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# Some Remarks on the Constant in the Strengthened C.B.S. Inequality: Application to $h$- and $p$-Hierarchical Finite Element Discretizations of Elasticity Problems

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**Abstract**

For a class of two-dimensional boundary value problems including diffusion and elasticity problems it is proved that the constants in the corresponding strengthened Cauchy-Buniakowski-Schwarz (C.B.S.) inequality in the cases of $h$-hierarchical and $p$-hierarchical finite element discretizations with triangular meshes differ by the factor 0.75.

For plane linear elasticity problems and triangulations with right isosceles triangles formulas are presented that show the dependence of the constant in the C.B.S. inequality on the Poisson's ratio. Furthermore, numerically determined bounds of the constant in the C.B.S. inequality are given for three-dimensional elasticity problems discretized by means of tetrahedral elements.

Finally, the robustness of iterative solvers for elasticity problems is discussed briefly.

**Key words:** finite elements, elasticity problems, hierarchical methods, multilevel methods

**AMS(MOS) subject classification:** 65N30, 65F10, 65N55, 73C02

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# 1 Introduction

The constant in the strengthened Cauchy-Buniakowski-Schwarz (C.B.S.) inequality

$$|a(u,v)| \leq \gamma \sqrt{a(u,u)}\sqrt{a(v,v)} \quad \forall u \in V_H,\, v \in T_h,\; V_h = V_H + T_h, \tag{1}$$

plays a basic role in the convergence analysis of iterative solvers for large scale systems of algebraic equations resulting from finite element discretizations of boundary value problems (b.v.p.), see, e.g., [7, 11].

We suppose that the finite element subspace $V_h \subset [H^1(\Omega)]^s$ is the direct sum of subspaces $V_H$ and $T_h$ spanned by two-level $h$- or $p$-hierarchical finite element ansatz functions (for more details see Section 2), and that $a(.,.)$ is a symmetric bilinear form arising in the variational formulation of the b.v.p.

The splitting of the finite element subspace $V_h$ into $V_H$ and $T_h$ is the basis of several iterative solvers. Examples for such solvers are conjugate gradient (cg) methods with two-level $h$- or $p$-hierarchical preconditioners proposed by Bank and Dupont [8], Axelsson and Gustafsson [2, 4] (see also [16]), the cg method with algebraic multilevel preconditioners developed by Axelsson and Vassilevski [5, 6], or multigrid methods of the projection type as described by Meis and Branca [22], Braess [9, 10], Verfürth [25], Jung [14, 15], Schieweck [23], and Thole [24]. The convergence rates of all these methods depend on the constant in the strengthened C.B.S. inequality. Therefore, it is of interest to give estimates of this constant.

In the present paper it is supposed that the bilinear form $a(.,.)$ can be written as a linear combination of terms of the type

$$\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\, dx\,, \quad i,j = 1,2,$$

where $\Omega$ is a two-dimensional bounded domain. General diffusion problems and plane linear elasticity problems can serve as examples for such bilinear forms.

In Section 2 we prove the relation $(\gamma^{\mathrm{l}})^2 = \frac{3}{4}(\gamma^{\mathrm{q}})^2$ for the constants $\gamma^{\mathrm{l}}$ and $\gamma^{\mathrm{q}}$ in the C.B.S. inequality in the two-level $h$- and $p$-hierarchical case, respectively, based on finite element discretizations with triangular elements. It is obvious that $(\gamma^{\mathrm{q}})^2 = 1$ is an upper bound in the $p$-hierarchical case. Therefore, we get for the $h$-hierarchical case $(\gamma^{\mathrm{l}})^2 = \frac{3}{4}$ for each bilinear form of the considered type and for triangulations with arbitrary triangles.

Estimates of the constant in the C.B.S. inequality for plane linear elasticity problems are given in some papers [1, 14, 15, 16, 20]. Margenov [20] shows for triangulations with right isosceles triangles that $(\gamma^{\mathrm{l}})^2 = \frac{3}{4}$ is an upper bound for all Poisson's ratios $\nu \in (0, \frac{1}{2})$. Achchab and Maitre [1] prove that $\frac{3}{4}$ is also an upper bound for triangulations with arbitrary triangles. In both papers the dependence of $\gamma^{\mathrm{l}}$ on $\nu$ is not studied. We analyse the constant in the C.B.S. inequality for plane linear elasticity problems more accurately, i.e. we show its dependence on the Poisson's ratio $\nu$. The estimates $(\gamma^{\mathrm{l}})^2(\nu)$ given in Section 3 are restricted to triangulations with right isosceles triangles.

The extension of the techniques used in [1, 20] and in this paper to three-dimensional elasticity problems is very complicated. Therefore, we give only some numerically determined estimates of $\gamma^{\mathrm{l}}$ and $\gamma^{\mathrm{q}}$, respectively, in dependence on $\nu$.

Finally, we discuss briefly the influence of the Poisson's ratio on the convergence properties of iterative methods.

# 2 A relation between the constants in the strengthened C.B.S. inequality in the two-level $h$- and $p$-hierarchical case

In this Section we consider symmetric bilinear forms $a(.,.) \ : \mathcal{V} \times \mathcal{V} \to \mathbb{R}^1$ arising in variational formulations of second-order elliptic b.v.p.'s. The space $\mathcal{V}$ is a subspace of the Sobolev space $[H^1(\Omega)]^s$ defined by the Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$. If not stated otherwise the case $s = 1$ is considered. The bilinear forms $a(.,.)$ are discretized by using finite elements. We suppose that a coarse triangulation $\mathcal{T}_H = \{\delta_H^{(r)}, r = 1, 2, \ldots, R_H\}$ ($R_H$ being the number of triangles) of the polygonally bounded plane domain $\Omega$ is given. The finer triangulation $\mathcal{T}_h = \{\delta_h^{(r)}, r = 1, 2, \ldots, R_h\}$ is constructed by connecting the midpoints of the edges of each triangle $\delta_H^{(r)}$. Corresponding to the triangulation $\mathcal{T}_H$ the finite element subspace

$$V_H = \operatorname{span}\{p_H^{(m)}(x), m = 1, 2, \ldots, N_H\} \subset \mathcal{V} \tag{2}$$

is defined. The functions $p_H^{(m)}(x)$ are continuous and piecewise linear, i.e. linear on each triangle $\delta_H^{(r)}$. Furthermore they satisfy the condition $p_H^{(m)}(x^{(n)}) = \delta_{mn}$. Here, $\delta_{mn}$ denotes the Kronecker symbol with $\delta_{mn} = 1$ for $m = n$ and $\delta_{mn} = 0$ for $m \neq n$, $m, n = 1, 2, \ldots, N_H$, $N_H$ is the number of nodes in $\Omega \cup \Gamma_N$ ($\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$), and $x^{(n)} = (x_1^{(n)}, x_2^{(n)})$ are the coordinates of the node $P^{(n)}$.

In the following we want to give upper bounds of the constant in the strengthened C.B.S. inequality. Here, two-level $h$-hierarchical and two-level $p$-hierarchical finite element discretizations are considered. For that reason we introduce the finite element subspaces

$$T_h^{\mathrm{l}} = \operatorname{span}\{p_h^{(m)}(x), m = N_H + 1, N_H + 2, \ldots, N_h\} \tag{3}$$

and

$$T_h^{\mathrm{q}} = \operatorname{span}\{q_H^{(m)}(x), m = N_H + 1, N_H + 2, \ldots, N_h\} \tag{4}$$

with continuous piecewise linear functions $p_h^{(m)}$ and continuous piecewise quadratic functions $q_H^{(m)}(x)$. The functions $p_h^{(m)}$ are linear on each triangle $\delta_h^{(r)} \in \mathcal{T}_h$, and the functions $q_H^{(m)}(x)$ are quadratic on each triangle $\delta_H^{(r)} \in \mathcal{T}_H$. For these functions the relations $p_h^{(m)}(x^{(n)}) = \delta_{mn}$ and $q_H^{(m)}(x^{(n)}) = \delta_{mn}$, $m, n = N_H + 1, N_H + 2, \ldots, N_h$, hold. The space $V_h^{\mathrm{l}} = V_H + T_h^{\mathrm{l}} \subset \mathcal{V}$ is a finite element subspace with a two-level h-hierarchical basis and $V_h^{\mathrm{q}} = V_H + T_h^{\mathrm{q}} \subset \mathcal{V}$ has a two-level $p$-hierarchical basis.

In Lemma 2.1 we give a relation between the constants in the strengthened C.B.S. inequality (1) for the pairs of subspaces $\{V_H, T_h\} = \{V_H, T_h^{\mathrm{l}}\}$ and $\{V_H, T_h\} = \{V_H, T_h^{\mathrm{q}}\}$. Here, the corresponding constant is denoted by $\gamma^{\mathrm{l}}$ and $\gamma^{\mathrm{q}}$, respectively.

**Lemma 2.1** *Let the bilinear form $a(.,.)$ be defined as a linear combination of terms of the type*

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx\,, \ \ i, j = 1, 2, \tag{5}$$

*with $u \in V_H$ and $v \in T_h^{\mathrm{l}}$ or $v \in T_h^{\mathrm{q}}$, respectively. Then*

$$(\gamma^{\mathrm{l}})^2 = \frac{3}{4}\, (\gamma^{\mathrm{q}})^2\,. \tag{6}$$

*Proof:* For a better readability of the proof we first formulate the two relations (7) which we need for proving (6), then we prove (6), and finally the relations (7) are proved.

Using the relations

$$a(u,v^{\mathrm{q}}) = \frac{4}{3}\,a(u,v^{\mathrm{l}}) \quad \text{and} \quad a(v^{\mathrm{q}},v^{\mathrm{q}}) = \frac{4}{3}\,a(v^{\mathrm{l}},v^{\mathrm{l}}) \tag{7}$$

with $u \in V_H$, $v^{\mathrm{q}} \in T_h^{\mathrm{q}}$, $v^{\mathrm{l}} \in T_h^{\mathrm{l}}$, and $v^{\mathrm{q}}(x^{(n)}) = v^{\mathrm{l}}(x^{(n)})$ for all $n = N_H+1, N_H+2, \ldots, N_h$ we get

$$\begin{array}{rlll} & (a(u,v^{\mathrm{q}}))^2 & \leq & (\gamma^{\mathrm{q}})^2\, a(u,u)\, a(v^{\mathrm{q}},v^{\mathrm{q}}) \quad \forall u \in V_H,\ \forall v^{\mathrm{q}} \in T_h^{\mathrm{q}} \\ \equiv & \left(\frac{4}{3}\,a(u,v^{\mathrm{l}})\right)^2 & \leq & (\gamma^{\mathrm{q}})^2\, a(u,u)\, \frac{4}{3}\,a(v^{\mathrm{l}},v^{\mathrm{l}}) \quad \forall u \in V_H,\ \forall v^{\mathrm{l}} \in T_h^{\mathrm{l}} \\ \equiv & (a(u,v^{\mathrm{l}}))^2 & \leq & \frac{3}{4}\,(\gamma^{\mathrm{q}})^2\, a(u,u)\, a(v^{\mathrm{l}},v^{\mathrm{l}}) \quad \forall u \in V_H,\ \forall v^{\mathrm{l}} \in T_h^{\mathrm{l}}\,, \end{array}$$

i.e. $(\gamma^{\mathrm{l}})^2 = \frac{3}{4}\,(\gamma^{\mathrm{q}})^2$.

Now we prove the relation

$$a(u,v^{\mathrm{q}}) = \frac{4}{3}\,a(u,v^{\mathrm{l}}) \tag{8}$$

with $u \in V_H$, $v^{\mathrm{q}} \in T_h^{\mathrm{q}}$, $v^{\mathrm{l}} \in T_h^{\mathrm{l}}$, and $v^{\mathrm{q}}(x^{(n)}) = v^{\mathrm{l}}(x^{(n)})$ for all $n = N_H+1, N_H+2, \ldots, N_h$. Since it is supposed that the bilinear form $a(.,.)$ is a linear combination of terms of the type (5) we have to prove that

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i}\frac{\partial v^{\mathrm{q}}}{\partial x_j}\,dx = \frac{4}{3}\int\limits_{\Omega} \frac{\partial u}{\partial x_i}\frac{\partial v^{\mathrm{l}}}{\partial x_j}\,dx\,,\ \ i,j = 1,2, \tag{9}$$

holds. We get for $v = v^{\mathrm{q}}$ and $v = v^{\mathrm{l}}$ the relations

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx = \sum_{r=1}^{R_H}\int\limits_{\delta_H^{(r)}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx = \sum_{r=1}^{R_H}\left[-\int\limits_{\delta_H^{(r)}} \frac{\partial^2 u}{\partial x_i \partial x_j}\,v\,dx + \int\limits_{\partial\delta_H^{(r)}} v\frac{\partial u}{\partial x_i}\,n_j\,dS\right], \tag{10}$$

where $n_j$ is the $j$-th component of the vector of the outer normal $\vec{n} = (n_1,n_2)^T$ on $\partial\delta_H^{(r)}$. Since $u$ is linear on $\delta_H^{(r)}$ the relation

$$\int\limits_{\Omega} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx = \sum_{r=1}^{R_H}\int\limits_{\partial\delta_H^{(r)}} v\frac{\partial u}{\partial x_i}\,n_j\,dS = \sum_{r=1}^{R_H}\sum_{\alpha=1}^{3}\frac{\partial u}{\partial x_i}\,n_j\int\limits_{e_{H,\alpha}^{(r)}} v\,dS \tag{11}$$

holds, where $e_{H,\alpha}^{(r)}$ are the edges of the triangle $\delta_H^{(r)}$. Denoting by $x^{(r,3+\alpha)}$ the coordinates of the midpoints of these edges (see Figure 1) one obtains

$$\int\limits_{e_{H,\alpha}^{(r)}} v\,dS = \begin{cases} v(x^{(r,3+\alpha)})\dfrac{\operatorname{meas} e_{H,\alpha}^{(r)}}{2} & \text{for} \quad v = v^{\mathrm{l}} \\[2ex] v(x^{(r,3+\alpha)})\dfrac{4\operatorname{meas} e_{H,\alpha}^{(r)}}{6} & \text{for} \quad v = v^{\mathrm{q}}\,. \end{cases} \tag{12}$$

Then, (8) follows immediately from (10), (11), and (12). It remains to prove $a(v^{\mathrm{q}},v^{\mathrm{q}}) = \frac{4}{3}a(v^{\mathrm{l}},v^{\mathrm{l}})$. We consider an arbitrary triangle $\delta_H^{(r)}$ of the triangulation $\mathcal{T}_H$ (see Figure 1).

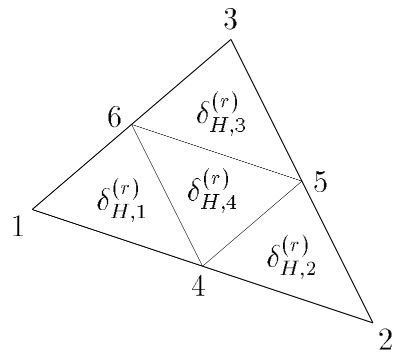


Figure 1: An arbitrary triangle $\delta_H^{(r)}$ with the local numbering of the nodes

Using barycentric coordinates $\lambda_k$ and a local numbering of the nodes as it is shown in Figure 1, the function $v^{\mathrm{q}}$ restricted to the triangle $\delta_H^{(r)}$ can be expressed by

$$v^{\mathrm{q}} = v^{\mathrm{q}}(x^{(r,4)})q_H^{(r,4)} + v^{\mathrm{q}}(x^{(r,5)})q_H^{(r,5)} + v^{\mathrm{q}}(x^{(r,6)})q_H^{(r,6)}$$

with

$$q_H^{(r,4)} = 4\lambda_1\lambda_2\,, \quad q_H^{(r,5)} = 4\lambda_2\lambda_3\,, \quad q_H^{(r,6)} = 4\lambda_3\lambda_1\,.$$

Therefore, the relation

$$\int\limits_{\delta_H^{(r)}} \frac{\partial v^{\mathrm{q}}}{\partial x_i}\frac{\partial v^{\mathrm{q}}}{\partial x_j}\,dx = \sum_{\alpha,\beta=1}^{3} v^{\mathrm{q}}(x^{(r,3+\alpha)})v^{\mathrm{q}}(x^{(r,3+\beta)}) \int\limits_{\delta_H^{(r)}} \frac{\partial q_H^{(r,3+\alpha)}}{\partial x_i}\frac{\partial q_H^{(r,3+\beta)}}{\partial x_j}\,dx \tag{13}$$

holds. In the following we distinguish the cases $\alpha = \beta$ and $\alpha \neq \beta$. We consider $\alpha = \beta = 1$ and $\alpha = 2$, $\beta = 3$ as examples. The computation of the integrals in (13) for other $\alpha$ and $\beta$ can be performed in an analogous manner. With

$$\frac{\partial q_H^{(r,4)}}{\partial x_i} = 4\Big(\frac{\partial \lambda_1}{\partial x_i}\lambda_2 + \frac{\partial \lambda_2}{\partial x_i}\lambda_1\Big)\,, \quad \frac{\partial q_H^{(r,5)}}{\partial x_i} = 4\Big(\frac{\partial \lambda_2}{\partial x_i}\lambda_3 + \frac{\partial \lambda_3}{\partial x_i}\lambda_2\Big)\,,$$
$$\frac{\partial q_H^{(r,6)}}{\partial x_i} = 4\Big(\frac{\partial \lambda_3}{\partial x_i}\lambda_1 + \frac{\partial \lambda_1}{\partial x_i}\lambda_3\Big)\,,$$

and the obvious relations

$$\int\limits_{\delta_H^{(r)}} \lambda_k\lambda_l\,dx = \frac{\operatorname{meas}\delta_H^{(r)}}{12}(1+\delta_{kl}) \ \ (\delta_{kl} \text{ is the Kronecker symbol})\,,$$

$$\sum_{k=1}^{3}\frac{\partial \lambda_k}{\partial x_i} = \sum_{k=1}^{3}\frac{\partial \lambda_k}{\partial x_j} = 0$$

it follows that

$$\begin{aligned} &\int\limits_{\delta_H^{(r)}} \frac{\partial q_H^{(r,5)}}{\partial x_i}\frac{\partial q_H^{(r,6)}}{\partial x_j}\,dx \\ &= \frac{16}{12}\operatorname{meas}\delta_H^{(r)}\Big[\frac{\partial \lambda_2}{\partial x_i}\frac{\partial \lambda_3}{\partial x_j} + 2\frac{\partial \lambda_2}{\partial x_i}\frac{\partial \lambda_1}{\partial x_j} + \frac{\partial \lambda_3}{\partial x_i}\frac{\partial \lambda_3}{\partial x_j} + \frac{\partial \lambda_3}{\partial x_i}\frac{\partial \lambda_1}{\partial x_j}\Big] \\ &= \frac{16}{12}\operatorname{meas}\delta_H^{(r)}\Big[\frac{\partial \lambda_2}{\partial x_i}\frac{\partial \lambda_1}{\partial x_j} + \frac{\partial \lambda_1}{\partial x_i}\frac{\partial \lambda_2}{\partial x_j}\Big] \end{aligned} \tag{14}$$

and

$$\int\limits_{\delta_H^{(r)}} \frac{\partial q_H^{(r,4)}}{\partial x_i}\frac{\partial q_H^{(r,4)}}{\partial x_j}\,dx$$

$$= \frac{16}{12}\operatorname{meas}\delta_H^{(r)}\left[2\frac{\partial\lambda_1}{\partial x_i}\frac{\partial\lambda_1}{\partial x_j}+\frac{\partial\lambda_1}{\partial x_i}\frac{\partial\lambda_2}{\partial x_j}+\frac{\partial\lambda_2}{\partial x_i}\frac{\partial\lambda_1}{\partial x_j}+2\frac{\partial\lambda_2}{\partial x_i}\frac{\partial\lambda_2}{\partial x_j}\right] \quad (15)$$

$$= \frac{16}{12}\operatorname{meas}\delta_H^{(r)}\left[\frac{\partial\lambda_1}{\partial x_i}\frac{\partial\lambda_1}{\partial x_j}+\frac{\partial\lambda_2}{\partial x_i}\frac{\partial\lambda_2}{\partial x_j}+\frac{\partial\lambda_3}{\partial x_i}\frac{\partial\lambda_3}{\partial x_j}\right].$$

For

$$v^1 = v^1(x^{(r,4)})p_h^{(r,4)} + v^1(x^{(r,5)})p_h^{(r,5)} + v^1(x^{(r,6)})p_h^{(r,6)},$$

i.e. the piecewise linear case, we get

$$\int\limits_{\delta_H^{(r)}} \frac{\partial v^1}{\partial x_i}\frac{\partial v^1}{\partial x_j}\,dx = \sum_{\alpha,\beta=1}^{3}\left[v^1(x^{(r,3+\alpha)})v^1(x^{(r,3+\beta)})\sum_{s=1}^{4}\int\limits_{\delta_{H,s}^{(r)}}\frac{\partial p_h^{(r,3+\alpha)}}{\partial x_i}\frac{\partial p_h^{(r,3+\beta)}}{\partial x_j}\,dx\right] \quad (16)$$

with $\delta_H^{(r)} = \bigcup_{s=1}^4 \delta_{H,s}^{(r)}$ (see also Figure 1). Again, we consider the cases $\alpha=\beta=1$ and $\alpha=2$, $\beta=3$. The functions $p_h^{(r,3+\alpha)}$ are defined on the triangles $\delta_{H,s}^{(r)}$ as given in Table 1. This table contains also the partial derivatives of the functions $p_h^{(r,3+\alpha)}$.

| | $\delta_{H,1}^{(r)}$ | $\delta_{H,2}^{(r)}$ | $\delta_{H,3}^{(r)}$ | $\delta_{H,4}^{(r)}$ | | $\delta_{H,1}^{(r)}$ | $\delta_{H,2}^{(r)}$ | $\delta_{H,3}^{(r)}$ | $\delta_{H,4}^{(r)}$ |
|---|---|---|---|---|---|---|---|---|---|
| $p_h^{(r,4)}$ | $2\lambda_2$ | $2\lambda_1$ | $0$ | $1-2\lambda_3$ | $\frac{\partial p_h^{(r,4)}}{\partial x_i}$ | $2\frac{\partial\lambda_2}{\partial x_i}$ | $2\frac{\partial\lambda_1}{\partial x_i}$ | $0$ | $-2\frac{\partial\lambda_3}{\partial x_i}$ |
| $p_h^{(r,5)}$ | $0$ | $2\lambda_3$ | $2\lambda_2$ | $1-2\lambda_1$ | $\frac{\partial p_h^{(r,5)}}{\partial x_i}$ | $0$ | $2\frac{\partial\lambda_3}{\partial x_i}$ | $2\frac{\partial\lambda_2}{\partial x_i}$ | $-2\frac{\partial\lambda_1}{\partial x_i}$ |
| $p_h^{(r,6)}$ | $2\lambda_3$ | $0$ | $2\lambda_1$ | $1-2\lambda_2$ | $\frac{\partial p_h^{(r,6)}}{\partial x_i}$ | $2\frac{\partial\lambda_3}{\partial x_i}$ | $0$ | $2\frac{\partial\lambda_1}{\partial x_i}$ | $-2\frac{\partial\lambda_2}{\partial x_i}$ |

Table 1: The definition of the functions $p_h^{(r,3+\alpha)}$ and their partial derivatives on the triangles $\delta_{H,s}^{(r)}$

Therefore, we obtain

$$\int\limits_{\delta_H^{(r)}} \frac{\partial p_h^{(r,5)}}{\partial x_i}\frac{\partial p_h^{(r,6)}}{\partial x_j}\,dx = \sum_{s=1}^{4}\int\limits_{\delta_{H,s}^{(r)}}\frac{\partial p_h^{(r,5)}}{\partial x_i}\frac{\partial p_h^{(r,6)}}{\partial x_j}\,dx = \frac{\operatorname{meas}\delta_H^{(r)}}{4}4\left[\frac{\partial\lambda_2}{\partial x_i}\frac{\partial\lambda_1}{\partial x_j}+\frac{\partial\lambda_1}{\partial x_i}\frac{\partial\lambda_2}{\partial x_j}\right] \quad (17)$$

and

$$\int\limits_{\delta_H^{(r)}} \frac{\partial p_h^{(r,4)}}{\partial x_i}\frac{\partial p_h^{(r,4)}}{\partial x_j}\,dx = \frac{\operatorname{meas}\delta_H^{(r)}}{4}4\left[\frac{\partial\lambda_1}{\partial x_i}\frac{\partial\lambda_1}{\partial x_j}+\frac{\partial\lambda_2}{\partial x_i}\frac{\partial\lambda_2}{\partial x_j}+\frac{\partial\lambda_3}{\partial x_i}\frac{\partial\lambda_3}{\partial x_j}\right]. \quad (18)$$

Since (14) and (17) as well as (15) and (18) differ by a factor $\frac{4}{3}$ also the integrals in (13) and (16) differ by a factor $\frac{4}{3}$. This completes the proof. □

**Remark 2.1**

(i) *Relations (7) were also proved by* JUNG *and* RÜDE *[17] for a bilinear form of the type*

$$a(u,v) = \int_{\Omega} (A(x)\nabla_x u, \nabla_x v)\, dx \tag{19}$$

*with a symmetric positive definite* $2 \times 2$ *matrix* $A(x) = [a_{ij}(x)]_{i,j=1}^{2}$, *where the functions* $a_{ij}(x)$ *are constant over the triangles* $\delta_H^{(r)}$. *The variable coefficient case is studied in [18].*

(ii) *From Lemma 2.1 we conclude that the relation (6) holds for example for bilinear forms of the type (19), or for the bilinear form corresponding to plane linear elasticity problems (see also Section 3).*

(iii) *The relation (8) holds also for three-dimensional problems discretized by means of tetrahedral elements. Here, we have to compute in (11) integrals over the surface of the tetrahedra. Denoting by* $f_{H,l}^{(r)}$, $l = 1, \ldots, 4$, *the faces on the surface of the tetrahedron* $\delta_H^{(r)}$ *and by* $x^{(r,3+k,l)}$ *the midpoints of the edges describing the face* $f_{H,l}^{(r)}$, *we get*

$$\int_{f_{H,l}^{(r)}} v^{\mathrm{q}}\, dF = \frac{1}{3}\, meas\, f_{H,l}^{(r)} \sum_{\alpha=1}^{3} v^{\mathrm{q}}(x^{(r,3+\alpha,l)})$$

*and*

$$\begin{aligned}\int_{f_{H,l}^{(r)}} v^{1}\, dF = \frac{1}{3}\,\frac{meas\, f_{H,l}^{(r)}}{4}\Big[&(v^1(x^{(r,4,l)}) + v^1(x^{(r,6,l)})) + (v^1(x^{(r,4,l)}) + v^1(x^{(r,5,l)})) \\ &+(v^1(x^{(r,5,l)}) + v^1(x^{(r,6,l)})) + (v^1(x^{(r,4,l)}) + v^1(x^{(r,5,l)}) + v^1(x^{(r,6,l)}))\Big] \\ = \frac{meas\, f_{H,l}^{(r)}}{4} \sum_{\alpha=1}^{3} v^1(x^{(r,3+\alpha,l)})\,.&\end{aligned}$$

*The relation* $a(v^{\mathrm{q}}, v^{\mathrm{q}}) = \frac{4}{3}a(v^1, v^1)$ *is not true in the three-dimensional case.*

Since $(\gamma^{\mathrm{q}})^2 = 1$ is a trivial upper bound of the constant in the C.B.S. inequality in the $p$-hierarchical case, Lemma 2.1 implies that $(\gamma^1)^2 = \frac{3}{4}$ is an upper bound in the $h$-hierarchical case for all bilinear forms $a(.,.)$ which are a linear combination of terms of the type (5) and for triangulations $\mathcal{T}_H$ with arbitrary triangles.

But sometimes it is of more interest whether one can show how the constant in the C.B.S. inequality depends on the geometry of the triangles or on the coefficients in the bilinear form. MAITRE and MUSY give in [19] for the bilinear form (19) with $a_{11} = a_{22} = 1$, and $a_{12} = 0$ the constants $\gamma^1$ and $\gamma^{\mathrm{q}}$ in dependence on the geometry of the triangles. They observe for this case the relation (6). AXELSSON [3] considers the bilinear form (19) with constant coefficients $a_{ij}$, $i,j = 1,2$, and shows that the constant $(\gamma^1)^2$ is bounded by $\frac{3}{4}$ for arbitrary triangles in the $h$-hierarchical case. Additionally, he shows how the constant depends on $a_{ij}$, $i,j = 1,2$, and on the geometry of the triangles. ACHCHAB and MAITRE prove in [1] for plane linear elasticity problems (state of plane strain) with arbitrary Poisson's ratio $\nu \in (0, \frac{1}{2})$ and triangulations with arbitrary triangles that $(\gamma^1)^2 = \frac{3}{4}$ is a sharp upper bound for the constant in the corresponding strengthened C.B.S. inequality in the $h$-hierarchical case. For the $p$-hierarchical case it is shown in [1] that $\gamma^{\mathrm{q}}$ tends to 1 for triangulations with right isosceles triangles and Poisson's ratio close to $\frac{1}{2}$. Using the estimate of $\gamma^{\mathrm{l}}$ in [1] and Lemma 2.1 we can prove a more general result, namely that the upper bound of $\gamma^{\mathrm{q}}$ can not be better than 1 for triangulations with arbitrary triangles and $\nu \to \frac{1}{2}$.

In the next Section we want to give some estimates of the constants $\gamma^{\mathrm{l}}$ and $\gamma^{\mathrm{q}}$ for linear elasticity problems We show how these constants depend on the Poisson's ratio, but unfortunately we need some restrictions on the triangulation.

# 3 The strengthened C.B.S. inequality for linear elasticity problems

In this Section upper bounds of the constant in the C.B.S. inequality for linear elasticity problems are presented. These upper bounds are given in dependence on the Poisson's ratio $\nu$. Here, it is supposed that the plane domain $\Omega$ can be decomposed into right isosceles triangles. Furthermore, we present some numerically determined bounds of the constant in the C.B.S. inequality for elasticity problems in three-dimensional domains.

The variational formulation of a plane linear elasticity problem is given by the following:

Find $u = (u_1, u_2)^T \in \mathcal{V} = \{u \in [H^1(\Omega)]^2 \,:\, u_1 = u_2 = 0 \text{ on } \Gamma_D\}$ such that

$$a(u, v) = \langle F, v\rangle \quad \forall v \in \mathcal{V} \tag{20}$$

holds with

$$a(u, v) = \int\limits_{\Omega} e^T(v) D e(u)\, dx \quad \text{and} \quad \langle F, v\rangle = \int\limits_{\Omega} v^T f\, dx + \int\limits_{\Gamma_N} v^T g_N\, ds. \tag{21}$$

Here $e(v) = (\varepsilon_{11}(v), \varepsilon_{22}(v), 2\varepsilon_{12}(v))^T$, $f$ and $g_N$ are the vectors of the volume and surface forces, respectively, The components $\varepsilon_{ij}$, $i, j = 1, 2$, of the strain tensor are defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and

$$D = E\frac{1 + (1-k)\nu}{(1+\nu)(1-k\nu)} \begin{pmatrix} 1 & \dfrac{\nu}{1+(1-k)\nu} & 0 \\ \dfrac{\nu}{1+(1-k)\nu} & 1 & 0 \\ 0 & 0 & \dfrac{1-k\nu}{2(1+(1-k)\nu)} \end{pmatrix}$$

with $k = 1$ for the state of plane stress and $k = 2$ for the state of plane strain.

We discretize problem (20) by means of $h$-hierarchical and $p$-hierarchical finite element ansatz functions. As described in Section 2 it is supposed that a coarse triangulation $\mathcal{T}_H$ and a fine triangulation $\mathcal{T}_h$ of the domain $\Omega$ are generated. Since the displacement $u$

is a vector function the basis functions of the finite element subspaces must be vector functions too. We define the spaces

$$\begin{aligned} V_H &= \operatorname{span}\{(p_H^{(m)},0)^T,\,(0,p_H^{(m)})^T,\ m=1,2,\ldots,N_H\}\,, \\ T_h^{\mathrm{l}} &= \operatorname{span}\{(p_h^{(m)},0)^T,\,(0,p_h^{(m)})^T,\ m=N_H+1,N_H+2,\ldots,N_h\}\,,\ \text{and} \\ T_h^{\mathrm{q}} &= \operatorname{span}\{(q_H^{(m)},0)^T,\,(0,q_H^{(m)})^T,\ m=N_H+1,N_H+2,\ldots,N_h\}\,. \end{aligned} \tag{22}$$

The functions $p_H^{(m)}(x)$, $p_h^{(m)}(x)$, and $q_H^{(m)}(x)$ are introduced in Section 2. Next we estimate the constant in the strengthened C.B.S. inequality with the bilinear form $a(.,.)$ defined in (21). First, we formulate some lemmas which allow us to perform the estimation of the constant locally. These lemmas and their proofs can also be found in other papers (see, e.g., [2, 12, 14, 19, 24]). Corresponding to the finite element triangulation $\mathcal{T}_H$ the bilinear form has the representation

$$a(u,v)=\sum_{r=1}^{R_H} a^{(r)}(u,v) \quad \forall u,v\in V_h,\ V_h=V_H+T_h, \tag{23}$$

where $a^{(r)}(u,v)$ is the restriction of $a(.,.)$ to the triangle $\delta_H^{(r)}$.

**Lemma 3.1** *If there exist numbers $\gamma^{(r)}\in[0,1]$ such that for all $r=1,2,\ldots R_H$*

$$|a^{(r)}(u,v)|\le \gamma^{(r)}\sqrt{a^{(r)}(u,u)}\sqrt{a^{(r)}(v,v)} \quad \forall u\in V_H,\ v\in T_h \tag{24}$$

*holds, then the strengthened C.B.S. inequality holds with*

$$\gamma=\max_{r=1,2,\ldots,R_H}\gamma^{(r)}.$$

Let us define matrices $A^{(r)}$, $B^{(r)}$, and $C^{(r)}$ in the following way:

$$\begin{aligned} (A^{(r)}\underline{u},\underline{u})_n &:= a^{(r)}(u,u)\,,\ \ \forall u\in V_{H|\delta_H^{(r)}}\leftrightarrow \underline{u}\in\mathbb{R}^n, \\ (B^{(r)}\underline{u},\underline{v})_m &:= a^{(r)}(u,v)\,,\ \ \forall u\in V_{H|\delta_H^{(r)}}\leftrightarrow \underline{u}\in\mathbb{R}^n,\ \forall v\in T_{h|\delta_H^{(r)}}\leftrightarrow \underline{v}\in\mathbb{R}^m, \\ (C^{(r)}\underline{v},\underline{v})_m &:= a^{(r)}(v,v)\,,\ \ \forall v\in T_{h|\delta_H^{(r)}}\leftrightarrow \underline{v}\in\mathbb{R}^m. \end{aligned} \tag{25}$$

In our application $n=m=6$ holds. We suppose that

$$\begin{aligned} &(1)\ \ a^{(r)}(v,v)\ge 0 \quad \forall v\in V_h, \\ &(2)\ \ a^{(r)}(u,v)=a^{(r)}(v,u) \quad \forall u,v\in V_h, \\ &(3)\ \ \ker\{C^{(r)}\}=\{\underline{v}\in\mathbb{R}^m\ :\ C^{(r)}\underline{v}=\underline{0}\}=\{\underline{0}\}, \\ &(4)\ \ \ker\{a^{(r)}\}=\left\{v\in V_{h|\delta_H^{(r)}}\ :\ a^{(r)}(v,z)=0\ \ \forall z\in V_{h|\delta_H^{(r)}}\right\}\subset V_{H|\delta_H^{(r)}}. \end{aligned} \tag{26}$$

Under these assumptions (26) the inequalities (24) can be written in the equivalent form

$$(\gamma^{(r)})^2=\sup_{\substack{u\in V_H\setminus\ker\{a^{(r)}\}\\ v\in T_h, v\neq 0}}\frac{|a^{(r)}(u,v)|^2}{a^{(r)}(u,u)a^{(r)}(v,v)}=\sup_{\substack{\underline{u}\in\mathbb{R}^n\setminus\ker\{A^{(r)}\}\\ \underline{v}\in\mathbb{R}^m,\underline{v}\neq 0}}\frac{(B^{(r)}\underline{u},\underline{v})_m^2}{(A^{(r)}\underline{u},\underline{u})_n(C^{(r)}\underline{v},\underline{v})_m}. \tag{27}$$

**Lemma 3.2** *Let the matrices $A^{(r)}$, $B^{(r)}$, and $C^{(r)}$ satisfy the following properties:*

$$\begin{array}{ll} (1) & A^{(r)} \textit{ is a symmetric, positive semidefinite } n \times n \textit{ matrix,} \\ (2) & B^{(r)} \textit{ is an arbitrary } m \times n \textit{ matrix,} \\ (3) & C^{(r)} \textit{ is a symmetric, positive definite } m \times m \textit{ matrix,} \\ (4) & \ker\{A^{(r)}\} \subseteq \ker\{B^{(r)}\}. \end{array} \tag{28}$$

*Then*

$$\sup_{\substack{\underline{u} \in \mathbb{R}^n \setminus \ker\{A^{(r)}\} \\ \underline{v} \in \mathbb{R}^m, \underline{v} \neq 0}} \frac{(B^{(r)}\underline{u}, \underline{v})_m^2}{(A^{(r)}\underline{u}, \underline{u})_n (C^{(r)}\underline{v}, \underline{v})_m} = \sup_{\underline{u} \in \mathbb{R}^n \setminus \ker\{A^{(r)}\}} \frac{((B^{(r)})^T (C^{(r)})^{-1} B^{(r)}\underline{u}, \underline{u})_n}{(A^{(r)}\underline{u}, \underline{u})_n}.$$

**Lemma 3.3** *Let the assumptions (28) and the two assumptions below be fulfilled:*

$$\begin{array}{ll} (1) & \dim \ker\{A^{(r)}\} = n - k \leq n \\ (2) & \textit{Let } \underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k \notin \ker\{A^{(r)}\} \textit{ be linearly independent vectors of } \mathbb{R}^n \\ & \textit{such that } \mathbb{R}^n = \ker\{A^{(r)}\} + \operatorname{span}\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k\}. \\ & \textit{Define the matrix } V^{(r)} = [\underline{v}_1 \ \underline{v}_2 \ \ldots \ \underline{v}_k]_{n \times k}. \end{array} \tag{29}$$

*Then the largest eigenvalue of the generalized eigenvalue problem*

$$(V^{(r)})^T (B^{(r)})^T (C^{(r)})^{-1} B^{(r)} V^{(r)} \underline{w} = \lambda (V^{(r)})^T A^{(r)} V^{(r)} \underline{w}, \quad \underline{w} \in \mathbb{R}^k \tag{30}$$

*is equal to $(\gamma^{(r)})^2$ from (27).*

Now we apply Lemma 3.1 – 3.3 to obtain upper bounds of the constant in the strengthened C.B.S. inequality in the case of plane linear elasticity problems. In Theorem 3.1 upper bounds in dependence on the Poisson's ratio $\nu$ are given for discretizations with $h$-hierarchical finite element ansatz functions.

**Theorem 3.1** *Suppose that the triangulation $\mathcal{T}_H$ consists of right isosceles triangles $\delta_H^{(r)}$ and that $h$-hierarchical finite element ansatz functions are used. Then*

$$(\gamma^1)^2 = \begin{cases} \dfrac{\nu + 4 + \sqrt{3\nu^2 - 4\nu + 2}}{8} & \textit{for the state of plane stress} \\ \dfrac{3\nu^2 - 7\nu + 4 - (\nu - 1)\sqrt{9\nu^2 - 8\nu + 2}}{8(\nu - 1)^2} & \textit{for the state of plane strain.} \end{cases} \tag{31}$$

*Proof:* We restrict the bilinear form $a(.,.)$ defined in (21) to a finite element $\delta_H^{(r)}$ as it is shown in Figure 2.

Define the abbreviations

$$\alpha = \frac{1}{1 - \nu^2}, \ \delta = \frac{\nu}{1 - \nu^2}, \ \beta = \frac{1}{2(1 + \nu)}$$

for the state of plane stress and

$$\alpha = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)}, \ \delta = \frac{\nu}{(1 + \nu)(1 - 2\nu)}, \ \beta = \frac{1}{2(1 + \nu)}$$

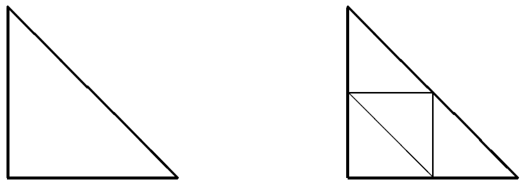

Figure 2: A right isosceles triangle of $\mathcal{T}_H$

for the state of plane strain. Using the finite element subspaces defined in (22), the corresponding matrices $A^{(r)}$, $B^{(r)}$, and $C^{(r)}$ have the following form:

$$A^{(r)} = \frac{E}{2}\begin{pmatrix} \alpha+\beta & \beta+\delta & -\alpha & -\beta & -\beta & -\delta \\ \beta+\delta & \alpha+\beta & -\delta & -\beta & -\beta & -\alpha \\ -\alpha & -\delta & \alpha & 0 & 0 & \delta \\ -\beta & -\beta & 0 & \beta & \beta & 0 \\ -\beta & -\beta & 0 & \beta & \beta & 0 \\ -\delta & -\alpha & \delta & 0 & 0 & \alpha \end{pmatrix},$$

$$B^{(r)} = \frac{E}{2}\begin{pmatrix} \beta & \beta & 0 & -\beta & -\beta & 0 \\ \delta & \alpha & -\delta & 0 & 0 & -\alpha \\ -(\alpha+\beta) & -(\delta+\beta) & \alpha & \beta & \beta & \delta \\ -(\delta+\beta) & -(\alpha+\beta) & \delta & \beta & \beta & \alpha \\ \alpha & \delta & -\alpha & 0 & 0 & -\delta \\ \beta & \beta & 0 & -\beta & -\beta & 0 \end{pmatrix},$$

$$C^{(r)} = \frac{E}{2}\begin{pmatrix} 2(\alpha+\beta) & \beta+\delta & -2\beta & -(\beta+\delta) & 0 & \beta+\delta \\ \beta+\delta & 2(\alpha+\beta) & -(\beta+\delta) & -2\alpha & \beta+\delta & 0 \\ -2\beta & -(\beta+\delta) & 2(\alpha+\beta) & \beta+\delta & -2\alpha & -(\beta+\delta) \\ -(\beta+\delta) & -2\alpha & \beta+\delta & 2(\alpha+\beta) & -(\beta+\delta) & -2\beta \\ 0 & \beta+\delta & -2\alpha & -(\beta+\delta) & 2(\alpha+\beta) & \beta+\delta \\ \beta+\delta & 0 & -(\beta+\delta) & -2\beta & \beta+\delta & 2(\alpha+\beta) \end{pmatrix}.$$

Obviously, we get

$$\ker\{A^{(r)}\} = \mathrm{span}\{(1,0,1,0,1,0)^T, (0,1,0,1,0,1)^T, (0,0,0,1,-1,0)^T\} \subseteq \ker\{B^{(r)}\}.$$

Furthermore, it is easy to show that the matrices $A^{(r)}$, $B^{(r)}$, and $C^{(r)}$ satisfy the properties (28). Since we want to apply Lemma 3.3 we have to choose a set of linearly independent vectors $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$ with $\underline{v}_i \notin \ker\{A^{(r)}\}$, $i = 1,2,3$, and $\ker\{A^{(r)}\} + \mathrm{span}\{\underline{v}_1, \underline{v}_2, \underline{v}_3\} = \mathbb{R}^6$. By means of the vectors

$$\underline{v}_1 = (1,0,0,0,0,0)^T\ ,\ \underline{v}_2 = (0,1,0,0,0,0)^T\ ,\ \text{and}\ \underline{v}_3 = (0,0,1,0,0,0)^T$$

we form the matrix $V^{(r)}$ introduced in Lemma 3.3. The corresponding generalized eigenvalue problem (30) is solved by means of the program MAPLE V [13], which calculates the eigenvalues (31).

Using the same idea for all other right isosceles triangles we get the same largest eigenvalue. Therefore, owing to Lemma 3.3, Lemma 3.2, and Lemma 3.1 we get the upper bounds (31) of the constant in the C.B.S. inequality. □

Theorem 3.2 follows immediately from Theorem 3.1 and Lemma 2.1.

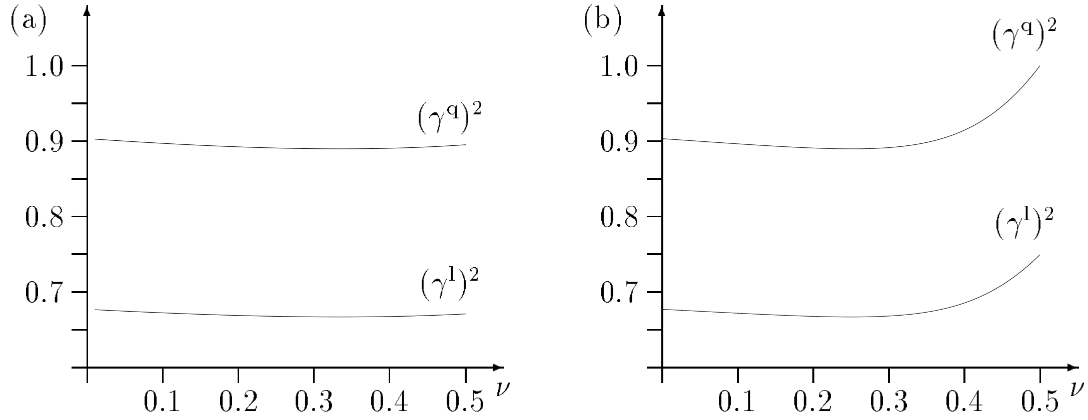


Figure 3: Plots of the functions $(\gamma^1)^2(\nu)$ and $(\gamma^q)^2(\nu)$ for the state of plane stress (a) and the state of plane strain (b)

**Theorem 3.2** *For a finite element triangulation with right isosceles triangles and p-hierarchical finite element ansatz functions ($p = 2$) the constant $(\gamma^q)^2$ in the strengthened C.B.S. inequality for plane linear elasticity problems is bounded by*

$$(\gamma^q)^2 = \frac{4}{3}(\gamma^1)^2$$

*with $\gamma^1$ from (31).*

Figure 3 shows plots of the functions $(\gamma^1)^2(\nu)$ and $(\gamma^q)^2(\nu)$.

**Remark 3.1**

*(i)* MARGENOV *[20] proves under the same assumptions as made in Theorem 3.1 that $(\gamma^1)^2 = \frac{3}{4}$ is an upper bound for all $\nu$. The dependence of the upper bound on $\nu$ is only shown by a table. Furthermore, he shows by numerical experiments that $\frac{3}{4}$ is also an upper bound for arbitrary right triangles.*

*(ii)* JUNG *considers in [15] a mesh refinement as it is shown in Figure 4.*

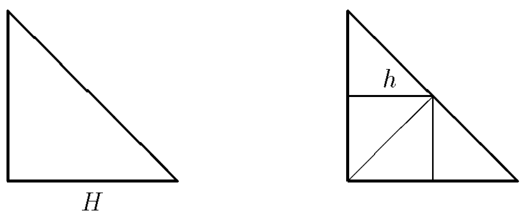


Figure 4: Non-standard refinement of a triangle

*In this case the following estimates for $(\gamma^1)^2$ are proved:*

$$(\gamma^1)^2 = \frac{13 - \nu^2 + \sqrt{(\nu^2 - 8\nu + 3)^2 + 16(1 - \nu^2)}}{8(3 - \nu)}$$

*for the state of plane stress and*

$$(\gamma^1)^2 = \frac{13 - 26\nu + 12\nu^2 + \sqrt{(12\nu^2 - 14\nu + 3)^2 + 16(1 - 2\nu)(1 - \nu)^2}}{8(3 - 4\nu)(1 - \nu)}$$

*for the state of plane strain. We remark that here* $(\gamma^1)^2$ *for the state of plane strain tends to 1 if* $\nu$ *tends to* $\frac{1}{2}$ *(see also Figure 5).*

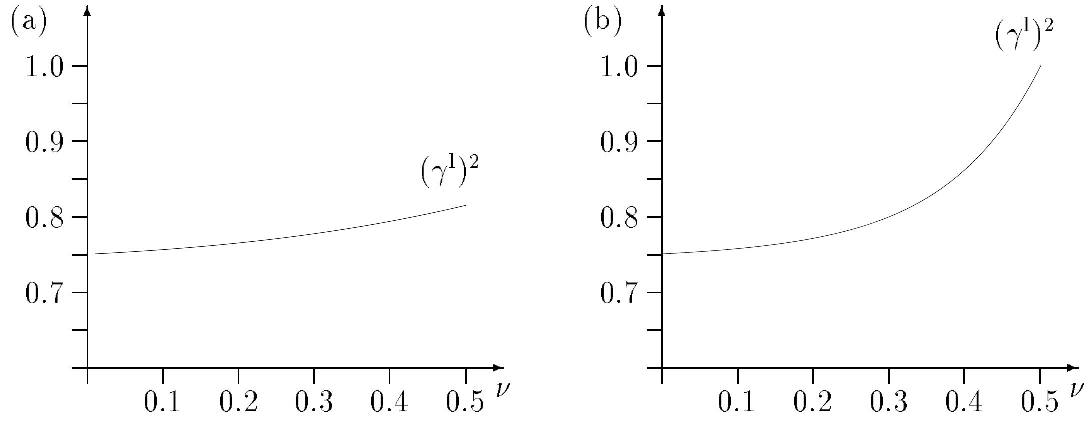


Figure 5: Plots of the functions $(\gamma^1)^2(\nu)$ for the state of plane stress (a) and the state of plane strain (b) in the case of non-standard mesh refinement

*(iii) In the paper [15],* Jung *studies the C.B.S. inequality for two-dimensional elasticity problems and discretizations with right isosceles triangles, where the finer triangulation is obtained by halving the triangles of the coarse triangulation (see Figure 6). For this case the following bounds of* $(\gamma^1)^2$ *are derived:*

$$(\gamma^1)^2 = \begin{cases} \dfrac{2}{3-\nu} & \textit{for the state of plane stress and} \\ \dfrac{2(1-\nu)}{3-4\nu} & \textit{for the state of plane strain.} \end{cases}$$

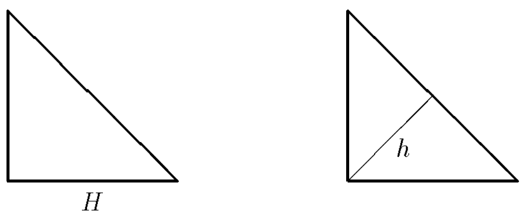


Figure 6: Halving of a triangle

*(iv)* Jung, Langer, *and* Semmler *give in [14, 16] for elasticity problems and finite element discretizations with triangular, rectangular, and hexahedral elements numerically determined bounds of* $(\gamma^q)^2$.

*(v) In Figure 7 we show the dependence of* $(\gamma^q)^2$ *(state of the plane strain) on the shape of the triangles. Here triangles with the vertices* $(0,0)$, $(1,0)$, *and* $(x_1^{(3)}, x_2^{(3)})$ *are considered (see Figure 7). The constant* $(\gamma^q)^2$ *is determined numerically. Obviously, we get the smallest constant for equilateral triangles.*

Unfortunately, relation (6) is not true for three-dimensional problems discretized by means of tetrahedral elements. In the following we want to give some first results concerning estimates of the constant in the C.B.S. inequality for three-dimensional elasticity

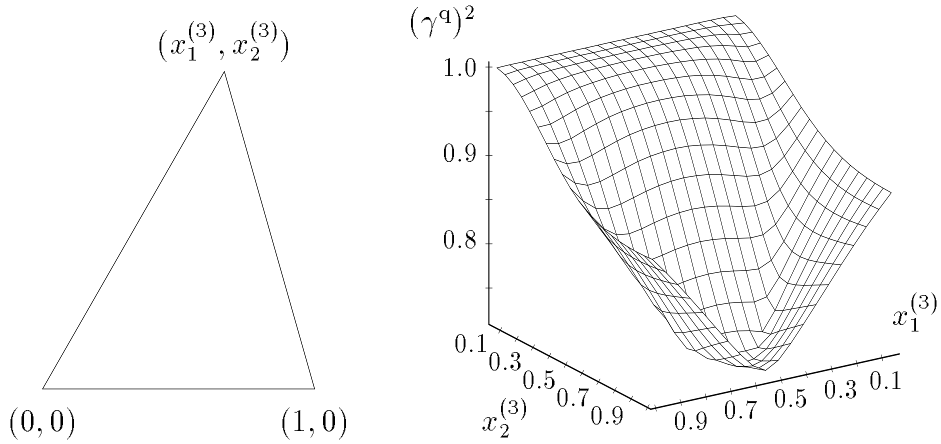


Figure 7: $(\gamma^q)^2$ in dependence on the shape of the triangle ($\nu = 0.3$)

problems. The extension of ACHCHAB's and MAITRE's proof for two-dimensional problems to the three-dimensional case is difficult. It is also very complicated to find a formula which indicates the dependence of $\gamma^1$ ($\gamma^q$) on the Poisson's ratio $\nu$ for special triangulations. Therefore, we are only able to present some numerical experiments. We consider a reference tetrahedron (see Figure 8). Figure 8 shows also how the constants $(\gamma^1)^2$ and $(\gamma^q)^2$ depend on $\nu$. We observe for the $h$-hierarchical case the upper bound 0.9 and for the $p$-hierarchical case the upper bound 1.

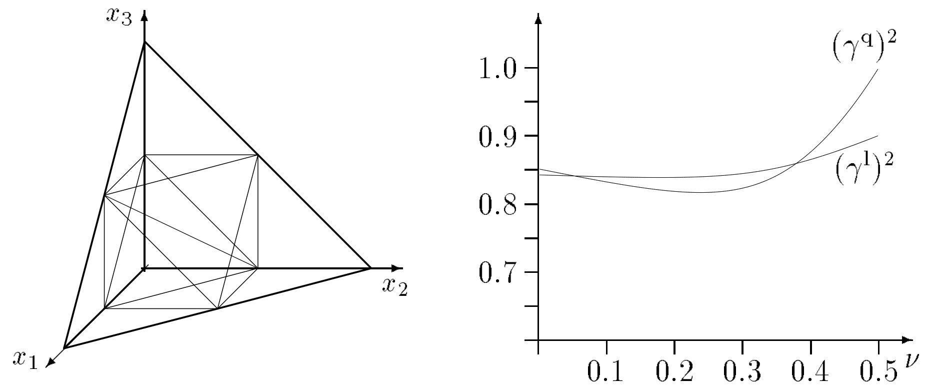


Figure 8: The reference tetrahedron and plots of the functions $(\gamma^1)^2(\nu)$ and $(\gamma^q)^2(\nu)$ for three-dimensional elasticity problems

# 4 A remark on multilevel iterative methods for elasticity problems

The knowledge of upper bounds of the constant in the C.B.S. inequality is of importance for computing optimal parameters in the algebraic multilevel preconditioner (AMLI) proposed by AXELSSON and VASSILEVSKI [5, 6] or in multigrid algorithms of the projection type described by MEIS and BRANCA [22] (see also [15, 24]). In these methods subproblems related to the subspaces $V_H$ and $T_h^1$, respectively, are to be solved. For the subproblem corresponding to $V_H$ usually a recursive multilevel strategy is used. The stiffness matrix resulting from the discretization of the subproblem on $T_h^1$ has a condition number $\kappa$ which is independent of the discretization parameter [2, 8, 15], but it is increasing with an increasing Poisson's ratio $\nu$. Numerically determined estimates of the condition number $\kappa(\nu)$ in the case of triangulations with right isosceles triangles and in the case of a tetrahedron as shown in Figure 8 are plotted in Figure 9 and in Figure 10, respectively. A good approximation for the function $\kappa(\nu)$ is the function $\frac{18.498-21.477\nu}{1-2\nu}$ in the case of the state of plane strain and the function $\frac{33.721-30.216\nu}{1-2\nu}$ in the three-dimensional case, respectively (see also a short remark in [21]).

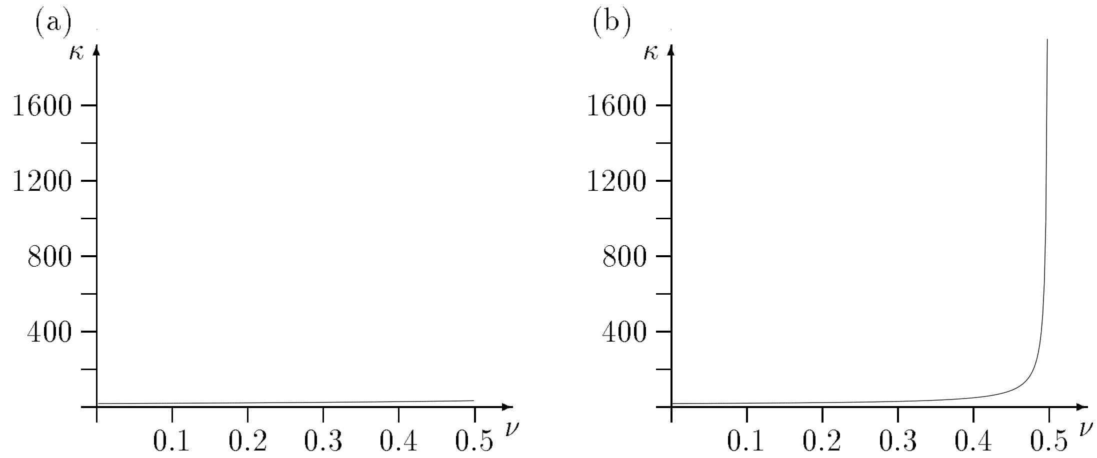


Figure 9: Plots of the functions $\kappa(\nu)$ for the state of plane stress (a) and the state of plain strain (b)

If one has a solver for the subproblem on $T_h^1$ which is robust with respect to $\nu$, then owing to Theorem 3.1 both the AMLI method and the multigrid algorithm of projection type are optimal robust iterative solvers for two-dimensional elasticity problems discretized by triangles with piecewise linear finite element ansatz functions. Up to now the construction of such robust subproblem solvers is still an open question. For other discretizations, as e.g. discretizations with quadrilateral elements and semi-coarsening, such solvers were constructed (see, e.g., [21]). Here, the special structure of the corresponding matrix is exploited such that an optimal direct solver can be used

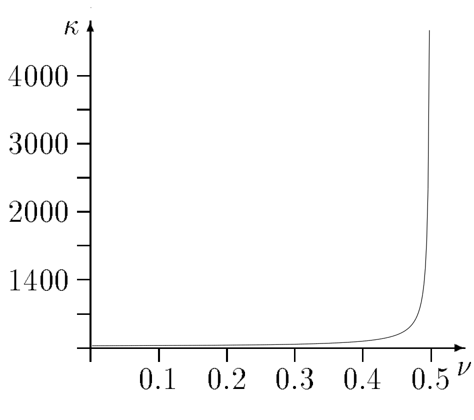


Figure 10: Plots of the function $\kappa(\nu)$ for a discretization with a tetrahedron

# 5 Conclusions

The bilinear forms considered in this paper cover a large class of practically relevant problems. For all these problems the upper bound $(\gamma^1)^2 = \frac{3}{4}$ in the strengthened C.B.S. inequality in the $h$-hierarchical case is now established. In the future work one can concentrate on showing the dependence of the constant on problem-describing parameters.

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