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PARTIAL FOURIER APPROXIMATION OF THE LAMÉ EQUATIONS IN AXISYMMETRIC DOMAINS

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Abstract

In this paper, we study the partial Fourier method for treating the Lamé equations in three-dimensional axisymmetric domains subjected to nonaxisymmetric loads. We consider the mixed boundary value problem of the linear theory of elasticity with the displacement $\hat{\mathbf{u}}$, the body force $\hat{\mathbf{f}} \in (L_2)^3$ and homogeneous Dirichlet and Neumann boundary conditions. The partial Fourier decomposition reduces, without any error, the three-dimensional boundary value problem to an infinite sequence of two-dimensional boundary value problems, whose solutions $\hat{\mathbf{u}}_n$ $(n = 0, 1, 2, \cdots)$ are the Fourier coefficients of $\hat{\mathbf{u}}$. This process of dimension reduction is described, and appropriate function spaces are given to characterize the reduced problems in two dimensions. The trace properties of these spaces on the rotational axis and some properties of the Fourier coefficients $\hat{\mathbf{u}}_n$ are proved, which are important for further numerical treatment, e.g. by the finite-element method. Moreover, generalized completeness relations are described for the variational equation, the stresses and the strains. The properties of the resulting system of two-dimensional problems are characterized. Particularly, a priori estimates of the Fourier coefficients $\hat{\mathbf{u}}_n$ and of the error of the partial Fourier approximation are given.

1 Introduction

The finite-element method has proved to be a very efficient and flexible numerical method for solving approximately problems in engineering and physics based on variational principles, see e.g. [1, 6, 21] . However, the application of the finite-element method for approximating three-dimensional boundary value problems (BVP), particularly in the theory of elasticity, involves the discretization of complex structures and the solution of large system of equations for which the cost, despite the advanced computational possibilities, may be extremly high. It is therefore still important to analyse approaches which simplify the solution process of three-dimensional problems, reduce the cost or admit an effective parallelization.
Considerable computational advantages can be achieved, if special geometrical and material properties of the elastic body are given and taken into account, as e.g. for axisymmetric solids subjected to nonaxisymmetric loads. Here, a dimension reduction can be obtained by applying partial Fourier analysis, which relies on the Fourier series expansion with respect to the rotational angle φ. By this means, the three-dimensional boundary value problem can be reduced to a sequence of two-dimensional boundary value problems, which do not depend on the rotational angle φ.
For the Lamé operator in three-dimensional axisymmetric domains with homogeneous and isotropic material properties, the reduced boundary value problems in two dimensions are posed on the meridian of the domain; they are decoupled and can be solved in parallel. In general, for boundary value problems in two dimensions, standard tools for pre- and post-processing as well as finite element algorithms and solvers are available and can be applied more easily than in three dimensions.

The application of the partial Fourier approximation (truncated partial Fourier series) for the dimension reduction, as a first step, and the subsequent discretization of the finite number of reduced problems by the finite-element method, as the second step, yields to the so-called Fourier-finite-element method. This method is often applied in engineering to approximate the solutions of boundary value problems in the theory of elasticity and in heat

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conduction, see e.g. [1, 3, 7, 9, 10, 12, 15, 16, 17, 18]; see also [4, 5, 11] for the approximating Fourier method. The mathematical framework of the Fourier-finite-element method for solving the Poisson equation and related problems is studied in [2, 8, 9, 10, 12, 18], with some extension to the Lamé equations in [14]. In papers of the engineering literature about the Fourier-finite-element method for the boundary value problems in the theory of elasticity (see e.g. [1, 3, 7, 15, 16, 17, 19, 20]) one finds mainly the practical implementation and experimental demonstration of the method. Particularly, further work is necessary to give some mathematical framework of the method also in the case of the Lamé equations.

In this paper, we consider the following model problem

$$\begin{aligned} -\mu\triangle\hat{\mathbf{u}}(\mathbf{x}) - (\lambda+\mu) grad\ div\ \hat{\mathbf{u}}(\mathbf{x}) &= \hat{\mathbf{f}}(\mathbf{x}) \quad \text{in} \quad \hat{\Omega}, \\ \hat{\mathbf{u}}(\mathbf{x}) &= \mathbf{0} \quad \text{on} \quad \hat{\Gamma}_D, \\ \sum_{j=1}^{3} \hat{\sigma}_{ij}(\hat{\mathbf{u}}(\mathbf{x}))\hat{\nu}_j &= 0 \quad (i=1,2,3) \quad \text{on} \quad \hat{\Gamma}_N, \end{aligned} \tag{1.1}$$

where $\hat{\mathbf{u}}$ is the displacement vector, $\hat{\mathbf{f}}$ the body force, λ and μ are the Lamé-coefficients, $\hat{\Gamma}_D$ and $\hat{\Gamma}_N$ are disjoint parts of the boundary $\hat{\Gamma}$ of $\hat{\Omega}$ with Dirichlet and Neumann conditions, respectively, $\hat{\sigma}_{ij}(\hat{\mathbf{u}})$ are the components of the stress tensor, and $\hat{\nu}_j$ the components of the unit outer normal on $\hat{\Gamma}_N$.

The main objective of this paper is to give the analytical framework of the partial Fourier decomposition and approximation for studying the boundary value problems of the linear theory of elasticity in three-dimensional axisymmetric domains with nonaxisymmetric data. Here, we describe the approach and define appropriate function spaces on the two-dimensional meridian domain Ω_a (Ω_a generates the three-dimensional axisymmetric domain $\hat{\Omega} \subset \mathbf{R}^3$) needed for the two-dimensional boundary value problems. Especially in the case where the boundary $\partial\Omega_a$ of Ω_a satisfies $\partial\Omega_a \cap \{r=0\} \neq \emptyset$ (i.e. $\hat{\Omega}$ contains the rotational axis), these spaces are weighted Sobolev spaces, where the powers of the radial coordinate r play the role of the weights.

In [3, 7, 15, 16] the trace properties of the Fourier coefficients on the rotational axis have been derived by mechanical considerations. In this paper, we shall confirm the assertions and give a rigorous mathematical proof for it.

Furthermore, functions over the three-dimensional domain $\hat{\Omega}$ are characterized by their Fourier coefficients via the generalized Parseval equations. The same is done for the functionals connected with the boundary value problem. Here we shall also give completeness relations which clarify the decomposition of function spaces related to $\hat{\Omega}$ into function spaces connected with the meridian domain Ω_a and which are needed to describe the behaviour of the Fourier coefficients. The properties of the two-dimensional boundary value problems determining the Fourier coefficients are studied, a priori estimates of the Fourier coefficients and of the error of the partial Fourier approximation are given, for $\hat{\mathbf{f}} \in (L_2(\hat{\Omega}))^3$ and without further regularity assumptions on the solution $\hat{\mathbf{u}}$.

The paper is divided into four Sections. In Section 2, the boundary value problem, its variational formulation as well as the corresponding function spaces are given, also in terms of cylindrical coordinates r, φ, z. In Section 3, the partial Fourier analysis is discussed and the corresponding function spaces on the plane meridian domain Ω_a of the axisymmetric domain $\hat{\Omega}$ are defined to characterize the two-dimensional problems for the Fourier coefficients. The trace properties of the solutions on the rotational axis are proved. Completeness relations are given to characterize the functions defined on $\hat{\Omega}$ by their Fourier coefficients on Ω_a. In Section 4, the dimension reduction of the three-dimensional problem as well as the associate boundary value problems in two dimensions are investigated and, finally, a priori estimates of their solutions and of the truncation error are given.

2 The variational formulation of the boundary value problem

Let $\hat{\Omega} \subset \mathbf{R}^3$ be a bounded domain, with Lipschitz continuous and piecewise twice continuously differentiable boundary $\hat{\Gamma} := \partial\hat{\Omega}$ ($\hat{\Gamma} \in C^{0,1} \cap PC^2$), and let $\hat{\Gamma}_D$ and $\hat{\Gamma}_N$ denote two disjoint open subsets of $\hat{\Gamma}$ such that $\hat{\Gamma} = \hat{\Gamma}_D \cup \hat{\Gamma}_N$ and meas$(\hat{\Gamma}_D) > 0$ (Lebesgue measure of $\hat{\Gamma}_D$). Let $(x_1,\ x_2,\ x_3)$ be the Cartesian coordinates of the point $x \in \mathbf{R}^3$. We assume that the domain $\hat{\Omega}$ is axisymmetric with respect to the x_3-axis, and that $\hat{\Omega} \setminus \Gamma_0$ is generated by rotation of a plane meridian domain Ω_a about the x_3-axis, where Γ_0 is the part of the x_3-axis contained in $\hat{\Omega}$. Define $\Gamma_a := \partial\Omega_a \setminus \bar{\Gamma}_0$ ($\bar{\Gamma}_0$: closure of Γ_0) and suppose, for simplicity, that Γ_0 and Γ_a form right angles at the points $P_i = \bar{\Gamma}_0 \cap \bar{\Gamma}_a$ $(i = 1, 2)$, and that Γ_a is straight-line in some neighbourhood of P_1 and P_2, see Figure 1. Thus, edges on $\partial\hat{\Omega}$ are admitted, but conical points of the domain $\hat{\Omega}$ are excluded.

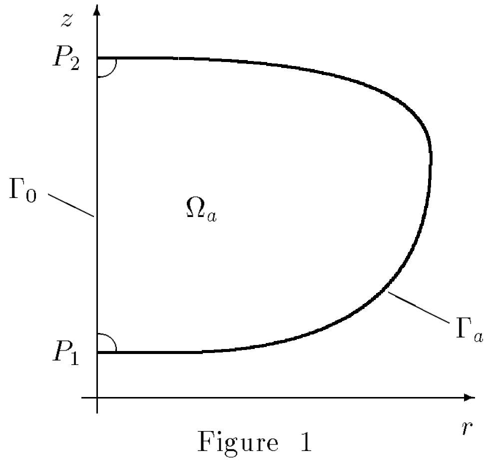

Figure 1

Let $W_2^k(\hat{\Omega})$ $(k = 0, 1, ...;\ W_2^0 = L_2)$ denote the usual Sobolev spaces, and let $\hat{W}(\hat{\Omega})$ be defined by $\hat{W}(\hat{\Omega}) := (W_2^1(\hat{\Omega}))^3$.
The variational solution of the boundary value problem (1.1) (cf. [6]) will be obtained by looking for $\hat{\mathbf{u}} \in \hat{V}_0(\hat{\Omega})$ such that

$$\hat{b}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \;=\; \hat{f}(\hat{\mathbf{v}}) \qquad \text{for any} \quad \hat{\mathbf{v}} \in \hat{V}_0(\hat{\Omega}), \tag{2.1}$$

where

$$\hat{b}(\hat{\mathbf{u}}, \hat{\mathbf{v}}) \;=\; \int_{\hat{\Omega}} \Big\{\lambda(div\,\hat{\mathbf{u}})(div\,\hat{\mathbf{v}}) + 2\mu \sum_{i,j=1}^{3} e_{ij}(\hat{\mathbf{u}})e_{ij}(\hat{\mathbf{v}})\Big\}dx,\ \ \hat{f}(\hat{\mathbf{v}}) \;=\; \int_{\hat{\Omega}} \sum_{i=1}^{3} \hat{f}_i\hat{v}_i dx \tag{2.2}$$

and

$$\hat{V}_0(\hat{\Omega}) \;=\; \{\hat{\mathbf{v}} \in \hat{W}(\hat{\Omega}) : \hat{\mathbf{v}} = \mathbf{0} \quad \text{on} \quad \hat{\Gamma}_D\}. \tag{2.3}$$

The components of the linearized strain tensor (deformation) are derived from the displacement vector field $\hat{\mathbf{u}}$ by

$$e_{ij}(\hat{\mathbf{u}}) = \frac{1}{2}\Big(\frac{\partial \hat{u}_i}{\partial x_j} + \frac{\partial \hat{u}_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3.$$

The corresponding components of the linearized stress tensor are, according to Hooke's law,

$$\hat{\sigma}_{ij}(\hat{\mathbf{u}}) = \lambda\Big(\sum_{k=1}^{3} e_{kk}(\hat{\mathbf{u}})\Big)\delta_{ij} + 2\mu e_{ij}(\hat{\mathbf{u}}), \quad i, j = 1, 2, 3, \tag{2.4}$$

where δ_{ij} denotes the Kronecker symbol, and λ, μ ($\lambda, \mu > 0$) are the Lamé coefficients. For the proof of the existence and uniqueness of the variational solution $\hat{\mathbf{u}} \in \hat{V}_0(\hat{\Omega})$ of (2.1), one employs Korn's inequality (cf. [13]),

$$\int_{\hat{\Omega}} \sum_{i,j=1}^{3} e_{ij}(\hat{\mathbf{v}}) e_{ij}(\hat{\mathbf{v}}) dx \geq C \|\hat{\mathbf{v}}\|^2_{(W_2^1(\hat{\Omega}))^3} \quad \text{for} \quad \hat{\mathbf{v}} \in \hat{V}_0(\hat{\Omega}). \tag{2.5}$$

Remark 2.1. For $\operatorname{meas} \Gamma_D > 0$, the mapping

$$\|\hat{\mathbf{v}}\| = \Big(\int_{\hat{\Omega}} \sum_{i,j=1}^{3} |e_{ij}(\hat{\mathbf{v}})|^2 dx \Big)^{1/2}$$

defines a norm on $\hat{V}_0(\hat{\Omega})$ which is equivalent to the norm $\|\hat{\mathbf{v}}\|_{\hat{V}_0(\hat{\Omega})} := \|\hat{\mathbf{v}}\|_{(W_2^1(\hat{\Omega}))^3}$, cf. [6]. That is, there are constants C_1 and C_2 such that the following inequalities hold,

$$C_1 \|\hat{\mathbf{v}}\|_{\hat{V}_0(\hat{\Omega})} \leq \|\hat{\mathbf{v}}\| \leq C_2 \|\hat{\mathbf{v}}\|_{\hat{V}_0(\hat{\Omega})} \quad \text{for} \quad \hat{\mathbf{v}} \in \hat{V}_0(\hat{\Omega}). \tag{2.6}$$

It is easy to prove that the bilinear and the linear forms defined by (2.2) are continuous over $V_0(\hat{\Omega}) \times V_0(\hat{\Omega})$ and $V_0(\hat{\Omega})$,respectively. To prove the $V_0(\hat{\Omega})$-ellipticity of the bilinear form, one uses Korn's inequality and relation (2.6). As a consequence of the Lax-Milgram lemma (see e.g. [6]), there is a unique solution $\hat{\mathbf{u}} \in V_0(\hat{\Omega})$ of the variational problem (2.1). Thus, the following theorem holds.

Theorem 2.1
Let $\hat{\mathbf{f}} \in (L_2(\hat{\Omega}))^3$ and $\operatorname{meas} \hat{\Gamma}_D > 0$ be satisfied. Further, let $\mu > 0$ and $\lambda > 0$ be fulfilled. Then, there exists a unique solution $\hat{\mathbf{u}} \in V_0(\hat{\Omega})$ of the variational problem (2.1). Moreover, the following a priori estimate holds,

$$\|\hat{\mathbf{u}}\|_{V_0(\hat{\Omega})} \leq C \|\hat{\mathbf{f}}\|_{(L_2(\hat{\Omega}))^3} \tag{2.7}$$

Proof: See e.g. [13].

Let r, φ, z denote the cylindrical coordinates, i.e. $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$. The domain $\hat{\Omega} \setminus \Gamma_0$ is transformed by the corresponding one-to-one mapping into the domain $\Omega := \Omega_a \times (-\pi, \pi]$, with cylindrical coordinates $(r, z) \in \Omega_a$ and $\varphi \in (-\pi, \pi]$. We now assume that $\hat{\Gamma}_D$ and $\hat{\Gamma}_N$ are such that they can be represented via the given mapping by $\Gamma_D := \Gamma_{aD} \times (-\pi, \pi]$ and $\Gamma_N := \Gamma_{aN} \times (-\pi, \pi]$, i.e. Γ_{aD} and Γ_{aN} are that parts of the boundary Γ_a of Ω_a which produce $\hat{\Gamma}_D$ and $\hat{\Gamma}_N$, respectively, by rotation about the x_3-axis. Consequently, for each function $\hat{u}(x)$, $x \in \hat{\Omega} \setminus \Gamma_0$, some function $u(r, \varphi, z)$ on Ω is defined uniquely by

$$u(r, \varphi, z) := \hat{u}(r\cos\varphi, r\sin\varphi, z), \tag{2.8}$$

and each vector field $\hat{\mathbf{u}}(x) = (\hat{u}_1(x), \hat{u}_2(x), \hat{u}_3(x))^T$, $x \in \hat{\Omega} \setminus \Gamma_0$, is transformed uniquely into a vector field $\mathbf{u} = (u_r(r, \varphi, z), u_\varphi(r, \varphi, z), u_z(r, \varphi, z))^T$ on Ω by (cf. [13])

$$\begin{aligned} u_r &= \hat{u}_1 \cos\varphi + \hat{u}_2 \sin\varphi, \\ u_\varphi &= -\hat{u}_1 \sin\varphi + \hat{u}_2 \cos\varphi, \\ u_z &= \hat{u}_3. \end{aligned} \tag{2.9}$$

Accordingly, mappings $L_2(\hat{\Omega}\setminus\Gamma_0) \to X^k_{1/2}(\Omega)$ and $\hat{W}(\hat{\Omega}\setminus\Gamma_0) \to W(\Omega)$ are defined, with $X^0_{1/2}(\Omega)$ and $W(\Omega)$ given by (2.11). Hence we have (see e.g. [8, 12, 13])

$$
\begin{aligned}
L_2^*(\Omega) &:= \{v = v(r,\varphi,z) : \int_\Omega |v|^2 dr d\varphi dz < \infty,\ 2\pi\text{-periodic with respect to } \varphi\},\\
X^0_{1/2}(\Omega) &:= \{v = v(r,\varphi,z) : r^{1/2} v \in L_2^*(\Omega)\},\\
W(\Omega) &:= \{\mathbf{v} = (v_r, v_\varphi, v_z) \in (X^0_{1/2}(\Omega))^3 : \frac{\partial v_r}{\partial r}, \frac{\partial v_\varphi}{\partial r}, \frac{\partial v_z}{\partial r}, \frac{\partial v_r}{\partial z}, \qquad (2.10)\\
&\qquad \frac{\partial v_\varphi}{\partial z}, \frac{\partial v_z}{\partial z}, \frac{1}{r}\frac{\partial v_r}{\partial \varphi} - \frac{1}{r} v_\varphi, \frac{1}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{1}{r} v_r, \frac{1}{r}\frac{\partial v_z}{\partial \varphi} \in X^0_{1/2}(\Omega)\},\\
V_0(\Omega) &:= \{\mathbf{v} \in W(\Omega) : \mathbf{v}|_{\Gamma_D} = \mathbf{0}\}.
\end{aligned}
$$

The norms in the above spaces are derived from the norms of the corresponding Sobolev spaces by using Cartesian coordinates. We get

$$
\begin{aligned}
\|v\|_{X^0_{1/2}(\Omega)} &= \Big\{\int_\Omega |v|^2 r dr d\varphi dz\Big\}^{1/2},\\
\|\mathbf{v}\|_{W(\Omega)} &= \Big\{\int_\Omega \Big(|v_r|^2 + |v_\varphi|^2 + |v_z|^2 + \Big|\frac{\partial v_r}{\partial r}\Big|^2 + \Big|\frac{\partial v_\varphi}{\partial r}\Big|^2 + \Big|\frac{\partial v_z}{\partial r}\Big|^2 + \Big|\frac{\partial v_r}{\partial z}\Big|^2 + \Big|\frac{\partial v_\varphi}{\partial z}\Big|^2\\
&\quad + \Big|\frac{\partial v_z}{\partial z}\Big|^2 + \Big|\frac{1}{r}(\frac{\partial v_r}{\partial \varphi} - v_\varphi)\Big|^2 + \Big|\frac{1}{r}(\frac{\partial v_\varphi}{\partial \varphi} + v_r)\Big|^2 + \Big|\frac{1}{r}\frac{\partial v_z}{\partial \varphi}\Big|^2\Big) r dr d\varphi dz\Big\}^{1/2},\\
\|\mathbf{v}\|_{V_0(\Omega)} &= \|\mathbf{v}\|_{W(\Omega)} \quad \text{for} \quad \mathbf{v} \in V_0(\Omega). \qquad (2.11)
\end{aligned}
$$

Further, the strain tensor $\varepsilon(\mathbf{u}) = (\varepsilon_{rr}, \varepsilon_{\varphi\varphi}, \varepsilon_{zz}, \gamma_{rz}, \gamma_{\varphi r}, \gamma_{z\varphi})^T$ can be expressed in terms of the displacement vector field $\mathbf{u}$ in Ω by

$$
\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \gamma_{\varphi r} \\ \gamma_{z\varphi} \end{bmatrix}
=
\begin{bmatrix}
\frac{\partial}{\partial r} & 0 & 0 \\
\frac{1}{r} & \frac{1}{r}\frac{\partial}{\partial \varphi} & 0 \\
0 & 0 & \frac{\partial}{\partial z} \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \\
\frac{1}{r}\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 \\
0 & \frac{\partial}{\partial z} & \frac{1}{r}\frac{\partial}{\partial \varphi}
\end{bmatrix}
\begin{bmatrix} u_r \\ u_\varphi \\ u_z \end{bmatrix}, \text{ or in matrix form by } \quad \varepsilon = \mathbf{D}\mathbf{u}. \qquad (2.12)
$$

The stress tensor $\sigma(\mathbf{u}) = (\sigma_{rr}, \sigma_{\varphi\varphi}, \sigma_{zz}, \tau_{rz}, \tau_{\varphi r}, \tau_{z\varphi})^T$ is given according to Hooke's law by

$$
\begin{bmatrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{zz} \\ \tau_{rz} \\ \tau_{\varphi r} \\ \tau_{z\varphi} \end{bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu
\end{bmatrix}
\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\varphi\varphi} \\ \varepsilon_{zz} \\ \gamma_{rz} \\ \gamma_{\varphi r} \\ \gamma_{z\varphi} \end{bmatrix} \qquad (2.13)
$$

or in matrix form by $\sigma = \mathbf{E}\varepsilon$, where E is Young's modulus of elasticity and ν is Poisson's ratio. The Lamé cofficients λ and μ can be expressed in terms of E and ν as follows (see e.g. [13]):

$$
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \qquad (2.14)
$$

One easily observes that the relations $\lambda > 0$, $\mu > 0$ correspond to $E > 0$, $0 < \nu < \frac{1}{2}$.
The variational solution of the boundary value problem (1.1) in cylindrical coordinates is obtained as follows (see e.g. [13, 15]). Find $\mathbf{u} \in V_0(\Omega)$ such that

$$b(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}) \quad \text{for any } \mathbf{v} \in V_0(\Omega), \tag{2.15}$$

where

$$b(\mathbf{u}, \mathbf{v}) = \int_\Omega (\varepsilon(\mathbf{u}))^T \sigma(\mathbf{v}) r dr d\varphi dz, \qquad f(\mathbf{v}) = \int_\Omega \mathbf{f}^T \mathbf{v} r dr d\varphi dz. \tag{2.16}$$

3 Partial Fourier decomposition of the solution

Let us consider the domain $\Omega := \Omega_a \times (-\pi, \pi]$, which is obtained from the axisymmetric domain $\hat{\Omega}$ (or more precisely: $\hat{\Omega} \setminus \Gamma_0$) by transformation in cylindrical coordinates. The system of trigonometric functions

$$1, \sin\varphi, \cos\varphi, \ldots, \cos n\varphi,\ \sin n\varphi, \ldots \tag{3.1}$$

is orthogonal and complete in $L_2(-\pi, \pi)$. Any function $w = w(\varphi)$, $w \in L_2(-\pi, \pi)$ and w being periodic with the period 2π, can be represented by a converging Fourier series with respect to the system (3.1). Let $u(r, \varphi, z) \in X^0_{1/2}(\Omega)$. Then, the function $u(r, \varphi, z)$ is, according to the definition of $X^0_{1/2}(\Omega)$, 2π-periodic with respect to φ, and $r^{1/2}u$ is square integrable on $\Omega = \Omega_a \times (-\pi, \pi]$. It follows that u is square integrable on $(-\pi, \pi)$ with respect to φ for almost any $(r, z) \in \Omega_a$. Consequently, $u(r, \varphi, z)$ can be represented by a Fourier series converging in $L_2(-\pi, \pi)$ for almost any $(r, z) \in \Omega_a$. For the components u_r, u_φ, u_z of the displacement vector field $\mathbf{u}$ we define the Fourier series as proposed in the engineering literature (see e.g. [1, 3, 7, 15, 17, 20]):

$$\begin{aligned}
u_r(r, \varphi, z) &= \sum_{n=0}^{\infty} (u^s_{rn}(r, z) \cos n\varphi + u^a_{rn}(r, z) \sin n\varphi), \\
u_\varphi(r, \varphi, z) &= \sum_{n=0}^{\infty} (u^s_{\varphi n}(r, z)(-\sin n\varphi) + u^a_{\varphi n}(r, z) \cos n\varphi), \\
u_z(r, \varphi, z) &= \sum_{n=0}^{\infty} (u^s_{zn}(r, z) \cos n\varphi + u^a_{zn}(r, z) \sin n\varphi),
\end{aligned} \tag{3.2}$$

with the symmetric and antisymmetric Fourier coefficients defined by (use $\mathbf{N} := \{1, 2, 3, \ldots\}$):

$$\begin{aligned}
u^s_{t0} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u_t(r, \varphi, z) d\varphi, \quad u^s_{tn} = \frac{1}{\pi} \int_{-\pi}^{\pi} u_t(r, \varphi, z) \cos n\varphi\ d\varphi, \quad n \in \mathbf{N},\ t = r, z, \\
u^a_{t0} &= 0, \qquad u^a_{tn} = \frac{1}{\pi} \int_{-\pi}^{\pi} u_t(r, \varphi, z) \sin n\varphi\ d\varphi, \quad n \in \mathbf{N}, \quad t = r, z, \\
u^s_{\varphi 0} &= 0, \qquad u^s_{\varphi n} = -\frac{1}{\pi} \int_{-\pi}^{\pi} u_\varphi(r, \varphi, z) \sin n\varphi\ d\varphi, \quad n \in \mathbf{N}, \\
u^a_{\varphi 0} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u_\varphi(r, \varphi, z) d\varphi, \qquad u^a_{\varphi n} = \frac{1}{\pi} \int_{-\pi}^{\pi} u_\varphi(r, \varphi, z) \cos n\varphi\ d\varphi, \quad n \in \mathbf{N}.
\end{aligned} \tag{3.3}$$

We note that the modified expansion of u_φ as given by (3.2) leads to a convenient decoupling of the two-dimensional problems. For $\mathbf{u} = (u_r(r, \varphi, z), u_\varphi(r, \varphi, z), u_z(r, \varphi, z))^T \in W(\Omega)$, it is not evident from the definition of $W(\Omega)$ that the partial derivatives of the components of $\mathbf{u}$ with respect to the coordinate variable φ are elements of the space $X^0_{1/2}(\Omega)$. Neverthelesss, we have

Lemma 3.1. Let $\mathbf{u} = (u_r(r,\varphi,z), u_\varphi(r,\varphi,z), u_z(r,\varphi,z))^T \in W(\Omega)$ be satisfied. Then relation $\frac{\partial u_t}{\partial\varphi} \in X^0_{1/2}(\Omega)$ holds for $t = r, \varphi, z$.

Proof: First, we show $\frac{\partial u_r}{\partial\varphi} \in X^0_{1/2}(\Omega)$. Using $r_{\max} := \max\{r : (r,z) \in \Omega_a\}$ and the definition of $W(\Omega)$ we obtain

$$\begin{aligned}\frac{1}{r^2_{\max}}\int_\Omega\Big|\frac{\partial u_r}{\partial\varphi}\Big|^2 r dr d\varphi dz &= \frac{1}{r^2_{\max}}\int_\Omega\Big|\frac{\partial u_r}{\partial\varphi} - u_\varphi + u_\varphi\Big|^2 r dr d\varphi dz \\ &\leq 2\int_\Omega\Big|\frac{1}{r}\Big(\frac{\partial u_r}{\partial\varphi} - u_\varphi\Big)\Big|^2 r dr d\varphi dz + \frac{2}{r^2_{\max}}\int_\Omega |u_\varphi|^2 r dr d\varphi dz < \infty.\end{aligned}$$

Consequently, $\frac{\partial u_r}{\partial\varphi} \in X^0_{1/2}(\Omega)$ is fulfilled. The remaining assertions can be proved similarly. ■

Thus, the first order derivatives $\frac{\partial u_t}{\partial r}, \frac{\partial u_t}{\partial\varphi}, \frac{\partial u_t}{\partial z}$ $(t = r,\varphi,z)$ of the components of the function $\mathbf{u} \in W(\Omega)$ can be represented by converging Fourier series according to (3.2). If $\left(\frac{\partial u_t}{\partial r}\right)^s_n$, $\left(\frac{\partial u_t}{\partial r}\right)^a_n$, ... are the Fourier coefficients of the derivatives $\frac{\partial u_t}{\partial r}$, ..., then we have for $t = r, z$

$$\begin{aligned}\frac{\partial u_t}{\partial r} &= \sum_{n=0}^\infty\left(\left(\frac{\partial u_t}{\partial r}\right)^s_n \cos n\varphi + \left(\frac{\partial u_t}{\partial r}\right)^a_n \sin n\varphi\right), \\ \frac{\partial u_t}{\partial z} &= \sum_{n=0}^\infty\left(\left(\frac{\partial u_t}{\partial z}\right)^s_n \cos n\varphi + \left(\frac{\partial u_t}{\partial z}\right)^a_n \sin n\varphi\right), \\ \frac{\partial u_t}{\partial\varphi} &= \sum_{n=1}^\infty\left(\left(\frac{\partial u_t}{\partial\varphi}\right)^s_n (-\sin n\varphi) + \left(\frac{\partial u_t}{\partial\varphi}\right)^a_n \cos n\varphi\right), \\ \frac{\partial u_\varphi}{\partial r} &= \sum_{n=0}^\infty\left(\left(\frac{\partial u_\varphi}{\partial r}\right)^s_n (-\sin n\varphi) + \left(\frac{\partial u_\varphi}{\partial r}\right)^a_n \cos n\varphi\right), \qquad (3.4) \\ \frac{\partial u_\varphi}{\partial z} &= \sum_{n=0}^\infty\left(\left(\frac{\partial u_\varphi}{\partial z}\right)^s_n (-\sin n\varphi) + \left(\frac{\partial u_\varphi}{\partial z}\right)^a_n \cos n\varphi\right), \\ \frac{\partial u_\varphi}{\partial\varphi} &= \sum_{n=1}^\infty\left(\left(\frac{\partial u_\varphi}{\partial\varphi}\right)^s_n \cos n\varphi + \left(\frac{\partial u_\varphi}{\partial\varphi}\right)^a_n \sin n\varphi\right),\end{aligned}$$

where the coefficients are given by $(n \in \mathbf{N})$

$$\begin{aligned}\Big(\frac{\partial u_t}{\partial\xi}\Big)^s_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\partial u_t(r,\varphi,z)}{\partial\xi}d\varphi, \quad \Big(\frac{\partial u_t}{\partial\xi}\Big)^s_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t(r,\varphi,z)}{\partial\xi}\cos n\varphi\, d\varphi, \\ \Big(\frac{\partial u_t}{\partial\xi}\Big)^a_0 &= 0, \qquad \Big(\frac{\partial u_t}{\partial\xi}\Big)^a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t(r,\varphi,z)}{\partial\xi}\sin n\varphi\, d\varphi, \quad \xi = r, z, \\ \Big(\frac{\partial u_t}{\partial\varphi}\Big)^s_n &= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t(r,\varphi,z)}{\partial\varphi}\sin n\varphi\, d\varphi, \\ \Big(\frac{\partial u_t}{\partial\varphi}\Big)^a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t(r,\varphi,z)}{\partial\varphi}\cos n\varphi\, d\varphi, \quad t = r, z, \qquad (3.5) \\ \Big(\frac{\partial u_\varphi}{\partial\xi}\Big)^s_0 &= 0, \qquad \Big(\frac{\partial u_\varphi}{\partial\xi}\Big)^s_n = -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_\varphi(r,\varphi,z)}{\partial\xi}\sin n\varphi\, d\varphi,\end{aligned}$$

$$\left(\frac{\partial u_\varphi}{\partial \xi}\right)^a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial u_\varphi(r,\varphi,z)}{\partial \xi} d\varphi, \quad \left(\frac{\partial u_t}{\partial \varphi}\right)^{s/a}_0 = 0 \quad \text{for} \quad t = r,\varphi,z,$$
$$\left(\frac{\partial u_\varphi}{\partial \xi}\right)^a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial u_\varphi(r,\varphi,z)}{\partial \xi} \cos n\varphi \, d\varphi, \quad \xi = r,z,$$
$$\left(\frac{\partial u_\varphi}{\partial \varphi}\right)^s_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial u_t(r,\varphi,z)}{\partial \varphi} \cos n\varphi \, d\varphi, \quad \left(\frac{\partial u_\varphi}{\partial \varphi}\right)^a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\partial u_t(r,\varphi,z)}{\partial \varphi} \sin n\varphi \, d\varphi.$$

Here and in the following, the notation $u^{s/a}$ means that some relation holds for u^s as well as u^a.

Lemma 3.2. Let $\mathbf{u} \in W(\Omega)$ be satisfied and the Fourier coefficients of the series (3.2) and (3.4) be defined according to (3.3) and (3.5), respectively. Then the following relations are satiesfied:

$$(i) \quad \left(\frac{\partial u_t}{\partial r}\right)^{s/a}_n = \frac{\partial u^{s/a}_{tn}}{\partial r}, \qquad (ii) \quad \left(\frac{\partial u_t}{\partial z}\right)^{s/a}_n = \frac{\partial u^{s/a}_{tn}}{\partial z} \quad \text{for} \quad t = r,\varphi,z, \quad n \in \mathbf{N}_0,$$

$$(iii) \quad \left(\frac{\partial u_t}{\partial \varphi}\right)^{s/a}_n = \begin{cases} n u^{s/a}_t & : \ t = r,z, \quad n \in \mathbf{N}, \\ -n u^{s/a}_{tn} & : \ t = \varphi, \quad n \in \mathbf{N}, \end{cases}$$

where $u^{s/a}_{tn}$ $(t = r,\varphi,z)$ are defined by (3.3), with $\mathbf{N}_0 := \{0,1,2,\cdots\}$.

Proof: Let $v(r,z) \in C_0^\infty(\Omega_a)$ (space of infinitely differentiable functions with compact support in Ω_a). Then, owing to the definition of the Fourier coefficients and to the integration by parts, we get for $t = r$ or $t = z$ the relations

$$\begin{aligned} \int_{\Omega_a} u^s_{tn}(r,z)\frac{\partial v(r,z)}{\partial r}\, drdz &= \frac{1}{\pi}\int_{\Omega_a}\Big(\int_{-\pi}^{\pi} u_t(r,\varphi,z)\cos n\varphi\, d\varphi\Big)\frac{\partial v(r,z)}{\partial r}\, drdz \\ &= -\frac{1}{\pi}\int_{\Omega_a}\Big(\int_{-\pi}^{\pi}\frac{\partial u_t}{\partial r}\cos n\varphi\, d\varphi\Big) v(r,z) drdz \\ &= -\int_{\Omega_a}\Big(\frac{\partial u_t}{\partial r}\Big)^s_n v(r,z)\, drdz, \quad n \in \mathbf{N}. \end{aligned} \tag{3.6}$$

Similarly, we get a corresponding relation for $n = 0$,

$$\int_{\Omega_a} u^s_{t0}(r,z)\frac{\partial v(r,z)}{\partial r}\, drdz = -\int_{\Omega_a}\Big(\frac{\partial u_t}{\partial r}\Big)^s_0 v(r,z)\, drdz. \tag{3.7}$$

Taking into consideration the definition of the generalized first order derivative, we obtain from (3.6) and (3.7) the relation $\frac{\partial u^s_{tn}}{\partial r} = \left(\frac{\partial u_t}{\partial r}\right)^s_n$ for $t = r,z$ and $n \in \mathbf{N}_0$. By analogous argumentation one proves the relations $\frac{\partial u^a_{tn}}{\partial r} = \left(\frac{\partial u_t}{\partial r}\right)^a_n$ and $\frac{\partial u^{s/a}_{tn}}{\partial z} = \left(\frac{\partial u_t}{\partial z}\right)^{s/a}_n$ for $t = r,z$, furthermore $\frac{\partial u^{s/a}_{\varphi n}}{\partial r} = \left(\frac{\partial u_\varphi}{\partial r}\right)^{s/a}_n$ and $\frac{\partial u^{s/a}_{\varphi n}}{\partial z} = \left(\frac{\partial u_\varphi}{\partial z}\right)^{s/a}_n$, $n \in \mathbf{N}_0$.

Applying integration by parts and the periodicity condition $u_t|_{-\pi} = u_t|_\pi$ $(t = r,\varphi,z)$, which holds for almost any $(r,z) \in \Omega_a$, we get the relations

$$\begin{aligned} \left(\frac{\partial u_t}{\partial \varphi}\right)^a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t}{\partial \varphi}\cos n\varphi d\varphi = \frac{n}{\pi}\int_{-\pi}^{\pi} u_t(r,\varphi,z)\sin n\varphi d\varphi = n u^a_{tn}, \\ \left(\frac{\partial u_t}{\partial \varphi}\right)^s_n &= -\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_t}{\partial \varphi}\sin n\varphi d\varphi = \frac{n}{\pi}\int_{-\pi}^{\pi} u_t\cos n\varphi d\varphi = n u^s_{tn}, \quad t = r,z, \\ \left(\frac{\partial u_\varphi}{\partial \varphi}\right)^s_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_\varphi}{\partial \varphi}\cos n\varphi d\varphi = \frac{n}{\pi}\int_{-\pi}^{\pi} u_\varphi\sin n\varphi d\varphi = -n u^s_{\varphi n}, \\ \left(\frac{\partial u_\varphi}{\partial \varphi}\right)^a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u_\varphi}{\partial \varphi}\sin n\varphi d\varphi = -\frac{n}{\pi}\int_{-\pi}^{\pi} u_\varphi\cos n\varphi d\varphi = -n u^a_{\varphi n}, \quad n \in \mathbf{N}, \end{aligned} \tag{3.8}$$

and assertion (iii) is proved too. ■

For the characterization of the Fourier coefficients, employ the following spaces of functions on Ω_a (see e.g. [8, 12]), with real α:

$$
\begin{aligned}
L_2(\Omega_a) &:= \{w = w(r,z) : \quad \int_{\Omega_a} |w|^2 dr dz < \infty\}, \\
L_{2,\alpha}(\Omega_a) &:= \{w = w(r,z) : \quad r^\alpha w \in L_2(\Omega_a)\}, \\
W^{1,2}_\alpha(\Omega_a) &:= \{w \in L_{2,\alpha}(\Omega_a) : \quad \frac{\partial w}{\partial r}, \frac{\partial w}{\partial z} \in L_{2,\alpha}(\Omega_a)\}, \\
W^{2,2}_{1/2}(\Omega_a) &:= \{w \in W^{1,2}_{1/2}(\Omega_a) : \quad \frac{\partial^2 w}{\partial r^2}, \frac{\partial^2 w}{\partial z^2}, \frac{\partial^2 w}{\partial r \partial z} \in L_{2,1/2}(\Omega_a)\}, \\
X^{1,2}_{1/2}(\Omega_a) &:= \{w \in W^{1,2}_{1/2}(\Omega_a) : \quad r^{-1} w \in L_{2,1/2}(\Omega_a)\}.
\end{aligned}
\tag{3.9}
$$

These spaces are equipped with the norms

$$
\begin{aligned}
\|w\|_{L_2(\Omega_a)} &= \Big\{\int_{\Omega_a} |w|^2 dr dz\Big\}^{1/2}, \quad \|w\|_{L_{2,\alpha}(\Omega_a)} = \Big\{\int_{\Omega_a} |r^\alpha w|^2 dr dz\Big\}^{1/2}, \\
\|w\|_{W^{1,2}_\alpha(\Omega_a)} &= \Big\{\|w\|^2_{L_{2,\alpha}(\Omega_a)} + \Big\|\frac{\partial w}{\partial r}\Big\|^2_{L_{2,\alpha}(\Omega_a)} + \Big\|\frac{\partial w}{\partial z}\Big\|^2_{L_{2,\alpha}(\Omega_a)}\Big\}^{1/2}, \\
|w|_{W^{2,2}_{1/2}(\Omega_a)} &= \Big\{\Big\|\frac{\partial^2 w}{\partial r^2}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial^2 w}{\partial z^2}\Big\|^2_{L_{2,1/2}(\Omega_a)} + 2\Big\|\frac{\partial^2 w}{\partial r \partial z}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\}^{1/2}, \\
\|w\|_{W^{2,2}_{1/2}(\Omega_a)} &= \Big\{|w|^2_{W^{2,2}_{1/2}(\Omega_a)} + \|w\|^2_{W^{1,2}_{1/2}(\Omega_a)}\Big\}^{1/2}, \\
\|w\|_{X^{1,2}_{1/2}(\Omega_a)} &= \Big\{\Big\|\frac{w}{r}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial w}{\partial r}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{\partial w}{\partial z}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\}^{1/2}.
\end{aligned}
\tag{3.10}
$$

In [12] it was shown that these spaces are suitable for the analysis of the Fourier coefficients of the solutions of boundary value problems of heat conduction in axisymmetric domains. However, for vector functions $\mathbf{u} \in W(\Omega)$, whose Fourier coefficients are also vector functions defined on Ω_a, the function spaces introduced previously are not sufficient to study their properties. For the mathematical analysis of the partial Fourier method for problems in elasticity further function spaces are needed.

Definition 3.1. Using $\mathbf{w} = (w_1(r,z), w_2(r,z), w_3(r,z))^T$, we introduce the following sets of functions:

$$
\begin{aligned}
V^a(\Omega_a) &:= \{\mathbf{w} = (w_1, w_2, w_3)^T \in \Big(W^{1,2}_{1/2}(\Omega_a)\Big)^3 : \ r^{-1}w_1, \ r^{-1}w_2 \in L_{2,1/2}(\Omega_a)\}, \\
W^a(\Omega_a) &:= \{\mathbf{w} = (w_1, w_2, w_3)^T \in \Big(W^{1,2}_{1/2}(\Omega_a)\Big)^3 : \\
&\qquad r^{-1}(w_1 - w_2), \ r^{-1}w_3 \in L_{2,1/2}(\Omega_a)\}, \\
V^a_0(\Omega_a) &:= \{\mathbf{w} \in V^a(\Omega_a) : \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma_{aD}\}, \\
W^a_0(\Omega_a) &:= \{\mathbf{w} \in W^a(\Omega_a) : \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma_{aD}\}.
\end{aligned}
\tag{3.11}
$$

It can easily be verified that these spaces are linear spaces.

Definition 3.2. In $V^a(\Omega_a)$, $W^a(\Omega_a)$, $V^a_0(\Omega_a)$ and $W^a_0(\Omega_a)$ we introduce the norms

$$
\|\mathbf{w}\|_{V^a(\Omega_a)} := \{\|w_1\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_2\|^2_{X^{1,2}_{1/2}(\Omega_a)} + \|w_3\|^2_{W^{1,2}_{1/2}(\Omega_a)}\}^{1/2},
$$

$$
\begin{aligned}
\|\mathbf{w}\|_{V_0^a(\Omega_a)} &:= \|\mathbf{w}\|_{V^a(\Omega_a)} \quad \text{for} \quad \mathbf{w} \in V_0^a(\Omega_a) \\
\|\mathbf{w}\|_{W^a(\Omega_a)} &:= \Big\{\sum_{i=1}^{3} \|w_i\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \Big\|\frac{1}{r}(w_1 - w_2)\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{w_3}{r}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\}^{1/2}, \\
\|\mathbf{w}\|_{W_0^a(\Omega_a)} &:= \|\mathbf{w}\|_{W^a(\Omega_a)} \quad \text{for} \quad \mathbf{w} \in W_0^a(\Omega_a).
\end{aligned}
$$

It is not difficult to show that the functionals given in (3.12) are indeed norms in the corresponding linear spaces. Now we state that these normed spaces are complete.

Lemma 3.3. The normed linear spaces $V^a(\Omega_a)$ and $W^a(\Omega_a)$ are Banach spaces.

Proof. Here, we have to show that any fundamental sequence (Cauchy sequence) in $V^a(\Omega_a)$ and $W^a(\Omega_a)$, repectively, converges. We will prove this assertion only for $W^a(\Omega_a)$, the proof for $V^a(\Omega_a)$ can be done likewise.
Let $\{\mathbf{w}_n = (w_{1n}(r,z), w_{2n}(r,z), w_{3n}(r,z))^T\}_{n=1}^{\infty}$ be any fundamental sequence in $W^a(\Omega_a)$. We associate with $\{\mathbf{w}_n\}$ another sequence $\{\mathbf{v}^n\}_{n=1}^{\infty}$ in $W(\Omega)$ and show that this is a fundamental sequence in $W(\Omega)$, and thus convergent, since $W(\Omega)$ is a Banach space. We define $\mathbf{v}^n$ by

$$
\begin{aligned}
\mathbf{v}^n &= (v_r^n(r,\varphi,z),\ v_\varphi^n(r,\varphi,z),\ v_z^n(r,\varphi,z))^T \\
&:= (w_{1n}(r,z)\sin\varphi,\ w_{2n}(r,z)\cos\varphi,\ w_{3n}(r,z)\sin\varphi)^T, \quad n \in \mathbf{N}.
\end{aligned} \tag{3.12}
$$

Using Fubini's theorem and the relation $\mathbf{w}_n = (w_{1n}, w_{2n}, w_{3n}) \in W^a(\Omega_a),\ n \in \mathbf{N}$, it can be shown that $\mathbf{v}^n \in W(\Omega)$ for every $n \in \mathbf{N}$. For example, we get

$$
\|v_r^n\|^2_{X^0_{1/2}(\Omega)} = \int_{\Omega_a}\int_{-\pi}^{\pi} |w_{1n}(r,z)\sin\varphi|^2 r dr d\varphi dz = \pi\|w_{1n}\|^2_{L_{2,1/2}(\Omega_a)} < \infty. \tag{3.13}
$$

In the same way, we derive that the $X^0_{1/2}(\Omega)$-norm of the rest of the terms in the definition of $W(\Omega)$ (see (2.11)) are bounded.
Let ε and $N(\varepsilon)$ be such that $\|\mathbf{w}_n - \mathbf{w}_m\|^2_{W^a(\Omega_a)} \le \varepsilon/(2\pi)$ for $n, m \ge N(\varepsilon)$ holds. Then we also get $\|\mathbf{v}^n - \mathbf{v}^m\|^2_{W(\Omega)} \le \varepsilon$ for $n, m \ge N(\varepsilon)$. Using the definition of $\|\cdot\|_{W(\Omega)}$, expressions of the type (3.13) and the triangle inequality, one easily proves the relation

$$
\begin{aligned}
\|\mathbf{v}^n - \mathbf{v}^m\|^2_{W(\Omega)} &= \pi\Big\{\sum_{i=1}^{3} \|w_{in} - w_{im}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \Big\|\frac{1}{r}(w_{3n} - w_{3m})\Big\|^2_{L_{2,1/2}(\Omega_a)} \\
&+ 2\Big\|\frac{1}{r}[((w_{1n} - w_{2n}) - \frac{1}{r}(w_{1m} - w_{2m})]\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} \\
&\le 2\pi\|\mathbf{w}_n - \mathbf{w}_m\|^2_{W^a(\Omega_a)} \le \varepsilon, \quad \text{for} \quad n, m > N(\varepsilon).
\end{aligned} \tag{3.14}
$$

It follows from (3.14) that the sequence $\{\mathbf{v}^n\}_{n=1}^{\infty}$ is a Cauchy sequence in $W(\Omega)$ and thus convergent. Let us denote by $\mathbf{v} = (w_1(r,z)\sin\varphi, w_2(r,z)\cos\varphi, w_3(r,z)\sin\varphi)^T$ its limit. In what follows we show that the fundamental sequence $\{\mathbf{w}_n\}_{n=1}^{\infty}$ in $W^a(\Omega_a)$ converges towards $\mathbf{w} = (w_1(r,z), w_2(r,z), w_3(r,z))^T$ and that $\mathbf{w} \in W^a(\Omega_a)$ holds. Using the definition of $\|\cdot\|_{W(\Omega)}$ and $\|\cdot\|_{W^a(\Omega_a)}$ one easily deduces the inequalities $\|\mathbf{w}\|^2_{W^a(\Omega_a)} \le \frac{1}{\pi}\|\mathbf{v}\|^2_{W(\Omega)} < \infty$ and $\|\mathbf{w}_n - \mathbf{w}\|^2_{W^a(\Omega_a)} \le \frac{1}{\pi}\|\mathbf{v}^n - \mathbf{v}\|^2_{W(\Omega)} < \infty$. This implies $\mathbf{w}_n \to \mathbf{w} \in W^a(\Omega_a)$, and the assertion is proved. ■

In the following, we characterize functions $\mathbf{u} \in W(\Omega)$ by means of their Fourier coefficients and some completeness relations.

Lemma 3.4. Let $\mathbf{u}_n^s = (u_{rn}^s, u_{\varphi n}^s, u_{zn}^s)^T$ and $\mathbf{u}_n^a = (u_{rn}^a, u_{\varphi n}^a, u_{zn}^a)^T$ (cf. (3.3)) represent the symmetric and the antisymmetric parts of the Fourier coefficients of the function $\mathbf{u} \in W(\Omega)$. For $t = r, \varphi, z$, we get the completeness relations

$$\|u_t\|^2_{X^0_{1/2}(\Omega)} = 2\pi\|u^s_{t0}\|^2_{L_{2,1/2}(\Omega_a)} + \pi\sum_{n=1}^{\infty}\Big(\|u^s_{tn}\|^2_{L_{2,1/2}(\Omega_a)} + \|u^a_{tn}\|^2_{L_{2,1/2}(\Omega_a)}\Big) < \infty, \quad (3.15)$$

moreover,

$$\begin{aligned}\|\mathbf{u}\|^2_{W(\Omega)} &= 2\pi\Big\{\|u^s_{r0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|u^s_{z0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|u^a_{\varphi 0}\|^2_{W^{1,2}_{1/2}(\Omega_a)} \\ &+ \Big\|\frac{1}{r}u^s_{r0}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{1}{r}u^a_{\varphi 0}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} + \pi\sum_{n=1}^{\infty}\sum_{e\in\{s,a\}}\Big\{\|u^e_{rn}\|^2_{W^{1,2}_{1/2}(\Omega_a)} \\ &+ \|u^e_{\varphi n}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \|u^e_{zn}\|^2_{W^{1,2}_{1/2}(\Omega_a)} + \Big\|\frac{1}{r}(u^e_{rn} - nu^e_{\varphi n})\Big\|^2_{L_{2,1/2}(\Omega_a)} \\ &+ \Big\|\frac{1}{r}(nu^e_{rn} - u^e_{\varphi n})\Big\|^2_{L_{2,1/2}(\Omega_a)} + n^2\Big\|\frac{1}{r}u^e_{zn}\|^2_{L_{2,1/2}(\Omega_a)}\Big\} < \infty, \end{aligned} \quad (3.16)$$

where $\sum_{e\in\{s,a\}}$ means summation over s and a. Furthermore, the inequality

$$\|\mathbf{u}\|^2_{W(\Omega)} \geq 2\pi\{\|\mathbf{u}^s_0\|^2_{V^a(\Omega_a)} + \|\mathbf{u}^a_0\|^2_{V^a(\Omega_a)}\} + \pi\sum_{n=1}^{\infty}\{\|\mathbf{u}^s_n\|^2_{W^a(\Omega_a)} + \|\mathbf{u}^a_n\|^2_{W^a(\Omega_a)}\} \quad (3.17)$$

holds.

Proof: Starting from

$$u^s_{rn} = \frac{1}{\pi}\int_{-\pi}^{\pi} u_r(r,\varphi,z)\cos n\varphi d\varphi, \quad n \in \mathbf{N},$$

and using the Cauchy-Schwarz inequality, we get the expression

$$|u^s_{rn}|^2 \leq \frac{1}{\pi^2}\int_{-\pi}^{\pi}|u_r(r,\varphi,z)|^2 d\varphi\int_{-\pi}^{\pi}|\cos n\varphi|^2 d\varphi = \frac{1}{\pi}\int_{-\pi}^{\pi}|u_r(r,\varphi,z)|^2 d\varphi, \ n \in \mathbf{N}, \quad (3.18)$$

and similarly one deduces the relation

$$|u_{r0}|^2 \leq \frac{1}{4\pi^2}\int_{-\pi}^{\pi}|u_r(r,\varphi,z)|^2 d\varphi\int_{-\pi}^{\pi}d\varphi = \frac{1}{2\pi}\int_{-\pi}^{\pi}|u_r(r,\varphi,z)|^2 d\varphi. \quad (3.19)$$

Multiplying the expressions (3.18) and (3.19) respectively with r ($r > 0$), integrating by parts over Ω_a, and applying the theorem of Fubini, we get the inequality

$$C\|u^s_{rn}\|^2_{L_{2,1/2}(\Omega_a)} \leq \|u_r\|^2_{X^0_{1/2}(\Omega)} < \infty, \quad (3.20)$$

with $C = 2\pi$ for $n = 0$ and $C = \pi$ for $n \in \mathbf{N}$. In the same way, one verifies the relation $C\|u^a_{rn}\|^2_{L_{2,1/2}(\Omega_a)} \leq \|u_r\|^2_{X^0_{1/2}(\Omega)} < \infty$ and similar relations for $u^{s/a}_{\varphi n}$, $u^{s/a}_{zn}$, i.e. we have

$$C\|u^{s/a}_{tn}\|^2_{L_{2,1/2}(\Omega_a)} \leq \|u_t\|^2_{X^0_{1/2}(\Omega)} < \infty, \quad n \in \mathbf{N}_0, \quad \text{for} \quad t = r, \varphi, z. \quad (3.21)$$

This implies $u^{s/a}_{rn}, u^{s/a}_{\varphi n}, u^{s/a}_{zn} \in L_{2,1/2}(\Omega_a)$. For the system of functions (3.1) we get the completeness relation

$$\int_{-\pi}^{\pi}|u_t(r,\varphi,z)|^2 d\varphi = 2\pi|u_{t0}(r,z)|^2 + \pi\sum_{n=1}^{\infty}\Big(|u^s_{tn}(r,z)|^2 + |u^a_{tn}(r,z)|^2\Big) < \infty \ (t = r,\varphi,z),$$

which holds for almost every $(r,z) \in \Omega_a$ (see e.g. [4]). Multiplying both sides of this equation with r $(r > 0)$ and integrating over Ω_a, we prove by applying the well-known theorems of Fubini and Lebesgue the identity (3.15). For $\mathbf{u} \in W(\Omega)$, we get u_r, u_φ, u_z, $\frac{\partial u_r}{\partial r}$, $\frac{\partial u_\varphi}{\partial r}$, $\frac{\partial u_z}{\partial r}$, $\frac{1}{r}(\frac{\partial u_\varphi}{\partial \varphi} + u_r)$, $\frac{1}{r}(\frac{\partial u_r}{\partial \varphi} - u_\varphi)$, $\frac{1}{r}\frac{\partial u_z}{\partial \varphi}$, $\frac{\partial u_r}{\partial z}$, $\frac{\partial u_\varphi}{\partial z}$, $\frac{\partial u_z}{\partial z} \in X^0_{1/2}(\Omega)$ (cf. (2.11)). Therefore, the Fourier coefficients of these functions fulfil relations of the form (3.20) and (3.21). Moreover, for each term quoted previously completeness relations of the type (3.15) can be shown. The relation (3.16) follows from the definition of $\|\mathbf{u}\|_{W(\Omega)}$ (cf. (2.11)) and Lemma 3.2. Inequality (3.17) follows immediately from the identity (3.16) and the definition of the norms $\|.\|_{V^a(\Omega_a)}$ and $\|.\|_{W^a(\Omega_a)}$, taking into consideration the relations $\mathbf{u}^s_0 = (u^s_{r0},\ 0,\ u^s_{z0})^T$ and $\mathbf{u}^a_0 = (0,\ u^a_{\varphi 0},\ 0)^T$. ∎

Previously, it was proved that by Definition 3.1 complete normed spaces are given. However, to show that these spaces are appropriate for the analysis of the Fourier coefficients $\mathbf{u}^{s/a}_n$, $n \in \mathbf{N}_0$ (cf. (3.3)), of the solution $\mathbf{u} \in V_0(\Omega)$ of the three-dimensional boundary value problem (2.15), we still have to prove that the Fourier coefficients are contained in these spaces. Moreover, these spaces contain additional weights of the form r^{-1} and in the case where $\Gamma_0 \neq \emptyset$ (see Figure 1), it is important to investigate the behaviour of $\mathbf{u}^{s/a}_n$ for $r = 0$, in particular in view of the numerical solution of the boundary value problems determining the Fourier coefficients on the meridian plane Ω_a. The engineers derived the trace properties of the Fourier coefficients $\mathbf{u}^{s/a}_n$ on the part Γ_0 of the rotational axis by mechanical considerations (see e.g. [3, 7, 15, 16]). Mercier/Raugel (cf. [12]) gave a basic assertion for functions $w \in X^{1,2}_{1/2}(\Omega_a)$, which is formulated in Lemma 3.5.

Lemma 3.5. Let $X^{1,2}_{1/2}(\Omega_a)$ be defined as in (3.9), with $\Gamma_0 \neq \emptyset$. Then, there is a continuous linear mapping $\gamma : X^{1,2}_{1/2}(\Omega_a) \to L_2(\Gamma_0)$ such that for all $w \in X^{1,2}_{1/2}(\Omega_a)$ the relation $\gamma w = w(0,z) = 0$ holds in $L_2(\Gamma_0)$.

Proof: See [12] Proposition 4.1. ∎

In the following, we prove some assertions on the traces of the Fourier coefficients $\mathbf{u}^{s/a}_n$ of $\mathbf{u}$, $n \in \mathbf{N}_0$.

Lemma 3.6. Let $\mathbf{u} \in V_0(\Omega)$ be satisfied and let $\mathbf{u}^s_n = (u^s_{rn},\ u^s_{\varphi n},\ u^s_{zn})^T$ and $\mathbf{u}^a_n = (u^a_{rn},\ u^a_{\varphi n},\ u^a_{zn})^T$, $n \in \mathbf{N}_0$, respectively denote the symmetric and antisymmetric Fourier coefficients defined according to (3.3). Then the following relations are fulfilled:

$$\begin{aligned} &(i) && \mathbf{u}^s_0,\ \mathbf{u}^a_0 \in V^a_0(\Omega_a), \quad \mathbf{u}^s_n,\ \mathbf{u}^a_n \in W^a_0(\Omega_a), \quad n \in \mathbf{N}, \\ &(ii) && u^{s/a}_{r0} = u^{s/a}_{\varphi 0} = 0 \quad \text{on} \quad \Gamma_0, \\ &(iii) && u^{s/a}_{r1} = u^{s/a}_{\varphi 1} \quad \text{and} \quad u^{s/a}_{z1} = 0 \quad \text{on} \quad \Gamma_0, \\ &&& u^{s/a}_{rn} = u^{s/a}_{\varphi n} = u^{s/a}_{zn} = 0 \quad \text{on} \quad \Gamma_0,\ \text{for } n \geq 2. \end{aligned} \tag{3.22}$$

Proof: (i) It follows from the definition of the Fourier coefficients of $\mathbf{u}$ (see (3.3)) that $\mathbf{u}^s_0 = (u^s_{r0}, 0, u^s_{z0})^T$ and $\mathbf{u}^a_0 = (0, u^a_{\varphi 0}, 0)^T$. With the help of completeness relation (3.16) and the definition of $V^a(\Omega_a)$, we obtain $\mathbf{u}^s_0,\ \mathbf{u}^a_0 \in V^a(\Omega_a)$. Moreover, identity (3.16) also implies the relations $\frac{1}{r}(u^{s/a}_{rn} - n u^{s/a}_{\varphi n})$, $\frac{1}{r}(n u^{s/a}_{rn} - u^{s/a}_{\varphi n}) \in L_{2,1/2}(\Omega_a)$, $n \in \mathbf{N}$. Since $\frac{1}{r}(u^{s/a}_{rn} - n u^{s/a}_{\varphi n}) + \frac{1}{r}(n u^{s/a}_{rn} - u^{s/a}_{\varphi n}) = (n+1)\frac{1}{r}(u^{s/a}_{rn} - u^{s/a}_{\varphi n})$ holds, we also have $\frac{1}{r}(u^{s/a}_{rn} - u^{s/a}_{\varphi n}) \in L_{2,1/2}(\Omega_a)$ for $n \in \mathbf{N}$. Together with $\frac{1}{r}u^{s/a}_{zn} \in L_{2,1/2}(\Omega_a)$, we get $\mathbf{u}^s_n,\ \mathbf{u}^a_n \in W^a(\Omega_a)$, $n \in \mathbf{N}$. The boundary condition $\mathbf{u}^s_n = \mathbf{u}^a_n = \mathbf{0}$ on Γ_{aD} $(n \in \mathbf{N}_0)$ follows from $\mathbf{u} = \mathbf{0}$ on $\Gamma_D = \Gamma_{aD} \times (-\pi, \pi]$ (see the definition of $V_0(\Omega)$).

(ii) Assertion (ii) obviously follows from (i), the definition of $V^a(\Omega_a)$ and Lemma 3.5.

(iii) In the following, we show that the relations $u^{s/a}_{r1} - u^{s/a}_{\varphi 1}$, $u^{s/a}_{z1} \in X^{1,2}_{1/2}(\Omega_a)$, and for

$n \in \mathbf{N} \setminus \{1\}$, $u_{rn}^{s/a}$, $u_{\varphi n}^{s/a}$, $u_{zn}^{s/a} \in X_{1/2}^{1,2}(\Omega_a)$ are satisfied. Owing to (3.16) and (i), the terms $\frac{1}{r}(u_{rn}^{s/a} - n u_{\varphi n}^{s/a})$, $\frac{\partial u_{rn}^{s/a}}{\partial r} - n\frac{\partial u_{\varphi n}^{s/a}}{\partial r}$, $\frac{\partial u_{rn}^{s/a}}{\partial z} - n\frac{\partial u_{\varphi n}^{s/a}}{\partial z}$, and $\frac{1}{r}(n u_{rn}^{s/a} - u_{\varphi n}^{s/a})$, $n\frac{\partial u_{rn}^{s/a}}{\partial r} - \frac{\partial u_{\varphi n}^{s/a}}{\partial r}$, $n\frac{\partial u_{rn}^{s/a}}{\partial z} - \frac{\partial u_{\varphi n}^{s/a}}{\partial z}$ are elements of the function space $L_{2,1/2}(\Omega_a)$. Moreover, repeating arguments from (i), we see that $u_{rn}^{s/a} - n u_{\varphi n}^{s/a}$, $n u_{rn}^{s/a} - u_{\varphi n}^{s/a} \in X_{1/2}^{1,2}(\Omega_a)$ holds. According to Lemma 3.5 the identities $u_{rn}^{s/a} - n u_{\varphi n}^{s/a} = 0$, $n u_{rn}^{s/a} - u_{\varphi n}^{s/a} = 0$ on Γ_0, for $n \in \mathbf{N}$, are obvious. From here, one easily sees that the relations $u_{r1}^{s/a} = u_{\varphi 1}^{s/a}$ on Γ_0 and $u_{rn}^{s/a} = u_{\varphi n}^{s/a} = 0$ on Γ_0 for $n = 2, 3, 4, \ldots$ are satisfied. For the components $u_{zn}^{s/a}$, $n \in \mathbf{N}$, one again uses (3.16) and Lemma 3.5 to complete the proof. ■

4 Partial Fourier decomposition of the BVP

In the previous section, we were mainly concerned with the analysis of the partial Fourier series expansion and with the study of the properties of the Fourier coefficients of the solution $\mathbf{u} \in W(\Omega)$ of the three-dimensional boundary value problem. The aim of this section consists in decomposing the three-dimensional variational problem (2.15) into a sequence of two-dimensional variational problems on the meridian plane Ω_a using partial Fourier analysis. Moreover, we show that there are unique solutions of the two-dimensional boundary value problems and that these solutions are the Fourier coefficients of the solution $\mathbf{u} \in W(\Omega)$ of (2.15). For simplicity, we introduce the following notations:

$$\begin{aligned}
\mathbf{u}_n^{s/a} &= (u_{rn}^{s/a}(r,z),\ u_{\varphi n}^{s/a}(r,z),\ u_{zn}^{s/a}(r,z))^T, \\
\mathbf{f}_n^{s/a} &= (f_{rn}^{s/a}(r,z),\ f_{\varphi n}^{s/a}(r,z),\ f_{zn}^{s/a}(r,z))^T, \\
\varepsilon_n^{s/a} &= (\varepsilon_{rrn}^{s/a},\ \varepsilon_{\varphi\varphi n}^{s/a},\ \varepsilon_{zzn}^{s/a},\ \gamma_{rzn}^{s/a},\ \gamma_{\varphi rn}^{s/a},\ \gamma_{z\varphi n}^{s/a})^T, \\
\sigma_n^{s/a} &= (\sigma_{rrn}^{s/a},\ \sigma_{\varphi\varphi n}^{s/a},\ \sigma_{zzn}^{s/a},\ \tau_{rzn}^{s/a},\ \tau_{\varphi rn}^{s/a},\ \tau_{z\varphi n}^{s/a})^T, \qquad (4.1)\\
\mathbf{R}_n^s &= diag[\cos n\varphi,\ -\sin n\varphi,\ \cos n\varphi], \\
\mathbf{R}_n^a &= diag[\sin n\varphi,\ \cos n\varphi,\ \sin n\varphi], \\
\mathbf{Q}_n^s &= diag[\cos n\varphi,\ \cos n\varphi,\ \cos n\varphi,\ \cos n\varphi,\ -\sin n\varphi,\ -\sin n\varphi], \\
\mathbf{Q}_n^a &= diag[\sin n\varphi,\ \sin n\varphi,\ \sin n\varphi,\ \sin n\varphi,\ \cos n\varphi,\ \cos n\varphi], \\
\mathbf{D}_n &= \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & -\frac{n}{r} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \\ \frac{n}{r} & \frac{\partial}{\partial r} - \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{n}{r} \end{bmatrix}.
\end{aligned}$$

Here, $\mathbf{R}_n^{s/a}$ and $\mathbf{Q}_n^{s/a}$ are diagonal matrices with the given diagonal elements.

Theorem 4.1. Let $\mathbf{u}, \mathbf{v} \in W(\Omega)$ and $\mathbf{f} \in (X_{1/2}^0(\Omega))^3$ be satisfied, and let the strain ε and the stress tensors σ be defined as in (2.12) and (2.13), respectively. Then the following assertions are satisfied:

(i) The strain and stress tensors can be represented by converging Fourier series of the form

$$\varepsilon = \sum_{n=0}^{\infty} \left(\mathbf{Q}_n^s \varepsilon_n^s + \mathbf{Q}_n^a \varepsilon_n^a\right), \quad \text{with} \quad \varepsilon_n^{s/a} = \mathbf{D}_n \mathbf{u}_n^{s/a}, \qquad (4.2)$$

$$\sigma = \sum_{n=0}^{\infty} \left(\mathbf{Q}_n^s \sigma_n^s + \mathbf{Q}_n^a \sigma_n^a \right), \quad \text{with} \quad \sigma_n^{s/a} = \mathbf{E} \varepsilon_n^{s/a} \quad \text{and} \quad \mathbf{E} \quad \text{from} \quad (2.13). \tag{4.3}$$

(ii) The strain tensor ε and its Fourier coefficients $\varepsilon_n^{s/a}$ satisfy the completeness relation

$$\begin{aligned} \|\varepsilon(\mathbf{v})\|^2_{(X^0_{1/2}(\Omega))^6} &= 2\pi\{\|\varepsilon_0^s(\mathbf{v}_0^s)\|^2_{(L_{2,1/2}(\Omega_a))^6} + \|\varepsilon_0^a(\mathbf{v}_0^a)\|^2_{(L_{2,1/2}(\Omega_a))^6}\} \\ &+ \pi \sum_{n=1}^{\infty} \{\|\varepsilon_n^s(\mathbf{v}_n^s)\|^2_{(L_{2,1/2}(\Omega_a))^6} + \|\varepsilon_n^a(\mathbf{v}_n^a)\|^2_{(L_{2,1/2}(\Omega_a))^6}\} < \infty. \end{aligned} \tag{4.4}$$

(iii) The bilinear form $b(\mathbf{u}, \mathbf{v})$ and the linear functional $f(\mathbf{v})$ from (2.16) can be represented in the form

$$b(\mathbf{u}, \mathbf{v}) = 2\pi\{b_0(\mathbf{u}_0^s, \mathbf{v}_0^s) + b_0(\mathbf{u}_0^a, \mathbf{v}_0^a)\} + \pi \sum_{n=1}^{\infty} \{b_n(\mathbf{u}_n^s, \mathbf{v}_n^s) + b_n(\mathbf{u}_n^a, \mathbf{v}_n^a)\}, \tag{4.5}$$

$$\mathbf{f}(\mathbf{v}) = 2\pi\{\mathbf{f}_0^s(\mathbf{v}_0^s) + \mathbf{f}_0^a(\mathbf{v}_0^a)\} + \pi \sum_{n=1}^{\infty} \{\mathbf{f}_n^s(\mathbf{v}_n^s) + \mathbf{f}_n^a(\mathbf{v}_n^a)\}. \tag{4.6}$$

The bilinear forms $b_n(\mathbf{u}_n^{s/a}, \mathbf{v}_n^{s/a})$ and the linear functionals $\mathbf{f}_n^{s/a}(\mathbf{v}_n^{s/a})$, $n \in \mathbf{N}_0$, from (4.5) and (4.6), in particular with $\mathbf{u}_0^s = (u_{r0}^s,\ 0,\ u_{z0}^s)^T$ and $\mathbf{u}_0^a = (0,\ u_{\varphi 0}^a,\ 0)^T$, are given by

$$\begin{aligned} b_n(\mathbf{u}_n^{s/a}, \mathbf{v}_n^{s/a}) &= \int_{\Omega_a} \left(\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\right)^T \sigma_n^{s/a}(\mathbf{v}_n^{s/a}) r\,dr\,dz \\ &= \frac{E}{(1+\nu)(1-2\nu)} \int_{\Omega_a} \Big\{ (1-\nu) \Big[\frac{\partial u_{rn}^{s/a}}{\partial r} \frac{\partial v_{rn}^{s/a}}{\partial r} + \frac{n^2}{r^2} u_{\varphi n}^{s/a} v_{\varphi n}^{s/a} \\ &- \frac{n}{r^2} u_{\varphi n}^{s/a} v_{rn}^{s/a} - \frac{n}{r^2} u_{rn}^{s/a} v_{\varphi n}^{s/a} + \frac{1}{r^2} u_{rn}^{s/a} v_{rn}^{s/a} + \frac{\partial u_{zn}^{s/a}}{\partial z} \frac{\partial v_{zn}^{s/a}}{\partial z} \Big] \\ &+ \nu \Big[\frac{1}{r} \frac{\partial u_{rn}^{s/a}}{\partial r} v_{rn}^{s/a} - \frac{n}{r} \frac{\partial u_{rn}^{s/a}}{\partial r} v_{\varphi n}^{s/a} + \frac{\partial u_{rn}^{s/a}}{\partial r} \frac{\partial v_{zn}^{s/a}}{\partial z} \\ &- \frac{n}{r} u_{\varphi n}^{s/a} \frac{\partial v_{rn}^{s/a}}{\partial r} - \frac{n}{r} u_{\varphi n}^{s/a} \frac{\partial v_{zn}^{s/a}}{\partial z} + \frac{1}{r} u_{rn}^{s/a} \frac{\partial v_{rn}^{s/a}}{\partial r} + \frac{1}{r} u_{rn}^{s/a} \frac{\partial v_{zn}^{s/a}}{\partial z} \\ &+ \frac{\partial u_{zn}^{s/a}}{\partial z} \frac{\partial v_{rn}^{s/a}}{\partial r} - \frac{n}{r} \frac{\partial u_{zn}^{s/a}}{\partial z} v_{\varphi n}^{s/a} + \frac{1}{r} \frac{\partial u_{zn}^{s/a}}{\partial z} v_{rn}^{s/a} \Big] \\ &+ \frac{1-2\nu}{2} \Big[\frac{\partial u_{\varphi n}^{s/a}}{\partial r} \frac{\partial v_{\varphi n}^{s/a}}{\partial r} + \frac{n}{r} \frac{\partial u_{\varphi n}^{s/a}}{\partial r} v_{rn}^{s/a} - \frac{1}{r} \frac{\partial u_{\varphi n}^{s/a}}{\partial r} v_{\varphi n}^{s/a} + \frac{n}{r} u_{rn}^{s/a} \frac{\partial v_{\varphi n}^{s/a}}{\partial r} \\ &+ \frac{n^2}{r^2} u_{rn}^{s/a} v_{rn}^{s/a} - \frac{n}{r^2} u_{rn}^{s/a} v_{\varphi n}^{s/a} - \frac{1}{r} u_{\varphi n}^{s/a} \frac{\partial v_{\varphi n}^{s/a}}{\partial r} - \frac{n}{r^2} u_{\varphi n}^{s/a} v_{rn}^{s/a} + \frac{1}{r^2} u_{\varphi n}^{s/a} v_{\varphi n}^{s/a} \\ &+ \frac{\partial u_{zn}^{s/a}}{\partial r} \frac{\partial v_{zn}^{s/a}}{\partial r} + \frac{\partial u_{zn}^{s/a}}{\partial r} \frac{\partial v_{rn}^{s/a}}{\partial z} + \frac{\partial u_{rn}^{s/a}}{\partial z} \frac{\partial v_{zn}^{s/a}}{\partial r} + \frac{\partial u_{rn}^{s/a}}{\partial z} \frac{\partial v_{rn}^{s/a}}{\partial z} + \frac{n^2}{r^2} u_{zn}^{s/a} v_{zn}^{s/a} \\ &+ \frac{n}{r} u_{zn}^{s/a} \frac{\partial v_{\varphi n}^{s/a}}{\partial z} + \frac{n}{r} \frac{\partial u_{\varphi n}^{s/a}}{\partial z} v_{zn}^{s/a} + \frac{\partial u_{\varphi n}^{s/a}}{\partial z} \frac{\partial v_{\varphi n}^{s/a}}{\partial z} \Big] \Big\} r\,dr\,dz, \end{aligned} \tag{4.7}$$

$$\mathbf{f}_n^{s/a}(\mathbf{v}_n^{s/a}) = \int_{\Omega_a} \mathbf{f}_n^{s/a^T} \mathbf{v}_n^{s/a} r\,dr\,dz = \int_{\Omega_a} \{f_{rn}^{s/a} v_{rn}^{s/a} + f_{\varphi n}^{s/a} v_{\varphi n}^{s/a} + f_{zn}^{s/a} v_{zn}^{s/a}\} r\,dr\,dz. \tag{4.8}$$

Proof: (i) Since $\mathbf{u} = (u_r,\ u_\varphi,\ u_z)^T \in W(\Omega)$ is satisfied, the functions $u_r,\ u_\varphi,\ u_z$ as well as their first order partial derivatives are contained in $X^0_{1/2}(\Omega)$ (cf. (2.11) and Lemma

3.1). Consequently, the components of the strain and stress tensors can be expressed by converging Fourier series using the series (3.2) and (3.4). Finally, using the relations given by Lemma 3.2, the definition of $\varepsilon_n^{s/a}$ and $\sigma_n^{s/a}$ and the notations (4.1), one easily shows that the expressions (4.2) and (4.3), respectively, represent the Fourier series expansion of ε and σ with respect to the system of trigonometric functions (3.1).
(ii) The proof of (4.4) is analogous to the proof of (iii) given next.
(iii) Let $b(\mathbf{u},\mathbf{v})$ be defined as in (2.16). Using the relations (4.2) and (4.3) we get by applying the results of the theorems of Lebesgue and Fubini the following identities:

$$\begin{aligned} b(\mathbf{u},\mathbf{v}) &= \int_\Omega (\varepsilon(\mathbf{u}))^T \sigma(\mathbf{v}) r dr d\varphi dz \\ &= \int_\Omega \Big\{ \sum_{n=0}^\infty \Big[(\mathbf{Q}_n^s \varepsilon_n^s(\mathbf{u}_n^s))^T + (\mathbf{Q}_n^a \varepsilon_n^a(\mathbf{u}_n^a))^T \Big] \sum_{m=0}^\infty [\mathbf{Q}_m^s \sigma_m^s(\mathbf{v}_m^s) + \mathbf{Q}_m^a \sigma_m^a(\mathbf{v}_m^a)] \Big\} r dr d\varphi dz \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \int_{\Omega_a} \int_{-\pi}^{\pi} \Big\{ (\varepsilon_n^s(\mathbf{u}_n^s))^T \mathbf{Q}_n^{sT} \mathbf{Q}_m^s \sigma_m^s(\mathbf{v}_m^s) + (\varepsilon_n^a(\mathbf{u}_n^a))^T \mathbf{Q}_n^{aT} \mathbf{Q}_m^a \sigma_m^a(\mathbf{v}_m^a) \\ &+ (\varepsilon_n^s(\mathbf{u}_n^s))^T \mathbf{Q}_n^{sT} \mathbf{Q}_m^a \sigma_m^a(\mathbf{v}_m^a) + (\varepsilon_n^a(\mathbf{u}_n^a))^T \mathbf{Q}_n^{aT} \mathbf{Q}_m^s \sigma_m^s(\mathbf{v}_m^s) \Big\} d\varphi r dr dz \\ &= \sum_{n=0}^\infty \sum_{m=0}^\infty \int_{\Omega_a} \Big\{ (\varepsilon_n^s(\mathbf{u}_n^s))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{sT} \mathbf{Q}_m^s \, d\varphi \, \sigma_m^s(\mathbf{v}_m^s) \\ &+ (\varepsilon_n^a(\mathbf{u}_n^a))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{aT} \mathbf{Q}_m^a \, d\varphi \, \sigma_m^a(\mathbf{v}_m^a) + (\varepsilon_n^s(\mathbf{u}_n^s))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{sT} \mathbf{Q}_m^a \, d\varphi \, \sigma_m^a(\mathbf{v}_m^a) \\ &+ (\varepsilon_n^a(\mathbf{u}_n^a))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{aT} \mathbf{Q}_m^s \, d\varphi \, \sigma_m^s(\mathbf{v}_m^s) \Big\} r dr dz. \end{aligned} \tag{4.9}$$

The integrals $\int_{-\pi}^{\pi} \mathbf{Q}_n^{sT} \mathbf{Q}_m^s d\varphi$, $\int_{-\pi}^{\pi} \mathbf{Q}_n^{aT} \mathbf{Q}_m^a d\varphi$, $\int_{-\pi}^{\pi} \mathbf{Q}_n^{sT} \mathbf{Q}_m^a d\varphi$, and $\int_{-\pi}^{\pi} \mathbf{Q}_n^{aT} \mathbf{Q}_m^s d\varphi$ contain expressions of the type (see (4.1))

$$\begin{aligned} \int_{-\pi}^{\pi} \cos n\varphi \cos m\varphi d\varphi &= \begin{cases} 0 & : \; n \neq m \\ \pi & : \; n = m \neq 0 \\ 2\pi & : \; n = m = 0 \end{cases} \\ \int_{-\pi}^{\pi} \sin n\varphi \sin m\varphi d\varphi &= \begin{cases} 0 & : \; n \neq m \\ \pi & : \; n = m \neq 0 \\ 0 & : \; n = m = 0 \end{cases} \\ \int_{-\pi}^{\pi} \sin n\varphi \cos n\varphi d\varphi &= 0, \quad n \in \mathbf{N}_0. \end{aligned} \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$\begin{aligned} b(\mathbf{u},\mathbf{v}) &= \sum_{n=0}^\infty \int_{\Omega_a} \Big\{ (\varepsilon_n^s(\mathbf{u}_n^s))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{sT} \mathbf{Q}_m^s \, d\varphi \, \sigma_m^s(\mathbf{v}_m^s) \\ &\quad + (\varepsilon_n^a(\mathbf{u}_n^a))^T \int_{-\pi}^{\pi} \mathbf{Q}_n^{aT} \mathbf{Q}_m^a \, d\varphi \, \sigma_m^a(\mathbf{v}_m^a) \Big\} r dr dz \\ &= 2\pi \int_{\Omega_a} \Big\{ (\varepsilon_0^s(\mathbf{u}_0^s))^T \sigma_0^s(\mathbf{v}_0^s) + (\varepsilon_0^a(\mathbf{u}_0^a))^T \sigma_0^a(\mathbf{v}_0^a) \Big\} r dr dz \\ &\quad + \pi \sum_{n=1}^\infty \int_{\Omega_a} \Big\{ (\varepsilon_n^s(\mathbf{u}_n^s))^T \sigma_n^s(\mathbf{v}_n^s) + (\varepsilon_n^a(\mathbf{u}_0^a))^T \sigma_n^a(\mathbf{v}_n^a) \Big\} r dr dz \\ &= 2\pi \{ b_0(\mathbf{u}_0^s, \mathbf{v}_0^s) + b_0(\mathbf{u}_0^a, \mathbf{v}_0^a) \} + \pi \sum_{n=1}^\infty \{ b_n(\mathbf{u}_n^s, \mathbf{v}_n^s) + b_n(\mathbf{u}_n^a, \mathbf{v}_n^a) \}. \end{aligned}$$

Similarly, using for $\mathbf{f}$ Fourier series expansion of the type (3.2), (3.3) and the notation in (4.1), one gets the relations

$$
\begin{aligned}
f(\mathbf{v}) &= \int_\Omega \mathbf{f}^T \mathbf{v} r dr d\varphi dz \\
&= \sum_{n=0}^{\infty} \int_{\Omega_a} \Big\{ \mathbf{f}_n^{sT} \int_{-\pi}^{\pi} \mathbf{R}_n^{sT} \mathbf{R}_n^s \, d\varphi \, \mathbf{v}_n^s + \mathbf{f}_n^{aT} \int_{-\pi}^{\pi} \mathbf{R}_n^{aT} \mathbf{R}_n^a \, d\varphi \, \mathbf{v}_n^a \Big\} r dr dz \\
&= 2\pi \Big\{ \int_{\Omega_a} \mathbf{f}_0^{sT} \mathbf{v}_0^s r dr dz + \int_{\Omega_a} \mathbf{f}_0^{aT} \mathbf{v}_0^a r dr dz \Big\} \\
&\quad + \pi \sum_{n=1}^{\infty} \Big\{ \int_{\Omega_a} \mathbf{f}_n^{sT} \mathbf{v}_n^s r dr dz + \int_{\Omega_a} \mathbf{f}_n^{aT} \mathbf{v}_n^a r dr dz \Big\} \\
&= 2\pi \{ \mathbf{f}_0^s(\mathbf{v}_0^s) + \mathbf{f}_0^a(\mathbf{v}_0^a) \} + \pi \sum_{n=1}^{\infty} \{ \mathbf{f}_n^s(\mathbf{v}_n^s) + \mathbf{f}_n^a(\mathbf{v}_n^a) \}. \blacksquare
\end{aligned}
$$

In Theorem 4.1 it is proved above all that the symmetric bilinear form $b(\mathbf{u}, \mathbf{v})$ and the linear functional $f(\mathbf{v})$ in three dimensions can be represented by converging series. For $n \in \mathbf{N}_0$, the terms $b_n(\mathbf{u}_n^{s/a}, \mathbf{v}_n^{s/a})$ are obviously symmetric bilinear forms, and the terms $\mathbf{f}_n^{s/a}(\mathbf{v}_n^{s/a})$, $n \in \mathbf{N}_0$, are linear forms in the corresponding function spaces on Ω_a. Moreover, from the representation of $b(\mathbf{u}, \mathbf{v})$ and $f(\mathbf{v})$ in Theorem 4.1 one can derive a sequence of two-dimensional variational problems, which are totally decoupled and can be solved independently from each other.

Theorem 4.2. Let $\mathbf{u}_n^s$ and $\mathbf{u}_n^a$ be the Fourier coefficients of the solution $\mathbf{u} \in V_0(\Omega)$ of the three-dimensional variational problem (2.15). Then $\mathbf{u}_n^s$ and $\mathbf{u}_n^a$, $n \in \mathbf{N}_0$, are unique solutions of the following two-dimensional variational problems.
Find functions

$$
\begin{aligned}
\mathbf{u}_0^s &= (u_{r0}^s, 0, u_{z0}^s)^T, \quad \mathbf{u}_0^a = (0, u_{\varphi 0}^a, 0)^T \in V_0^a(\Omega_a) \quad \text{and} \\
\mathbf{u}_n^s &= (u_{rn}^s, u_{\varphi n}^s, u_{zn}^s)^T, \quad \mathbf{u}_n^a = (u_{rn}^a, u_{\varphi n}^a, u_{zn}^a)^T \in W_0^a(\Omega_a), \quad n \in \mathbf{N},
\end{aligned}
$$

satisfying the equations

$$
\begin{aligned}
b_0(\mathbf{u}_0^s, \mathbf{w}) &= f_0^s(\mathbf{w}), && (4.11)\\
b_0(\mathbf{u}_0^a, \mathbf{w}) &= f_0^a(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in V_0^a(\Omega_a), && (4.12)\\
b_n(\mathbf{u}_n^s, \mathbf{w}) &= f_n^s(\mathbf{w}), && (4.13)\\
b_n(\mathbf{u}_n^a, \mathbf{w}) &= f_n^a(\mathbf{w}) \quad \text{for} \quad \mathbf{w} \in W_0^a(\Omega_a), \quad n \in \mathbf{N}, && (4.14)
\end{aligned}
$$

where the bilinear forms $b_n(\mathbf{u}_n^{s/a}, \mathbf{w})$ and the linear forms $f_n^{s/a}(\mathbf{w})$, $n \in \mathbf{N}_0$, are defined by (4.7) and (4.8), respectively.

Proof: We consider the variational equation (2.15), with $b(\mathbf{u}, \mathbf{v})$ and $f(\mathbf{v})$ according to the completeness relations (4.5) and (4.6). For proving the decoupling, we define some special test functions as follows:

$$
\begin{aligned}
\mathbf{v}^{(0)} &:= (w_r(r,z),\ 0,\ w_z(r,z))^T \in V_0^a(\Omega_a), \\
\mathbf{v}^{(n)} &:= (w_r(r,z)\cos n\varphi,\ w_\varphi(r,z)(-\sin n\varphi),\ w_z(r,z)\cos n\varphi)^T \quad \text{for} \quad n \in \mathbf{N}, \qquad (4.15)\\
&\quad\ \text{with} \quad (w_r(r,z),\ w_\varphi(r,z),\ w_z(r,z))^T \in W_0^a(\Omega_a).
\end{aligned}
$$

It is clear that $\mathbf{v}^{(n)} \in V_0(\Omega)$ for all $n \in \mathbf{N}_0$, with $V_0(\Omega)$ from (2.11). For these functions, Fourier series expansion of the type (3.2) can be proved and their corresponding Fourier coefficients are given according to (3.3) by

$$\begin{aligned}\left(\mathbf{v}^{(0)}\right)_k^s &= (w_r(r,z),\ 0, w_z(r,z))^T\,\delta_{0k}, \quad \left(\mathbf{v}^{(0)}\right)_k^a = \mathbf{0},\ k\in\mathbf{N}_0,\\ \left(\mathbf{v}^{(n)}\right)_k^s &= (w_r(r,z), w_\varphi(r,z), w_z(r,z))^T\,\delta_{nk}, \quad \left(\mathbf{v}^{(n)}\right)_k^a = \mathbf{0},\ k\in\mathbf{N}_0,\ n\in\mathbf{N},\end{aligned}$$

where δ_{nk} is the Kronecker symbol. We get the equations (4.11) and (4.13) by inserting these test functions in (2.15) and (2.16), and by taking into consideration the completeness relations (4.5) and (4.6). On the other hand, the equations (4.12) and (4.14) are derived by using the test functions

$$\begin{aligned}\mathbf{v}^{(0)} &= (0,\ w_\varphi(r,z),\ 0)^T \in V_0^a(\Omega_a),\\ \mathbf{v}^{(n)} &= (w_r(r,z)\sin n\varphi,\ w_\varphi(r,z)\cos n\varphi,\ w_z(r,z)\sin n\varphi)^T \quad \text{for } n\in\mathbf{N},\\ &\quad \text{with } (w_r(r,z),\ w_\varphi(r,z),\ w_z(r,z))^T \in W_0^a(\Omega_a),\end{aligned}\tag{4.16}$$

in (2.15) and (2.16), together with the representation (4.5) and (4.6).
Subsequently, we show that the equations (4.11) – (4.14) have unique solutions, which are the Fourier coefficients of the three-dimensional solution. To do this, we have to prove that the assumptions of the lemma of Lax-Milgram (see e.g. [6]) are fulfilled.
The continuity of the linear forms $f_n^{s/a}(.)$ follows immediately from the Cauchy-Schwarz inequality and further estimates, viz.

$$\begin{aligned}\left|f_0^{s/a}(\mathbf{w})\right| &= \left|\int_{\Omega_a} \mathbf{f}_0^{s/a^T}\mathbf{w}\,r\,dr\,dz\right| \le \|\mathbf{f}_0^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{w}\|_{(L_{2,1/2}(\Omega_a))^3}\\ &\le C\|\mathbf{f}_0^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{w}\|_{V_0^a(\Omega_a)} \quad \text{for} \quad \mathbf{w}\in V_0^a(\Omega_a) \qquad (4.17)\\ \left|f_n^{s/a}(\mathbf{w})\right| &= \left|\int_{\Omega_a} \mathbf{f}_n^{s/a^T}\mathbf{w}\,r\,dr\,dz\right| \le \|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{w}\|_{(L_{2,1/2}(\Omega_a))^3}\\ &\le \|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{w}\|_{W_0^a(\Omega_a)} \quad \text{for} \quad \mathbf{w}\in W_0^a(\Omega_a),\ n\in\mathbf{N}, \qquad (4.18)\end{aligned}$$

with some positive constant C. From Korn's inequality (2.5), we get the relation

$$\|\mathbf{v}\|^2_{V_0(\Omega)} \le K\|\varepsilon(\mathbf{v})\|^2_{\left(X^0_{1/2}(\Omega)\right)^6} \quad \text{for} \quad \mathbf{v}\in V_0(\Omega),\ \text{with } K>0. \tag{4.19}$$

Inserting the test functions defined by (4.15) and (4.16) in (4.19), and taking into consideration the inequality (3.17) and the completeness relation (4.4), we derive the a priori estimates

$$\begin{aligned}\|\mathbf{v}_0^s\|^2_{V_0^a(\Omega_a)} &\le K\|\varepsilon_0^s(\mathbf{v}_0^s)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6},\ \|\mathbf{v}_0^a\|^2_{V_0^a(\Omega_a)} \le K\|\varepsilon_0^a(\mathbf{v}_0^a)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6}, \qquad (4.20)\\ \|\mathbf{v}_n^s\|^2_{W_0^a(\Omega_a)} &\le K\|\varepsilon_n^s(\mathbf{v}_n^s)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6},\ \|\mathbf{v}_n^a\|^2_{W_0^a(\Omega_a)} \le K\|\varepsilon_n^a(\mathbf{v}_n^a)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6},\ n\in\mathbf{N},\end{aligned}$$

where $\mathbf{v}_n^s, \mathbf{v}_n^a$ ($n\in\mathbf{N}_0$) are the Fourier coefficients of $\mathbf{v}^{(n)}$, $n\in\mathbf{N}_0$. It follows from (2.6) (see Remark 2.1), the definition of $\|\cdot\|_{V_0(\Omega)}$, $\|\cdot\|_{V_0^a(\Omega_a)}$, $\|\cdot\|_{W_0^a(\Omega_a)}$ and the completeness relation (4.4) that the following inequalities hold:

$$\begin{aligned}\|\varepsilon_0^s(\mathbf{v}_0^s)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} &\le C\|\mathbf{v}_0^s\|^2_{V_0^a(\Omega_a)},\ \|\varepsilon_0^a(\mathbf{v}_0^a)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \le C\|\mathbf{v}_0^a\|^2_{V_0^a(\Omega_a)}, \qquad (4.21)\\ \|\varepsilon_n^s(\mathbf{v}_n^s)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} &\le C\|\mathbf{v}_n^s\|^2_{W_0^a(\Omega_a)},\ \|\varepsilon_n^a(\mathbf{v}_n^a)\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \le C\|\mathbf{v}_n^a\|^2_{W_0^a(\Omega_a)},\ n\in\mathbf{N},\end{aligned}$$

whith some $C > 0$. Denoting by λ_1 the largest eigenvalue of the matrix $\mathbf{E}$ from (2.13) and using (4.7) as well as the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |b_n(\mathbf{u}_n^{s/a}, \mathbf{v}_n^{s/a})| &= \Big| \int_{\Omega_a} \left(\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a}) \right)^T \mathbf{E}\, \varepsilon_n^{s/a}(\mathbf{v}_n^{s/a}) r dr dz \Big| \\ &\leq \lambda_1 \|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|_{\left(L_{2,1/2}(\Omega_a)\right)^6} \|\varepsilon_n^{s/a}(\mathbf{v}_n^{s/a})\|_{\left(L_{2,1/2}(\Omega_a)\right)^6}, \; n \in \mathbf{N}_0. \quad (4.22) \end{aligned}$$

Finally, the continuity of $b_n(.,.)$ follows from (4.22) and (4.21). If λ_2 denotes the smallest eigenvalue of $\mathbf{E}$, we get the V_0^a- and W_0^a-ellipticity condition, respectively, by considering

$$b_n(\mathbf{u}_n^{s/a}, \mathbf{u}_n^{s/a}) = \int_{\Omega_a} \left(\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a}) \right)^T \mathbf{E}\, \varepsilon_n^{s/a}(\mathbf{u}_n^{s/a}) r dr dz \geq \lambda_2 \|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6}, \quad (4.23)$$

where $n \in \mathbf{N}_0$, together with relation (4.20). Thus, it is proved that the assumptions of the Lax-Milgram lemma are satisfied. The uniqueness of the solutions of the variational problems (4.11) – (4.14) implies that $\mathbf{u}_n^{s/a}$, $n \in \mathbf{N}_0$, are the Fourier coefficients of the solution $\mathbf{u}$ of the three-dimensional variational problem (2.15). ■

Theorem 4.3. Let $\mathbf{u}_n^{s/a} = (u_{rn}^{s/a},\ u_{\varphi n}^{s/a},\ u_{zn}^{s/a})^T$ and $\mathbf{f}_n^{s/a} = (f_{rn}^{s/a},\ f_{\varphi n}^{s/a},\ f_{zn}^{s/a})^T$, $n \in \mathbf{N}_0$, be the Fourier coefficients of $\mathbf{u}$ and $\mathbf{f}$ from (2.15). Then, for $\mathbf{u}_n^{s/a}$ we get the following a priori estimates:

$$\|\mathbf{u}_0^{s/a}\|^2_{V_0^a(\Omega_a)} \leq K \|\varepsilon_0^{s/a}(\mathbf{u}_0^{s/a})\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \leq C \|\mathbf{f}_0^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3}, \quad (4.24)$$

$$\|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)} \leq K \|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \leq C \|\mathbf{f}_n^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3}, \quad n \in \mathbf{N}, \quad (4.25)$$

$$\begin{aligned} \|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)} &\leq M \Big\{ \|\mathbf{u}_n^{s/a}\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2 \Big\| \frac{1}{r} \mathbf{u}_n^{s/a} \|^2_{(L_{2,1/2}(\Omega_a))^3} \Big\} \\ &\leq \frac{C}{n^2} \|\mathbf{f}_n^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3} \quad \text{for } n \geq 2, \quad (4.26) \end{aligned}$$

with some positive constants C, K and M.

Proof: For brevity, we utilize in the following C as a generic positive constant; i. e., C has different values at different places, but C is independent of n and of the corresponding Fourier coefficients. By analogy to (4.17), (4.18), we get

$$\left| f_0^{s/a}(\mathbf{u}_0^{s/a}) \right| \leq C \|\mathbf{f}_0^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3} \|\mathbf{u}_0^{s/a}\|_{V_0^a(\Omega_a)}, \quad (4.27)$$

$$\left| f_n^{s/a}(\mathbf{u}_n^{s/a}) \right| \leq \|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3} \|\mathbf{u}_n^{s/a}\|_{W_0^a(\Omega_a)}, \; n \in \mathbf{N}. \quad (4.28)$$

Starting from (4.11)–(4.14) for $\mathbf{w} = \mathbf{u}^{s/a}$ and using (4.23), the relations

$$\left| f_n^{s/a}(\mathbf{u}_n^{s/a}) \right| \geq \lambda_2 \|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6}, \; n \in \mathbf{N}_0, \quad (4.29)$$

hold. Combining (4.27), (4.28) and (4.29), one derives the inequalities

$$\|\varepsilon_0^{s/a}\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \leq C \|\mathbf{f}_0^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3} \|\mathbf{u}_0^{s/a}\|_{V_0^a(\Omega_a)}, \quad (4.30)$$

$$\|\varepsilon_n^{s/a}\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \leq C \|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3} \|\mathbf{u}_n^{s/a}\|_{W_0^a(\Omega_a)}, \quad n \in \mathbf{N}. \quad (4.31)$$

Using (4.23) and (4.20), we get

$$b_0(\mathbf{u}_0^{s/a},\mathbf{u}_0^{s/a}) \geq C\|\mathbf{u}_0^{s/a}\|^2_{V_0^a(\Omega_a)}, \quad b_n(\mathbf{u}_n^{s/a},\mathbf{u}_n^{s/a}) \geq C\|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)}, \quad n\in\mathbf{N}. \quad (4.32)$$

Finally, owing to (4.27), (4.28) and (4.32), we derive the relations

$$b_0(\mathbf{u}_0^{s/a},\mathbf{u}_0^{s/a}) = f_0^{s/a}(\mathbf{u}_0^{s/a}) \leq C\|\mathbf{f}_0^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{u}_0^{s/a}\|_{V_0^a(\Omega_a)}, \quad (4.33)$$

$$b_n(\mathbf{u}_n^{s/a},\mathbf{u}_n^{s/a}) = f_n^{s/a}(\mathbf{u}_n^{s/a}) \leq C\|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|_{W_0^a(\Omega_a)}, \ n\in\mathbf{N}. \quad (4.34)$$

Taking into account (4.30), (4.33) and (4.31), (4.34), respectively, we derive the relations

$$\|\varepsilon_n^{s/a}\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6} \leq C\|\mathbf{f}_n^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3}, \quad n\in\mathbf{N_0}. \quad (4.35)$$

Finally, the assertions (4.24) and (4.25) follow from (4.20) and (4.35).
In order to show assertion (4.26), we first of all prove the following equivalence for some norm terms:

$$\begin{aligned}
&\Big\{n^2\Big\|\frac{1}{r}u_{rn}^{s/a}\|^2_{L_{2,1/2}(\Omega_a)} + n^2\Big\|\frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\}\\
\leq\ & C\Big\|\frac{1}{r}u_{rn}^{s/a} - \frac{n}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \Big\|\frac{n}{r}u_{rn}^{s/a} - \frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)}\\
\leq\ & C\Big\{n^2\Big\|\frac{1}{r}u_{rn}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)} + n^2\Big\|\frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)}\Big\} \quad \text{for } n\geq 2. \quad (4.36)
\end{aligned}$$

Obviously, for $n \geq 2$ the inequality

$$n^2 \leq \frac{16}{9}\Big(n-\frac{1}{n}\Big)^2 \quad (4.37)$$

holds, and from

$$\Big(n-\frac{1}{n}\Big)\frac{u_{rn}^{s/a}}{r} = \Big(\frac{n}{r}u_{rn}^{s/a} - \frac{1}{r}u_{\varphi n}^{s/a}\Big) - \frac{1}{n}\Big(\frac{1}{r}u_{rn}^{s/a} - \frac{n}{r}u_{\varphi n}^{s/a}\Big),$$

one obtains the inequality

$$\Big(n-\frac{1}{n}\Big)^2\Big\|\frac{u_{rn}^{s/a}}{r}\Big\|^2_{L_{2,1/2}(\Omega)} \leq 2\Big\|\frac{n}{r}u_{rn}^{s/a} - \frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \frac{2}{n^2}\Big\|\frac{1}{r}u_{rn}^{s/a} - \frac{n}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega)}. \quad (4.38)$$

Similarly, one shows the relation

$$\Big(n-\frac{1}{n}\Big)^2\Big\|\frac{u_{\varphi n}^{s/a}}{r}\Big\|^2_{L_{2,1/2}(\Omega)} \leq 2\Big\|\frac{1}{r}u_{rn}^{s/a} - \frac{n}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)} + \frac{2}{n^2}\Big\|\frac{n}{r}u_{rn}^{s/a} - \frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega)}. \quad (4.39)$$

The first inequality of (4.36) follows from (4.37), (4.38) and (4.39). The application of the triangle inequality leads to the second one.
Taking the definition of the norm of $W_0^a(\Omega_a)$, inequality (4.19), and the completeness relations (4.4) and (3.16), we derive the estimate

$$\begin{aligned}
\|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)} &\leq C\{\|\mathbf{u}_n^{s/a}\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + \Big\|\frac{1}{r}u_{rn}^{s/a} - \frac{n}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)}\\
&+ \Big\|\frac{n}{r}u_{rn}^{s/a} - \frac{1}{r}u_{\varphi n}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)} + n^2\Big\|\frac{1}{r}u_{zn}^{s/a}\Big\|^2_{L_{2,1/2}(\Omega_a)}\}\\
&\leq C\|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|^2_{\left(L_{2,1/2}(\Omega_a)\right)^6}. \quad (4.40)
\end{aligned}$$

Furthermore, using (4.36), (4.29) and (4.40), we get

$$\begin{aligned}\|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)} &\le C\{\|\mathbf{u}_n^{s/a}\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\Big\|\frac{1}{r}\mathbf{u}_n^{s/a}\Big\|^2_{(L_{2,1/2}(\Omega_a))^3}\} \\ &\le C\|\varepsilon_n^{s/a}(\mathbf{u}_n^{s/a})\|^2_{(L_{2,1/2}(\Omega_a))^6} \le C\|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\|\mathbf{u}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3} \\ &\le C\|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\Big\|\frac{1}{r}\mathbf{u}_n^{s/a}\Big\|_{(L_{2,1/2}(\Omega_a))^3}. \end{aligned} \tag{4.41}$$

By means of the inequality $2ab \le a^2/n^2 + n^2b^2$, for $n \ge 2$ the relation

$$\|\mathbf{f}_n^{s/a}\|_{(L_{2,1/2}(\Omega_a))^3}\Big\|\tfrac{1}{r}\mathbf{u}_n^{s/a}\Big\|_{(L_{2,1/2}(\Omega_a))^3} \le \tfrac{1}{2}\Big\{\tfrac{1}{n^2}\|\mathbf{f}_n^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3} + n^2\Big\|\tfrac{1}{r}\mathbf{u}_n^{s/a}\Big\|^2_{(L_{2,1/2}(\Omega_a))^3}\Big\}. \tag{4.42}$$

is obvious. Taking (4.42) and (4.41), we get the completion of the proof by

$$\|\mathbf{u}_n^{s/a}\|^2_{W_0^a(\Omega_a)} \le \frac{C}{n^2}\|\mathbf{f}_n^{s/a}\|^2_{(L_{2,1/2}(\Omega_a))^3}. \quad \blacksquare \tag{4.43}$$

We now consider a finite number of two-dimensional problems (4.11)–(4.14), here for $n = 0, 1, ..., N$, and their solutions which are the corresponding Fourier coefficients $\mathbf{u}_n^{s/a}$ of $\mathbf{u}$. By means of $\mathbf{R}_n^{s/a}$ from (4.1), the solution $\mathbf{u} = (u_r, u_\varphi, u_z)^T$ of the three-dimensional boundary value problem (2.15) and its partial Fourier approximation (truncated partial Fourier series) $\mathbf{u}_N = (u_{rN}, u_{\varphi N}, u_{zN})^T$ can be written as follows:

$$\mathbf{u} = \sum_{n=0}^{\infty}(\mathbf{R}_n^s\mathbf{u}_n^s + \mathbf{R}_n^a\mathbf{u}_n^a) \quad \text{and} \quad \mathbf{u_N} = \sum_{n=0}^{N}(\mathbf{R}_n^s\mathbf{u}_n^s + \mathbf{R}_n^a\mathbf{u}_n^a)\,. \tag{4.44}$$

It is well-known that the approximation of functions in $L_2(-\pi, \pi)$ by means of the trigonometric functions $\{1, ..., \sin N\varphi, \cos N\varphi\}$ leads to the corresponding truncated Fourier series of the order N, see e. g. [4, 5, 11], with the obvious extension to partial Fourier series. Now we shall estimate the norm of the error $\mathbf{u} - \mathbf{u}_N$ in the space $W(\Omega)$.

Theorem 4.4. Let $\mathbf{u} \in V_0(\Omega)$ be the solution of the variational problem (2.15), with $\mathbf{f} \in (X^0_{1/2}(\Omega))^3$, and $\mathbf{u}_N$ its Fourier approximation defined by (4.44). Then there is a constant C independent of N and $\mathbf{f}$ such that the following estimate holds:

$$\|\mathbf{u} - \mathbf{u}_N\|_{W(\Omega)} \le CN^{-1}\|\mathbf{f}\|_{(X^0_{1/2}(\Omega))^3} \quad \text{for} \quad N \ge 1. \tag{4.45}$$

Proof: Using (4.44), the completeness relation (3.16), the triangle inequality and the a priori estimate (4.26), we get the inequalities

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_N\|^2_{W(\Omega)} &\le CN^{-2}\sum_{n=N+1}^{\infty}\sum_{e\in\{s,a\}} n^2\Big\{\|\mathbf{u}_n^e\|^2_{(W^{1,2}_{1/2}(\Omega_a))^3} + n^2\Big\|\frac{1}{r}\mathbf{u}_n^e\Big\|^2_{(L_{2,1/2}(\Omega_a))^3}\Big\} \\ &\le CN^{-2}\sum_{n=2}^{\infty}\sum_{e\in\{s,a\}}\|\mathbf{f}_n^e\|^2_{(L_{2,1/2}(\Omega_a))^3} \le CN^{-2}\|\mathbf{f}\|^2_{(X^0_{1/2}(\Omega))^3}, \end{aligned} \tag{4.46}$$

and, thus, Theorem 4.4 is proved. ■

Obviously, the error estimate (4.45) is valid without additional smoothness requirements like $\hat{\mathbf{u}} \in (W_2^2(\hat{\Omega}))^3$. This is due to the refined a priori estimate (4.26) which indicates some additional regularity of $\mathbf{u}$ with respect to the angle φ.

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