Application of random matrix theory to quasiperiodic systems

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Abstract

We study statistical properties of energy spectra of a tight-binding model on the two-dimensional quasiperiodic Ammann-Beenker tiling. Taking into account the symmetries of finite approximants, we find that the underlying universal level-spacing distribution is given by the Gaussian orthogonal random matrix ensemble, and thus differs from the critical level-spacing distribution observed at the metal-insulator transition in the three-dimensional Anderson model of disorder. Our data allow us to see the difference to the Wigner surmise.

In a recent paper [1], we investigated energy spectra of quasiperiodic tight-binding models, concentrating on the case of the octagonal Ammann-Beenker tiling [2] shown in Fig. 1. The Hamiltonian is restricted to constant hopping matrix elements along the edges of the tiles in Fig. 1. Previous studies of the same model had led to diverging results on the level statistics: For periodic approximants, level repulsion was observed [3,4], and the level-spacing distribution $P(s)$ was argued to follow a log-normal distribution [4]. On the other hand, for octagonal patches with an exact eightfold symmetry and free boundary conditions, level clustering was found [5]. On the basis of our numerical results for $P(s)$ and the spectral rigidity Δ_3 [6], compiled in Ref. [1], we concluded that the underlying universal level-spacing distribution of this system is given by the Gaussion orthogonal random matrix ensemble (GOE) [6,7]. Concerning the contradictory results of previous investigations, we attribute these to the non-trivial symmetry properties of the octagonal tiling. The periodic approximants studied in Refs. [3,4] show, besides an exact reflection symmetry, an “almost symmetry” under rotation by 90 degrees which may influence the level statistics [6], whereas the octagonal patches used in Ref. [5] possess the full D_8-symmetry of the regular octagon. Hence the level statistics observed in this case is that of a superposition of seven completely independent subspectra, and therefore rather close to a Poisson law.

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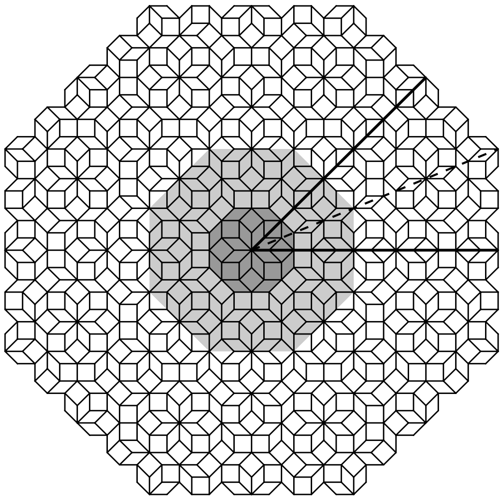

Fig. 1. Octagonal cluster of the Ammann-Beenker tiling with 833 vertices and exact D_8 symmetry around the central vertex as indicated by the solid and dashed lines. Shadings indicate successive inflation steps of the central octagon.

To arrive at this conclusion, we considered in Ref. [1] different patches that approximate the infinite quasiperiodic tiling, both with free and periodic boundary conditions. Exact symmetries were either exploited to block-diagonalize the Hamiltonian, thus splitting the spectrum into its irreducible parts, or avoided altogether by choosing patches without any symmetries.

Here, we concentrate on the D_8-symmetric octagonal patch shown in Fig. 1. For this case, the Hamiltonian matrix splits into ten blocks according to the

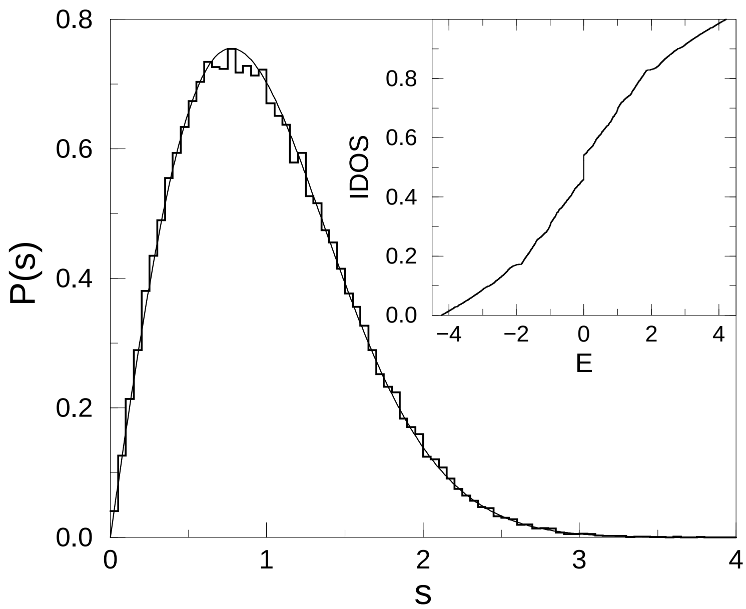

Fig. 2. IDOS (inset) for the D_8-symmetric patch with $N = 157369$ vertices, and $P(s)$ averaged over the three largest sectors. The smooth line denotes $P_{\mathrm{GOE}}(s)$.

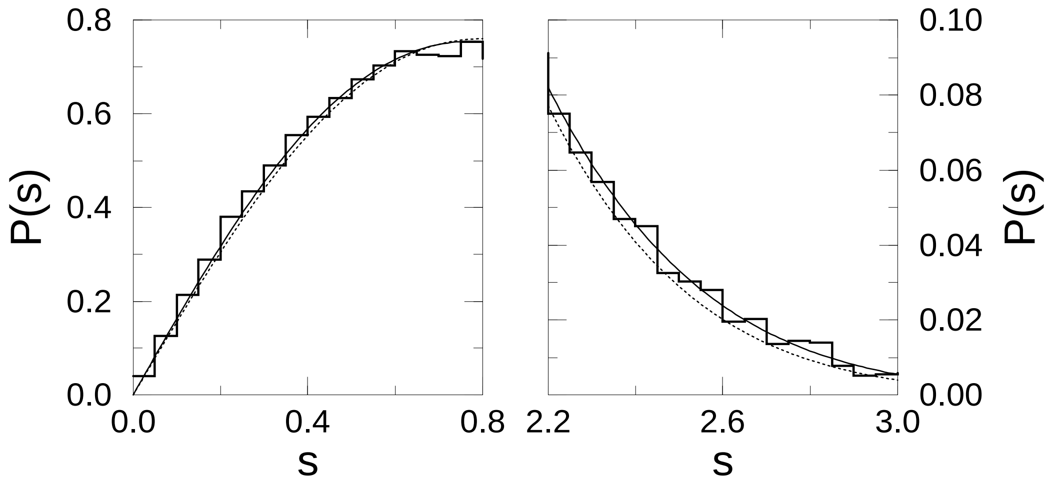

Fig. 3. Small- and large-s behaviour of $P(s)$ of Fig. 2, compared to $P_{\mathrm{GOE}}(s)$ (solid line) and $P_{\mathrm{W}}(s)$ (dashed line).

irreducible representations of the dihedral group D_8, resulting in seven different independent subspectra as there are three pairs of identical spectra. In Fig. 2, we show the integrated density of states (IDOS) for a patch, which contains $N = 157369$ vertices and corresponds to three more inflation steps performed on the patch of Fig. 1. Apparently, the IDOS is rather smooth, and the only prominent feature that shows up, apart from a few small gaps, is the huge fraction (13077 of 157369, hence about 8.3%) of exactly degenerate eigenvalues in the band center. For the level-spacing distribution $P(s)$, these do not matter as they would only contribute to $P(0)$, wherefore we can neglect them completely. The IDOS shown in Fig. 2 is fitted to a cubic spline which is then used to “unfold” the spectrum [8], i.e., to correct for the non-constant density of states, what is necessary if we want to compare to results of random matrix theory.

The level-spacing distribution $P(s)$ for the unfolded spectra is shown in Figs. 2 and 3, measured in units of the mean level spacing. Here, we averaged over the three largest subspectra, each of which contains 18043 levels after removing the degenerate states in the band center. The resulting histogram is compared to the GOE distribution $P_{\mathrm{GOE}}(s)$ (solid lines in Figs. 2 and 3) and, focusing on the small- and large-s behaviour, also to the Wigner surmise $P_{\mathrm{W}}(s)$ (dashed lines in Fig. 3). Apparently, the level-spacing distribution of the quasiperiodic Hamiltonian is well described by random matrix theory, and one can clearly see that $P_{\mathrm{GOE}}(s)$ fits the numerical data even better than $P_{\mathrm{W}}(s)$.

Fig. 4 shows the corresponding Σ_2 statistics [6], compared to the exact GOE result. The Σ_2 statistics measures the fluctuation in the number of energy levels n in an energy range L, i.e., $\Sigma_2 = \langle n^2\rangle - \langle n\rangle^2$ where $\langle . \rangle$ denotes the spectral average. Again, the agreement with our numerical results is good, supporting the conclusion that the underlying universal level statistics is described by the

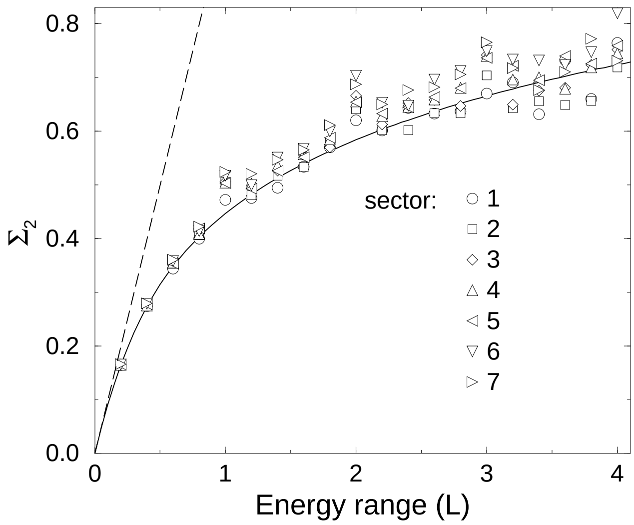

Fig. 4. Σ_2 statistics for the seven independent subspectra of the D_8-symmetric octagonal patch with 157369 vertices. The lines indicate the GOE (solid) and Poisson (dashed) behaviour.

GOE. Because typical eigenstates in our model are expected to be multifractal, one might have expected that one finds a “critical” level-spacing distribution as observed at the metal-insulator transition in the three-dimensional Anderson model of disorder [9] — however, this is clearly not the case.

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