat{A}^H\mathcal{B}$. So we can determine a nonsingular matrix $\mathcal{U}$ such that

$$\mathcal{U}^H\mathcal{B}\mathcal{U}=\mathcal{D}_b, \quad \mathcal{U}^{-1}\hat{\mathcal{A}}\mathcal{U}=\mathcal{D}_a.$$

Taking the product form of $\hat{\mathcal{A}}$ we have

$$\mathcal{U}^H\mathcal{B}\mathcal{U}=\mathcal{D}_b, \quad \mathcal{U}^H\mathcal{A}\mathcal{U}=\mathcal{D}_b\mathcal{D}_a,$$

which is just the result of Thompson [18] or Uhlig for the real case [20]. We can also easily obtain the canonical forms for all the pencils with $\mathcal{A}=\pm\mathcal{A}^H$, $\mathcal{B}=\pm\mathcal{B}^H$.

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