Technische Universität Chemnitz-Zwickau

Sonderforschungsbereich 393

Numerische Simulation auf massiv parallelen Rechnern

Günther Windisch

Two-point boundary value problems with piecewise constant coefficients: weak solution and exact discretization

Preprint SFB393/96-21

Abstract. For two-point boundary value problems in weak formulation with piecewise constant coefficients and piecewise continuous right-hand side functions we derive a representation of its weak solution by local Green's functions. Then we use it to generate exact three-point discretizations by Galerkin's method on essentially arbitrary grids. The coarsest possible grid is the set of points at which the piecewise constant coefficients and the right-hand side functions are discontinuous. This grid can be refined to resolve any solution properties like boundary and interior layers much more correctly. The proper basis functions for the Galerkin method are entirely defined by the local Green's functions. The exact discretizations are of completely exponentially fitted type and stable. The system matrices of the resulting tridiagonal systems of linear equations are in any case irreducible M-matrices with a uniformly bounded norm of its inverse.

AMS(MOS) subject classification: 65L10

Key words: piecewise constant coefficients, weak solutions, Galerkin method, exact discretizations, M-matrices, stability

Preprint-Reihe des Chemnitzer SFB 393

SFB393/96-21 **Dezember 1996**

1. Introduction

The aim of the paper is to construct exact discretizations of two-point boundary value problems

$$u \in U: \quad \int_0^1 (u' \ v' + b_\omega(x) \ u' \ v + c_\omega(x) \ u \ v) \ dx = \int_0^1 f \ v \ dx, \quad \forall v \in V, \quad (*)$$

where

$$U = \{u(x) \in W_2^1(0,1): \quad u(0) = u_0, \ u(1) = u_1\}, \quad V = \overset{\circ}{W}{}_2^1 (0,1),$$

with piecewise constant coefficients $b_\omega(x), c_\omega(x)$, $c_\omega(x) \geq 0$, and piecewise continuous right-hand side functions $f(x)$ on essentially arbitrary grids $\omega = \{x_i\}_{i=1}^n$,

$$0 = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = 1, \qquad h_i = x_i - x_{i-1} > 0.$$

To do this, the coarsest possible grid ω must contain all points of the interval $(0,1)$ at which at least one of the functions $b_\omega(x), c_\omega(x)$ and $f(x)$ is discontinuous. Such grids, which can be refined arbitrary, are then called essentially arbitrary grids ω.

In our paper, we shall present the following results for essentially arbitrary grids.

(a) We generate a representation of the weak solution $u(x) \in W_2^2(0,1)$ of Problem (*) by local Green's functions.

(b) Making use of the representation of the $W_2^2(0,1)$-solution $u(x)$, we then explicitly show that Problem (*) is a two-point boundary value problem of inverse isotone type.

(c) We construct exact discretizations of Problem (*) on essentially arbitrary grids ω by Galerkin's method approach. For this, the proper basis functions are defined by the local Green's functions, which yield completely exponentially fitted discretizations. Furthermore, the tridiagonal system matrices A of the resulting systems of linear equations $Ay = r$ are in any case irreducable M-matrices. In this way, the exact discretizations preserve the inverse isotonicity of Problem (*).

(d) We show that $||u(x) - u_h(x)||_{C[0,1]} \leq \gamma \max\limits_{1 \leq i \leq n} h_i$, where $u_h(x)$ is the ansatz of the Galerkin method.

(e) We derive bounds for $||(D^{-1}A)^{-1}||_\infty$ independent of the grids ω, which show the stability of the derived exact discretizations. The positive diagonal matrices D are defined by the basis functions.

The paper is organized as follows.

In Section 2 we derive a representation of the weak solution of the two-point boundary value Problem (*) with piecewise constant coefficients $b_\omega(x)$, $c_\omega(x)$ and piecewise continuous functions $f(x)$ on ω by applying local Green's functions. We show that the weak solutions may exhibit interior layers at points where its coefficient $b_\omega(x)$ of the first order derivative term is discontinuous, see [4], [6]. As a conclusion, we get that Problem (*) is boundary value problem of inverse isotone type.

In Section 3 we generate exact discretizations on essentially arbitrary grids ω by Galerkin's method approach using proper basis functions. The resulting discretizations are of completely exponentially fitted type. This approach is known from discretization methods for singularly perturbed boundary value problems to derive uniformly convergent discretization methods, see [9], [8].

In Section 4 we compare the weak solutions $u(x)$ with the ansatz for the Galerkin method $u_h(x)$ to show that the ansatz converges to the weak solutions. In Section 5 we investigate the stability of the exact discretizations. It will be shown that the row sum norm of the inverses of the tridiagonal system matrices $D^{-1}A$ are uniformly bounded.

We begin by reviewing some ideas of the application of Green's functions $G(x,\xi)$ to represent the classical solution of two-point boundary value problems.

For $b, c \in \mathbb{R}$ with $c \geq 0$, $f(x) \in C[0,1]$ and $u_0, u_1 \in \mathbb{R}$, consider the two-point boundary value problem

$$\begin{aligned} &Lu = -u'' + b\ u' + c\ u = f(x), \quad 0 < x < 1, \\ &u(0) = u_0, \quad u(1) = u_1. \end{aligned} \tag{1.1}$$

The characteristic equation $-\lambda^2 + b\,\lambda + c = 0$ for the differential equation $Lu = 0$ has the real roots

$$\lambda_2 = \tfrac{b-\sqrt{b^2+4c}}{2} \quad \leq 0 \leq \quad \lambda_1 = \tfrac{b+\sqrt{b^2+4c}}{2}.$$

It is obvious that

$$\begin{aligned} &\kappa = \max\{|b|, c\} > 0 \iff \lambda_2 < \lambda_1, \\ &\kappa = \max\{|b|, c\} = 0 \iff \lambda_2 = \lambda_1 = 0. \end{aligned} \tag{1.2}$$

Let $w_0(x), w_1(x)$ be the solution of the two boundary value problems

$$\begin{array}{lll} Lw_0 = 0, & w_0(0) = 1, & w_0(1) = 0, \\ Lw_1 = 0, & w_1(0) = 0, & w_1(1) = 1, \end{array} \tag{1.3}$$

respectively, which are explicitly given by

$$\begin{array}{lll} \kappa > 0: & w_0(x) = \frac{e^{\lambda_1+\lambda_2 x} - e^{\lambda_1 x+\lambda_2}}{e^{\lambda_1} - e^{\lambda_2}}, & w_1(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{e^{\lambda_1} - e^{\lambda_2}}, \\ \kappa = 0: & w_0(x) = 1 - x, & w_1(x) = x. \end{array} \tag{1.4}$$

Observe that for $\kappa \geq 0$

$$\begin{array}{ll} (a) & w_0(x) \geq 0, \quad w_0'(x) < 0, \quad \forall x \in [0,1], \\ & w_1(x) \geq 0, \quad w_1'(x) > 0, \quad \forall x \in [0,1], \\ (b) & w_0'(0) + w_1'(0) \leq 0, \\ & w_0'(1) + w_1'(1) \geq 0, \\ (c) & W(x) = \begin{cases} \frac{(\lambda_1-\lambda_2)e^{(\lambda_1+\lambda_2)x}}{e^{\lambda_1}-e^{\lambda_2}} > 0, & \kappa > 0, \\ 1, \quad \kappa = 0, \end{cases} \end{array} \tag{1.5}$$

where $W(x) = w_0(x)w_1'(x) - w_0'(x)w_1(x)$ is the Wronskian of $w_0(x), w_1(x)$.

Then, with the Green's function

$$G(x,\xi) = \begin{cases} \frac{1}{W(\xi)}\ w_0(x)w_1(\xi), & \xi < x, \\ \frac{1}{W(\xi)}\ w_0(\xi)w_1(x), & \xi \geq x, \end{cases} \qquad (x,\xi) \in [0,1]\times[0,1], \tag{1.6}$$

the unique solution of the boundary value problem (1.1) has the representation

$$\begin{aligned} u(x) &= u_0 w_0(x) + u_1 w_1(x) + \int_0^1 G(x,\xi)\ f(\xi)\ d\xi \\ &= \left[u_0 + \int_0^x \frac{w_1(\xi)f(\xi)}{W(\xi)}d\xi\right] w_0(x) + \left[u_1 + \int_x^1 \frac{w_0(\xi)f(\xi)}{W(\xi)}d\xi\right] w_1(x), \end{aligned} \tag{1.7}$$

where

$$u'(x) = \left[u_0 + \int_0^x \frac{w_1(\xi)f(\xi)}{W(\xi)}d\xi\right] w_0'(x) + \left[u_1 + \int_x^1 \frac{w_0(\xi)f(\xi)}{W(\xi)}d\xi\right] w_1'(x),$$

$$u''(x) = \left[u_0 + \int_0^x \frac{w_1(\xi)f(\xi)}{W(\xi)}d\xi\right] w_0''(x) + \left[u_1 + \int_x^1 \frac{w_0(\xi)f(\xi)}{W(\xi)}d\xi\right] w_1''(x) - f(x).$$

Hence, $u(x) \in C^2[0,1]$ for $\forall f(x) \in C[0,1]$.

We introduce the weak formulation of Problem (1.1). For this let

$$U = \{u(x) \in W_2^1(0,1): \quad u(0) = u_0,\ u(1) = u_1\},$$
$$V = \overset{\circ}{W}{}_2^1(0,1), \tag{1.8}$$

where $W_2^1(0,1)$, $\overset{\circ}{W}{}_2^1(0,1)$ are the Sobolev spaces

$$W_2^1(0,1) = \{u(x) \in L_2(0,1) \quad \text{with generalized } u'(x) \in L_2(0,1)\},$$
$$\overset{\circ}{W}{}_2^1(0,1) = \{u(x) \in W_2^1(0,1):\ u(0) = u(1) = 0\}.$$

This is a proper pair of sufficiently general function spaces which imply unique solvable weak formulations of second order boundary value problems, see [3].

Problem (1.1) then takes the form: Find $u \in U$ such that

$$a(u,v) = (f,v), \quad \forall v \in V, \tag{1.9}$$

where

$$\begin{aligned} a(u,v) &= \int_0^1 (u'\ v' + b\ u'\ v + c\ u\ v)\ dx, \\ (f,v) &= \int_0^1 f\ v\ dx. \end{aligned} \tag{1.10}$$

We remark that for $b \neq 0$ the bilinear form $a(u,v)$ is not symmetric and not coercive.

Problem (1.9) has a unique solution $u(x) \in U$ and a straightforward calculation shows that $u(x)$ is given by formula (1.7).

Because of

$$w_0(x) \geq 0,\ w_1(x) \geq 0, \quad \forall x \in [0,1],$$

$$G(x,\xi) \geq 0, \quad \forall (x,\xi) \in [0,1] \times [0,1],$$

the boundary value problem (1.1) is of inverse isotone type, see [2]. Its weak formulation (1.9) thus also exhibits this important property.

2. Representation of the weak solution of two-point boundary value problems with piecewise constant coefficients

Using local Green's functions, in the present section we shall derive a representation of the weak solution of two-point boundary value problems with piecewise constant coefficients and for piecewise continuous right-hand side functions $f(x)$. The coarsest set of grid points ω for the representation of the weak solution is defined by the set of points $\{x_i\}_{i=1}^{n} \subset (0,1)$ at which the coefficients $b_\omega(x), c_\omega(x)$ and $f(x)$ are discontinuous. From the construction it is clear then that this set of grid points can be extended to an essentially arbitrary irregular grid ω.

Let $\overline{\omega} = \{x_i\}_{i=0}^{n+1}$ be an arbitrary grid, where

$$\begin{aligned} &0 = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = 1, \\ &h_i = x_i - x_{i-1} > 0, \qquad i = 1, \ldots, n+1, \end{aligned} \tag{2.1}$$

and

$$\omega = \overline{\omega} \setminus \{x_0, x_{n+1}\} = \overline{\omega} \setminus \{0, 1\}.$$

For the piecewise constant coefficients $b_\omega(x), c_\omega(x)$, assume that

$$\begin{aligned} b_\omega(x)|_{(x_{i-1},x_i)} &= b_i \in \mathbb{R}, \\ c_\omega(x)|_{(x_{i-1},x_i)} &= c_i \in \mathbb{R}, \qquad c_i \geq 0, \end{aligned} \tag{2.2}$$

for $i = 1, \ldots, n+1$.

Furthermore, let

$$f(x)|_{(x_{i-1},x_i)} \in C[x_{i-1}, x_i], \quad i = 1, \ldots, n+1.$$

Consider the following two-point boundary value problem with piecewise constant coefficients, where U, V are the function spaces introduced by (1.8).

Find $u \in U$ such that

$$a_\omega(u, v) = (f, v), \quad \forall v \in V, \tag{2.3}$$

where

$$
\begin{aligned}
a_\omega(u,v) &= \int_0^1 (u' \ v' + b_\omega(x) \ u' \ v + c_\omega(x) \ u \ v) \ dx, \\
(f,v) &= \int_0^1 f \ v \ dx.
\end{aligned}
$$

The assumptions imply that Problem (2.3) has an unique solution $u \in W_2^2(0,1)$, where $W_2^2(0,1) = \{u(x) \in L_2(0,1): \text{ with generalized } u''(x) \in L_2(0,1)\}$, see [3].

Because the weak solution $u(x) \in W_2^2(0,1)$ of Problem (2.3) is a continuous function on [0,1], let

$$
u(x)|_{[x_{i-1},x_i]} = u_i(x), \qquad i = 1,\dots,n+1. \tag{2.4}
$$

In analogy to (1.7), we shall use the following ansatz

$$
u_i(x) = u(x_{i-1}) \ w_{i0}(x) + u(x_i) \ w_{i1}(x) + \int_{x_{i-1}}^{x_i} G_i(x,\xi) \ f(\xi) \ d\xi, \tag{2.5}
$$

$i = 1,\dots,n+1$, where $G_i(x,\xi)$ for $(x,\xi) \in [x_{i-1},x_i] \times [x_{i-1},x_i]$ are the local Green's functions.

We must determine the unknown values $u(x_i)$ for $i = 1,\dots,n$, where $u(x_0) = u(0) = u_0$, $u(x_{n+1}) = u(1) = u_1$ are the given boundary values.

Define

$$
L_i w = -w'' + b_i \ w' + c_i \ w, \quad i = 1,\dots,n+1.
$$

The functions $w_{i0}(x), w_{i1}(x)$ are then the solutions of the two boundary value problems

$$
\begin{array}{lll}
L_i w_{i0} = 0, & w_{i0}(x_{i-1}) = 1, & w_{i0}(x_i) = 0, \\
L_i w_{i1} = 0, & w_{i1}(x_{i-1}) = 0, & w_{i1}(x_i) = 1,
\end{array} \tag{2.6}
$$

respectively, which are easily available and generate the local Green's functions $G_i(x,\xi)$.

Under the assumptions (2.2), the characteristic equations

$$
-(\lambda^i)^2 + b_i \ \lambda^i + c_i = 0, \tag{2.7}
$$

for the differential equations $L_i w = 0$ has the real roots

$$\lambda_2^i = \frac{b_i - \sqrt{b_i^2 + 4c_i}}{2} \quad \leq 0 \leq \quad \lambda_1^i = \frac{b_i + \sqrt{b_i^2 + 4c_i}}{2}, \qquad i = 1, \ldots, n+1.$$

Let $\kappa_i = \max\{|b_i|, c_i\}$, we have

$$\kappa_i > 0 \iff \lambda_2^i < \lambda_1^i,$$

$$\kappa_i = 0 \iff \lambda_2^i = \lambda_1^i = 0.$$

For $x \in [x_{i-1}, x_i]$, we get

$$\begin{aligned}
\kappa_i > 0: \qquad & w_{i0}(x) = \frac{e^{x_i\lambda_1^i + x\lambda_2^i} - e^{x\lambda_1^i + x_i\lambda_2^i}}{e^{x_i\lambda_1^i + x_{i-1}\lambda_2^i} - e^{x_{i-1}\lambda_1^i + x_i\lambda_2^i}}, \\
& w_{i1}(x) = \frac{e^{x\lambda_1^i + x_{i-1}\lambda_2^i} - e^{x_{i-1}\lambda_1^i + x\lambda_2^i}}{e^{x_i\lambda_1^i + x_{i-1}\lambda_2^i} - e^{x_{i-1}\lambda_1^i + x_i\lambda_2^i}}, \\
\kappa_i = 0: \qquad & w_{i0}(x) = \frac{x_i - x}{h_i}, \\
& w_{i1}(x) = \frac{x - x_{i-1}}{h_i},
\end{aligned} \tag{2.8}$$

and remark that $\kappa_i > 0$ implies

$$h_i\lambda_1^i > h_i\lambda_2^i \iff x_i\lambda_1^i + x_{i-1}\lambda_2^i > x_{i-1}\lambda_1^i + x_i\lambda_2^i.$$

For $\kappa_i \geq 0$, $i = 1, \ldots, n+1$, it is then immediately verified that

$$\begin{aligned}
(a) \quad & w_{i0}(x) \geq 0, \quad w_{i0}'(x) < 0, \quad \forall x \in [x_{i-1}, x_i], \\
& w_{i1}(x) \geq 0, \quad w_{i1}'(x) > 0, \quad \forall x \in [x_{i-1}, x_i], \\
(b) \quad & w_{i0}'(x_{i-1}) + w_{i1}'(x_{i-1}) \leq 0, \\
& w_{i0}'(x_i) + w_{i1}'(x_i) \geq 0, \\
(c) \quad & W_i(x) = w_{i0}(x)w_{i1}'(x) - w_{i0}'(x)w_{i1}(x) \\
& = \begin{cases} \frac{(\lambda_1^i - \lambda_2^i)e^{(x - x_{i-1})(\lambda_1^i + \lambda_2^i)}}{e^{h_i\lambda_1^i} - e^{h_i\lambda_2^i}} > 0, & \kappa_i > 0, \\ \frac{1}{h_i} > 0, & \kappa_i = 0, \end{cases} \qquad \forall x \in [x_{i-1}, x_i].
\end{aligned} \tag{2.9}$$

Now we define the local Green's functions $G_i(x, \xi)$ by

$$G_i(x,\xi) = \begin{cases} \frac{1}{W_i(\xi)}\ w_{i0}(x)w_{i1}(\xi), & x_{i-1} \le \xi < x, \\ \\ \frac{1}{W_i(\xi)}\ w_{i0}(\xi)w_{i1}(x), & x \le \xi \le x_i, \end{cases} \tag{2.10}$$

for $\forall (x,\xi) \in [x_{i-1}, x_i] \times [x_{i-1}, x_i]$, $i = 1, \ldots, n+1$.

We see that

$$G_i(x,\xi) \ge 0 \quad \text{on} \quad [x_{i-1}, x_i] \times [x_{i-1}, x_i],$$

$$G_i(x,\xi) = G_i(\xi, x) \quad \Leftrightarrow \quad b_i = 0.$$

Thus, (2.5) becomes

$$u_i(x) = \left[u(x_{i-1}) + \int\limits_{x_{i-1}}^{x} \frac{w_{i1}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i0}(x) + \left[u(x_i) + \int\limits_{x}^{x_i} \frac{w_{i0}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i1}(x), \tag{2.11}$$

and, furthermore, we have

$$u_i^{'}(x) = \left[u(x_{i-1}) + \int\limits_{x_{i-1}}^{x} \frac{w_{i1}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i0}^{'}(x) + \left[u(x_i) + \int\limits_{x}^{x_i} \frac{w_{i0}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i1}^{'}(x), \tag{2.12}$$

$$u_i^{''}(x) = \left[u(x_{i-1}) + \int\limits_{x_{i-1}}^{x} \frac{w_{i1}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i0}^{''}(x) + \left[u(x_i) + \int\limits_{x}^{x_i} \frac{w_{i0}(\xi)f(\xi)}{W_i(\xi)} d\xi\right] w_{i1}^{''}(x)$$

$$-f(x)|_{[x_{i-1}, x_i]}.$$

Our assumptions now imply for $u(x)$ from (2.4), (2.5) that

$$\begin{array}{ll} u(x) \in C[0,1], & \\ u_i(x) \in C^2[x_{i-1}, x_i], & \text{i = 1,\ldots,n+1.} \\ u_i(x_{i-1}) = u(x_{i-1}), \quad u_i(x_i) = u(x_i), & \end{array} \tag{2.13}$$

Substituting (2.5) in (2.3) gives for $\forall v \in V$

$$
\begin{aligned}
a_\omega(u,v) &= \sum_{i=1}^{n+1} \int_{x_{i-1}}^{x_i} (u_i' \; v' + b_i \; u_i' \; v + c_i \; u_i \; v) \; dx \\
&= \sum_{i=1}^{n+1} \left(\int_{x_{i-1}}^{x_i} (-u_i'' + b_i \; u_i' + c_i \; u_i) \; v \; dx + u_i'(x)v(x)|_{x_{i-1}}^{x_i} \right) \\
&= \sum_{i=1}^{n+1} \left(\int_{x_{i-1}}^{x_i} f \; v \; dx + u_i'(x_i)v(x_i) - u_i'(x_{i-1})v(x_{i-1}) \right) \\
&= \int_0^1 f \; v \; dx + \sum_{i=1}^{n} \left(u_i'(x_i) - u_{i+1}'(x_i) \right) \; v(x_i),
\end{aligned}
\tag{2.14}
$$

because of $v(x_0) = v(0) = 0, \quad v(x_{n+1}) = v(1) = 0.$

Thus, (2.14) implies the system of linear equations

$$
u_i'(x_i) - u_{i+1}'(x_i) = 0, \qquad i = 1, \ldots, n. \tag{2.15}
$$

according to the unknowns $u(x_i)$.

It follows from the conditions (2.15) that $u(x) \in C^1[0,1]$. With $u_i(x) \in C^2[x_{i-1}, x_i]$ we finally get $u(x) \in W_2^2(0,1)$, see Lemma 4.2 from [4].

With (2.12) the i-th equation of (2.15) takes the form

$$
w_{i0}'(x_i) \; u(x_{i-1}) + \left[w_{i1}'(x_i) - w_{i+1,0}'(x_i) \right] \; u(x_i) - w_{i+1,1}'(x_i) \; u(x_{i+1}) = \hat{r}_i, \tag{2.16}
$$

where

$$
\hat{r}_i = -w_{i0}'(x_i) \int_{x_{i-1}}^{x_i} \frac{w_{i1}(\xi) \; f(\xi)}{W_i(\xi)} d\xi + w_{i+1,1}'(x_i) \int_{x_i}^{x_{i+1}} \frac{w_{i+1,0}(\xi) \; f(\xi)}{W_{i+1}(\xi)} d\xi. \tag{2.17}
$$

The right-hand side expression of (2.17) is the reason for the definition of the following set of functions for $i = 1, \ldots, n$

$$
\phi_i(\xi) = \begin{cases} \frac{e^{-\lambda_2^i(\xi - x_{i-1})} - e^{-\lambda_1^i(\xi - x_{i-1})}}{e^{-\lambda_2^i h_i} - e^{-\lambda_1^i h_i}}, & \kappa_i > 0, \\ & \qquad x_{i-1} \le \xi \le x_i, \\ \frac{\xi - x_{i-1}}{h_i}, & \kappa_i = 0, \\ \frac{e^{\lambda_1^{i+1}(x_{i+1} - \xi)} - e^{\lambda_2^{i+1}(x_{i+1} - \xi)}}{e^{\lambda_1^{i+1} h_{i+1}} - e^{\lambda_2^{i+1} h_{i+1}}}, & \kappa_{i+1} > 0, \\ & \qquad x_i < \xi \le x_{i+1}, \\ \frac{x_{i+1} - \xi}{h_{i+1}}, & \kappa_{i+1} = 0, \\ 0, & else. \end{cases} \tag{2.18}
$$

Using (2.8), (2.9)(c), we obtain now from (2.17) that

$$
\hat{r}_i = \int_{x_{i-1}}^{x_{i+1}} f \ \phi_i \ dx, \qquad i = 1, \ldots, n. \tag{2.19}
$$

We introduce next a matrix form of the tridiagonal system of linear equations (2.15) to go into details of its main qualitative properties.

Define

$$
A = tridiag(a_{i,i-1}, a_{ii}^- + a_{ii}^+, a_{i,i+1})_{n \times n},
$$

$$
\underline{u} = (u(x_1), u(x_2), \ldots, u(x_n))^T,
$$

$$
r = (r_1, r_2, \ldots, r_n)^T.
$$

Then we identify (2.15) with

$$
A\underline{u} = r. \tag{2.20}
$$

Using (2.9), we get from (2.16) that

$$
\begin{aligned}
a_{i,i-1} &= w'_{i0}(x_i) < 0, \quad i = 2, \ldots, n, \\
a_{ii}^- &= w'_{i1}(x_i) > 0, \quad i = 1, \ldots, n, \\
a_{ii}^+ &= -w'_{i+1,0}(x_i) > 0, \quad i = 1, \ldots, n, \\
a_{i,i+1} &= -w'_{i+1,1}(x_i) < 0, \quad i = 1, \ldots, n-1.
\end{aligned} \tag{2.21}
$$

Thus, under condition (2.2), the matrix A is an L-matrix, [12]. This means we have $a_{ii} = a_{ii}^- + a_{ii}^+ > 0$ for $i = 1, \ldots, n$ and $a_{ij} \leq 0$ for $i \neq j$, see [12]. Moreover, the tridiagonal matrix A is irreducible because all of its codiagonal entries $a_{i,i-1}$ and $a_{i,i+1}$ are nonzero, [12].

Furthermore, by (2.9)(b), it follows that

$$\begin{aligned} &a_{i,i-1} + a_{ii}^- = w_{i0}'(x_i) + w_{i1}'(x_i) \geq 0, \\ &a_{ii}^+ + a_{i,i+1} = -(w_{i+1,0}'(x_i) + w_{i+1,1}'(x_i)) \geq 0, \end{aligned} \qquad \text{i = 2,...,n-1,} \qquad (2.22)$$

and

$$\begin{aligned} &a_{11}^- + a_{11}^+ + a_{12} > 0, \\ &a_{n,n-1} + a_{nn}^- + a_{nn}^+ > 0. \end{aligned} \qquad (2.23)$$

Hence, by a usual diagonal dominance argument, the L-matrix A is an irreducible M-matrix, see [12]. This implies that (2.20) is uniquely solvable and $A^{-1} > 0$ entrywise.

The entries of the right-hand side vector r of the system (2.20) are given by

$$\begin{aligned} &r_1 = a_{10}u_0 + \hat{r}_1, \\ &r_i = \hat{r}_i, \quad i = 2, \ldots, n, \\ &r_n = a_{n,n+1}u_1 + \hat{r}_n, \end{aligned} \qquad (2.24)$$

where $a_{10} = -w_{10}'(0) > 0$ and $a_{n,n+1} = w_{n+1,n}'(1) > 0$.

The right-hand side vector $r = r(u_0, u_1, f(x))$ of the system of linear equations (2.20) depends isotonically on its arguments. This means, if we replace u_k by v_k with $u_k \leq v_k$, $k = 0, 1$, and $f(x)$ by $g(x)$ such that $f(x) \leq g(x)$ on [0,1] then

$$r(u_0, u_1, f(x)) \leq r(v_0, v_1, g(x)).$$

Hence, by $A^{-1} > 0$, we conclude that

$$A^{-1}r(u_0, u_1, f(x)) \leq A^{-1}r(v_0, v_1, g(x)).$$

The latter inequality shows that the solution

$$\underline{u} = (u(x_1), u(x_2), \ldots, u(x_n))^T = A^{-1}r(u_0, u_1, f(x))$$

of the system of linear equations (2.20) depends isotonically on the boundary values u_0, u_1 and on the right-hand side function $f(x)$ of Problem (2.3).

The Hermite interpolation of the points $(x_{i-1}, u(x_{i-1}))$ and $(x_i, u(x_i))$ by $u_i(x)$ for $i = 1, \dots, n+1$ generates the weak solution $u(x) \in C^1[0,1]$. By ansatz (2.5), $u_i(x)$ depends isotonically on $u(x_{i-1}), u(x_i)$ and on $f(x)|_{[x_{i-1},x_i]}$. This proves the fact that under condition (2.2) the weak solution $u(x)$ of Problem (2.3) depends isotonically on the boundary values u_0, u_1 and on the right-hand side function $f(x)$. Thus, Problem (2.3) is of inverse isotone type, see also [3].

Next we derive explicit formulas for the entries of the matrix A on the basis of the roots λ_k^i of the characteristic equations (2.7). This means we assume that all of the roots $\lambda_k^i,\ k = 1,2,\ i = 1, \dots, n+1$ are computed from $\{b_i, c_i\}_{i=1}^{n+1}$.

For $\kappa_i > 0$, define the auxiliary values

$$\begin{aligned} &p_i = e^{\lambda_1^i h_i} - e^{\lambda_2^i h_i} > 0, \qquad q_i = e^{-\lambda_2^i h_i} - e^{-\lambda_1^i h_i} = \tfrac{p_i}{e^{b_i h_i}} > 0, \\ &\rho_i = \tfrac{\lambda_1^i - \lambda_2^i}{2(\cosh((\lambda_1^i - \lambda_2^i)h_i) - 1)} > 0. \end{aligned} \tag{2.25}$$

Now (2.8), (2.21) imply

$$\begin{aligned} a_{i,i-1} &= w'_{i0}(x_i) = -\frac{\lambda_1^i - \lambda_2^i}{e^{-h_i\lambda_2^i} - e^{-h_i\lambda_1^i}} = -\rho_i p_i < 0, \\ a_{ii}^- &= w'_{i1}(x_i) = \frac{\lambda_1^i e^{h_i\lambda_1^i} - \lambda_2^i e^{h_i\lambda_2^i}}{e^{h_i\lambda_1^i} - e^{h_i\lambda_2^i}} = \frac{\lambda_1^i - \lambda_2^i}{2}\coth\left(\frac{\lambda_1^i - \lambda_2^i}{2} h_i\right) + \frac{b_i}{2} > 0. \end{aligned} \tag{2.26}$$

Clearly, for $\kappa_i = 0$ we have

$$a_{i,i-1} = -\tfrac{1}{h_i}, \qquad a_{ii}^- = \tfrac{1}{h_i}. \tag{2.27}$$

Assuming $\kappa_{i+1} > 0$, we get

$$\begin{aligned} a_{i,i+1} &= -w'_{i+1,1}(x_i) = -\rho_{i+1} q_{i+1} < 0, \\ a_{ii}^+ &= -w'_{i+1,0}(x_i) = \frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{2}\coth\left(\frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{2} h_{i+1}\right) - \frac{b_{i+1}}{2} > 0, \end{aligned} \tag{2.28}$$

and for $\kappa_{i+1} = 0$ we get

$$a_{i,i+1} = -\tfrac{1}{h_{i+1}}, \qquad a_{ii}^+ = \tfrac{1}{h_{i+1}}. \tag{2.29}$$

Remark

We note that $A = A^T$ if and only if $b_\omega(x) \equiv 0$. To see this, let $b_i = 0$ for $i = 1, \ldots, n+1$. If $\kappa_i = 0$, we have $a_{i,i-1} = a_{i-1,i} = -\frac{1}{h_i}$. Otherwise, if $\kappa_i > 0$ then $a_{i,i-1} = -\rho_i p_i$, $a_{i-1,i} = -\rho_i q_i$, and, $p_i = q_i$, see (2.25), implies that $a_{i,i-1} = a_{i-1,i}$. Conversely, $A = A^T$ means then $a_{i,i-1} = a_{i-1,i}$ for $i = 2, \ldots, n$, which automatically holds if $\kappa_i = 0$. On the other hand, for $\kappa_i > 0$, we get $a_{i,i-1} = -\rho_i p_i = -\rho_i q_i = a_{i-1,i}$. Thus, $p_i = q_i$ implies $b_i = 0$, see (2.25).

Much more general, $b_i = 0$ implies $a_{i,i-1} = a_{i-1,i}$ independend of the behaviour of $b_\omega(x)$ at the other subintervals of the grid ω.

Remark

The behaviour of some types of weak solutions for different piecewise constant coefficients $b_\omega(x)$ and $c_\omega(x)$ is illustrated in **Appendix 1** of the paper. The examples demonstrate for which situations weak solutions exhibit boundary and interior layers.

3. Exact discretization by Galerkins's method approach

In this section, we shall apply Galerkin's method to derive an exact discretization of Problem (2.3). A discretization method on the grid $\omega = \{x_i\}_{i=1}^n$ is said to be "exact" if it generates a system of algebraic equations $Ay = r$ whose solution is $\{y_i = u(x_i)\}_{i=1}^n$, where $u(x)$ is the solution of the problem under consideration. For examples, see [4], [6], [11]. In our case $u(x)$ is the weak solution of Problem (2.3). This means, we shall show that Galerkin's method with the set of basis functions (2.18) is exact for Problem (2.3) on essentially arbitrary grids ω.

To include the boundary conditions $u(0) = u_0$, $u(1) = u_1$ in the ansatz for the Galerkin's method, we first supplement the set of functions (2.18) with

$$\phi_0(\xi) = \begin{cases} \frac{e^{\lambda_1^1(x_1-\xi)} - e^{\lambda_2^1(x_1-\xi)}}{e^{\lambda_1^1 h_1} - e^{\lambda_2^1 h_1}}, & \kappa_1 > 0, \\ \frac{x_1-\xi}{h_1}, & \kappa_1 = 0, \\ 0, & else, \end{cases} \qquad 0 \le \xi \le x_1,$$

$$\phi_{n+1}(\xi) = \begin{cases} \frac{e^{-\lambda_2^{n+1}(\xi - x_n)} - e^{-\lambda_1^{n+1}(\xi - x_n)}}{e^{-\lambda_2^{n+1} h_{n+1}} - e^{-\lambda_1^{n+1} h_{n+1}}}, & \kappa_{n+1} > 0, \\ \frac{\xi - x_n}{h_{n+1}}, & \kappa_{n+1} = 0, \\ 0, & else, \end{cases} \quad x_n \leq \xi \leq 1.$$

Define

$$U_h = \{u_h(x) = u_0\phi_0(x) + \sum_{i=1}^{n} y_i\phi_i(x) + u_1\phi_{n+1}(x),\ y_i \in \mathbb{R}\} \subset U,$$
$$V_h = span\{\phi_i(x)\}_{i=1}^{n} \subset V. \tag{3.1}$$

Applying now Galerkin's method to solve Problem (2.3), we get the following finite dimensional problem.

Find $u_h \in U_h$ such that

$$a_\omega(u_h, \phi_i) = (f, \phi_i), \qquad i = 1, \ldots, n. \tag{3.2}$$

We shall show that the tridiagonal system of linear equations resulting from (3.2) is identical with the system of linear equations (2.20). First of all, we have the following theorem.

Theorem 1 Under assumptions (2.2) it follows that

$$A = tridiag(a_\omega(\phi_{i-1}, \phi_i), a_\omega(\phi_i, \phi_i), a_\omega(\phi_{i+1}, \phi_i))_{n\times n},$$

where A is the system matrix of the linear system (2.20).

Proof. The proof of Theorem 1 is given in **Appendix 2**.

Remark The right-hand side vector of the system of linear equations (3.2) is identical with the vector $r = (r_1, \ldots, r_n)^T$ defined by (2.24).

We have thus proved the following theorem.

Theorem 2 The Galerkin method (3.2) yields an exact discretization of problem (2.3), i.e. it holds $y_i = u(x_i), \forall x_i \in \omega$.

4. Comparison of the weak solution with the approximate solution

The results of Section 3 imply that the Galerkin method (3.2) is super convergent because $y_i = u(x_i)$, $\forall x_i \in \omega$, means that the order of convergence at the grid points is infinity.

We thus consider next the behaviour of the difference

$$d_i(x) = |u_i(x) - u_h(x)|, \quad x \in [x_{i-1}, x_i], \qquad i = 1, \ldots, n+1, \tag{4.1}$$

where $d_i(x_{i-1}) = d_i(x_i) = 0$.

The restriction $u_i(x) = u(x)|_{[x_{i-1},x_i]}$ of the weak solution of Problem (2.3) is given by (2.11). For the restriction of the ansatz $u_h(x)$ we find from (3.1) with the basis functions (2.18)

$$u_h(x)|_{[x_{i-1},x_i]} = y_{i-1}\phi_{i-1}(x)|_{[x_{i-1},x_i]} + y_i\phi_i(x)|_{[x_{i-1},x_i]}$$

$$= \begin{cases} y_{i-1}\frac{e^{\lambda_1^i(x_i-x)}-e^{\lambda_2^i(x_i-x)}}{e^{\lambda_1^i h_i}-e^{\lambda_2^i h_i}} + y_i\frac{e^{-\lambda_2^i(x-x_{i-1})}-e^{-\lambda_1^i(x-x_{i-1})}}{e^{-\lambda_2^i h_i}-e^{-\lambda_1^i h_i}}, & \kappa_i > 0, \\ y_{i-1}\frac{x_i-x}{h_i} + y_i\frac{x-x_{i-1}}{h_i}, & \kappa_i = 0. \end{cases} \tag{4.2}$$

For $\kappa_i \geq 0$, we get from (2.8), (2.9), (2.25) that

$$\begin{aligned} \frac{e^{\lambda_1^i(x_i-x)}-e^{\lambda_2^i(x_i-x)}}{e^{\lambda_1^i h_i}-e^{\lambda_2^i h_i}} &= \frac{w_{i1}'(x_{i-1})}{W_i(x)}w_{i0}(x) = e^{-(x-x_{i-1})b_i}w_{i0}(x), \\ \frac{e^{-\lambda_2^i(x-x_{i-1})}-e^{-\lambda_1^i(x-x_{i-1})}}{e^{-\lambda_2^i h_i}-e^{-\lambda_1^i h_i}} &= -\frac{w_{i0}'(x_i)}{W_i(x)}w_{i1}(x) = e^{(x_i-x)b_i}w_{i1}(x). \end{aligned} \tag{4.3}$$

Hence, for $x \in [x_{i-1}, x_i]$ holds that

$$u_i(x) - u_h(x) = u(x_{i-1})\left[1 - \frac{w_{i1}'(x_{i-1})}{W_i(x)}\right]w_{i0}(x) + u(x_i)\left[1 + \frac{w_{i0}'(x_i)}{W_i(x)}\right]w_{i1}(x)$$

$$+ \int_{x_{i-1}}^{x_i} G_i(x,\xi)\ f(\xi)\ d\xi$$

$$= u(x_{i-1})\left[1 - e^{-(x-x_{i-1})b_i}\right]w_{i0}(x) + u(x_i)\left[1 - e^{(x_i-x)b_i}\right]w_{i1}(x)$$

$$+ \int_{x_{i-1}}^{x_i} G_i(x,\xi)\ f(\xi)\ d\xi$$

$$= u(x_{i-1})b_i e^{-\theta_{i0}(x-x_{i-1})b_i}(x - x_{i-1})w_{i0}(x) + u(x_i)b_i e^{\theta_{i1}(x_i-x)b_i}(x - x_i)w_{i1}(x)$$

$$+ \int_{x_{i-1}}^{x_i} G_i(x,\xi)\ f(\xi)\ d\xi, \tag{4.4}$$

where $0 < \theta_{i0}, \theta_{i1} < 1$ follow from the Taylor series expansion of the coefficients $[1 - e^{-(x-x_{i-1})b_i}]$ and $[1 - e^{(x_i-x)b_i}]$ at x_{i-1} and x_i, respectively.

We remark that $d_i(x) \equiv 0$ if $b_i = 0$ and $f(x)|_{[x_{i-1},x_i]} \equiv 0$.

It is immediate from (4.4) that

$$d_i(x) \leq |u(x_{i-1})||b_i||x - x_{i-1}|e^{-\theta_{i0}(x-x_{i-1})b_i} + |u(x_i)||b_i||x - x_i|e^{\theta_{i1}(x_i-x)b_i}$$

$$+ \int_{x_{i-1}}^{x_i} G_i(x,\xi)\ |f(\xi)|\ d\xi \leq C_i h_i, \tag{4.5}$$

where

$$C_i = (\Theta_{i0}|u(x_{i-1})| + \Theta_{i1}|u(x_i)|)|b_i| + \hat{G}_i \max_{[x_{i-1},x_i]} |f(x)|,$$

$$\Theta_{i0} = \begin{cases} 1, & b_i > 0, \\ e^{|b_i|h_i}, & b_i < 0, \end{cases} \qquad \Theta_{i1} = \begin{cases} e^{|b_i|h_i}, & b_i > 0, \\ 1, & b_i < 0, \end{cases}$$

$$\hat{G}_i = \max G_i(x,\xi), \quad (x,\xi) \in [x_{i-1}, x_i] \times [x_{i-1}, x_i],$$

$$||u(x)||_{C[0,1]} = \max_{x\in[0,1]} |u(x)| \leq C.$$

We have thus proved the following theorem.

Theorem 3 For the ansatz $u_h(x)$ of the exact discretization (3.2) of Problem (2.3) there exists a constant $\gamma > 0$ such that

$$||u - u_h||_{C[0,1]} = \max_{x\in[0,1]} |u(x) - u_h(x)| \leq \gamma \max_{1\leq i\leq n+1} h_i.$$

5. Stability

The next task is to proof that our exact discretization is stable for essentially arbitrary grids ω. In a fist step we change the system of linear equations $Ay = r$, see (2.20), such that all of the components r_i of the right-hand side vector r are of order $O(1)$. For this aim we multiply the i-th equation of $Ay = r$ by the reciprocial of $d_i > 0$, where

$$d_i = \int_0^1 \phi_i(\xi)\ d\xi = \int_{x_{i-1}}^{x_{i+1}} \phi_i(\xi)\ d\xi > 0. \tag{5.1}$$

Let D be the following diagonal matrix with all of its diagonal entries positive

$$D = diag(d_1, d_2, \ldots, d_n), \tag{5.2}$$

then, for stability properties consider

$$D^{-1}Ay = D^{-1}r. \tag{5.3}$$

To show that the row sum norm of the inverses $(D^{-1}A)^{-1} > 0$ are uniformly bounded for essentially arbitrary grids ω, we shall use the following Theorem.

Theorem (see [1], [13])

Let B be a monotone matrix (det $B \neq 0,\ B^{-1} \geq 0$ entrywise) and let $v > 0$ with $Bv > 0$. Then

$$||B^{-1}||_\infty \leq \frac{||v||_\infty}{\min\limits_{1 \leq i \leq n} (Bv)_i}. \tag{5.4}$$

Letting now B be the M-matrix

$$B = D^{-1}A, \tag{5.5}$$

To apply the estimate (5.4), we shall construct solutions v of (5.3) such that

$$Bv = D^{-1}Av = (1, \ldots, 1)^T.$$

Theorem 4 Let $u(x) > 0,\ x \in (0,1)$ be the nonnegative solution of Problem (2.3) for $f(x) \equiv 1,\ u_0 = u_1 = 0$. Putting

$$v = (u(x_1), u(x_2), \ldots, u(x_n))^T > 0, \tag{5.6}$$

then there holds

$$||(D^{-1}A)^{-1}||_\infty \leq ||v||_\infty \leq ||u(x)||_{C[0,1]} = \max_{x\in[0,1]} u(x), \qquad (5.7)$$

for essentially arbitrary grids ω.

Proof. The two-point boundary value problem

$$\text{find } u \in V: \qquad a_\omega(u,v) = (1,v), \quad \forall v \in V, \qquad (5.8)$$

has a unique weak solution $u(x)$ with $u(x) > 0$ for $x \in (0,1)$ by the maximum principle, see [3], [7].

Thus, with $f(x) \equiv 1$, $u_0 = u_1 = 0$, we get from (2.24)

$$D^{-1}r = (1,\ldots,1)^T.$$

Hence, for an arbitrary grid ω, our exact discretization implies now that

$$D^{-1}Av = ((1,\ldots,1)^T \qquad \text{for} \quad v = (u(x_1), u(x_2), \ldots, u(x_n))^T > 0,$$

which proves the theorem. □

Remarks

The bound of $||(D^{-1}A)^{-1}||_\infty$, given by (5.7), simplifies for constant coefficients $b_\omega(x), c_\omega(x)$ as follows.

a) $b_\omega(x) \equiv 0,\ c_\omega(x) \equiv 0,\ x \in [0,1]:$

$-u'' = 1, \quad u(0) = u(1) = 0,$ is equivalent to (5.8),

solution: $u(x) = \frac{1}{2}x(1-x)$,

$d_i = \frac{h_i + h_{i+1}}{2} > 0, \quad i = 1,\ldots,n,$

$D^{-1}A = tridiag\left(-\frac{2}{h_i(h_i+h_{i+1})}, \frac{2}{h_ih_{i+1}}, -\frac{2}{h_{i+1}(h_i+h_{i+1})}\right)_{n\times n},$

$$||(D^{-1}A)^{-1}||_\infty \leq \max_{x\in[0,1]} \frac{1}{2}x(1-x) = \frac{1}{8}, \quad \forall\, \omega.$$

b) $b_\omega(x) \equiv 0,\ c_\omega(x) \equiv c > 0,\ x \in [0,1]$:

$$-u'' + cu = 1, \quad u(0) = u(1) = 0, \quad \text{is equivalent to (5.8)},$$

$$\text{solution:} \quad u(x) = \frac{2\sinh(\frac{\sqrt{c}}{2}(1-x))\sinh(\frac{\sqrt{c}}{2}x)}{c\ \cosh(\frac{\sqrt{c}}{2})},$$

$$d_i = \frac{\sinh(\frac{\sqrt{c}}{2}(h_i+h_{i+1}))}{\sqrt{c}\cosh(\frac{\sqrt{c}}{2}h_i)\cosh(\frac{\sqrt{c}}{2}h_{i+1})} > 0, \quad i = 1, \ldots, n,$$

$$D^{-1}A = tridiag\,(a_{i,i-1}, a_{ii}, a_{i,i+1})_{n\times n}\ ,$$

$$a_{i,i-1} = -\frac{c\ \cosh(\frac{\sqrt{c}}{2}h_{i+1})}{2\sinh(\frac{\sqrt{c}}{2}h_i)\sinh(\frac{\sqrt{c}}{2}(h_i+h_{i+1}))},$$

$$a_{ii} = \frac{c\ \cosh(\frac{\sqrt{c}}{2}(h_i+h_{i+1}))}{2\sinh(\frac{\sqrt{c}}{2}h_i)\sinh(\frac{\sqrt{c}}{2}h_{i+1})},$$

$$a_{i,i+1} = -\frac{c\ \cosh(\frac{\sqrt{c}}{2}h_i)}{2\sinh(\frac{\sqrt{c}}{2}h_{i+1})\sinh(\frac{\sqrt{c}}{2}(h_i+h_{i+1}))},$$

$$||(D^{-1}A)^{-1}||_\infty \leq \max_{x\in[0,1]} \frac{2\sinh(\frac{\sqrt{c}}{2}(1-x))\sinh(\frac{\sqrt{c}}{2}x)}{c\ \cosh(\frac{\sqrt{c}}{2})} \leq \frac{2\sinh^2(\frac{\sqrt{c}}{4})}{c\ \cosh(\frac{\sqrt{c}}{2})} \leq \frac{1}{8}, \quad \forall\ \omega.$$

$c \ \rightarrow\ +0$ implies case a).

c) $b_\omega(x) \equiv b,\ c_\omega(x) \equiv c > 0,\ x \in [0,1]$:

$\max\{|b|, c\} > 0$ implies $\lambda_2 = \frac{b-\sqrt{b^2+4c}}{2} < \lambda_1 = \frac{b+\sqrt{b^2+4c}}{2}$,

$-u'' + b\ u' + c\ u = 1, \quad u(0) = u(1) = 0,$ is equivalent to (5.8),

solution: $u(x) > 0,\ x \in (0,1), \quad \max_{x\in[0,1]} u(x) \leq \dfrac{2\sinh^2(\frac{\sqrt{c}}{4})}{c\ \cosh(\frac{\sqrt{c}}{2})} \leq \dfrac{1}{8},\ \forall b \in \mathbb{R}$,

$d_i > 0, \quad i = 1, \ldots, n,$

$p_i = e^{\lambda_1 h_i} - e^{\lambda_2 h_i} > 0, \quad q_i = \frac{p_i}{e^{bh_i}} > 0, \quad \rho_i = \frac{\lambda_1-\lambda_2}{2(\cosh((\lambda_1-\lambda_2)h_i)-1)} > 0,$

$D^{-1}A = tridiag\left(-\frac{\rho_i p_i}{d_i}, \frac{\lambda_1-\lambda_2}{2d_i}\left[\coth(\frac{\lambda_1-\lambda_2}{2}h_i) + \coth(\frac{\lambda_1-\lambda_2}{2}h_{i+1})\right], -\frac{\rho_{i+1}q_{i+1}}{d_i}\right)_{n\times n},$

$$||(D^{-1}A)^{-1}||_\infty \leq \max_{x\in[0,1]} u(x) \leq \frac{2\sinh^2(\frac{\sqrt{c}}{4})}{c\ \cosh(\frac{\sqrt{c}}{2})} \leq \frac{1}{8}, \quad \forall\omega.$$

$c > 0$, and $b \rightarrow 0$ implies case b).

6. Final remarks

The presented exact discretization methods splits Problem (2.3) into $n+1$ separat boundary value problems with constant coefficients as follows

$$\begin{array}{l} L_i v = -v'' + b_i\ v' + c_i\ v = f(x), \quad x \in (x_{i-1}, x_i), \\ v(x_{i-1}) = u(x_{i-1}), \quad v(x_i) = u(x_i), \\ i = 1, \ldots, n+1, \end{array} \tag{6.1}$$

where $u(x_{i-1}) = y_{i-1}, u(x_i) = y_i$. We can solve each of the boundary problems (6.1) by exact discretizations indepentend of each other to get more information on the behaviour of $u_i(x)$.

Appendix 1

In the following figures we shall illustrate some typical behaviours of weak solutions for special constellations of the coefficients $b_\omega(x), c_\omega(x)$. In Examples 2 and 3 we focus attention on interior layers around grid points where $b_\omega(x)$ changes sign from plus to minus for growing values of x. For simplicity, we assume in each of the examples an uniform grid ω.

Data of the Examples

In all of the Examples we have chosen $u_0 = 1, \quad u_1 = 2, \quad f(x) \equiv 0,\ h = \frac{1}{10}$.

Example 1:

first: $b_\omega(x) \equiv 0,$ $\qquad c_\omega(x) = \begin{cases} c_4 = 50, \\ c_i = 0, \quad i \neq 4, \end{cases}$

second: $b_\omega(x) \equiv 0,$ $\qquad c_\omega(x) = \begin{cases} c_4 = 50, \\ c_6 = 200, \\ c_i = 0, \quad i \neq 4,6 \end{cases}$

Example 2:

first: $b_\omega(x) = \begin{cases} b_4 = -50, \\ b_5 = 50, \\ b_i = 0, \quad i \neq 4,5, \end{cases}$ $\qquad c_\omega(x) \equiv 0,$

second: $b_\omega(x) = \begin{cases} b_4 = 50, \\ b_5 = -50, \\ b_i = 0, \quad i \neq 4,5, \end{cases}$ $\qquad c_\omega(x) \equiv 0,$

Example 3:

first: $b_\omega(x):\ b_i = 100(-1)^i,$ $\qquad c_\omega(x) \equiv 0,$

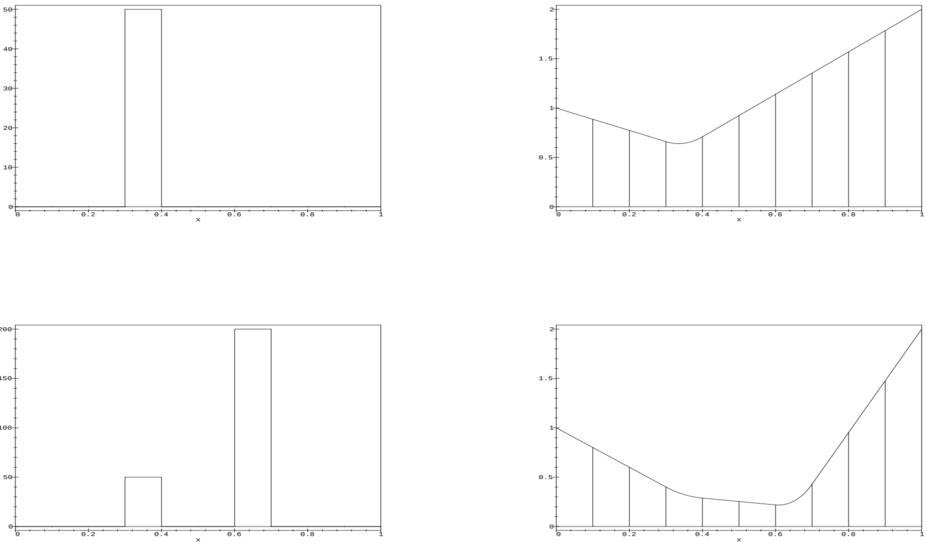
second: $b_\omega(x):\ b_i = 100(-1)^i,$ $\qquad c_\omega(x) \equiv 1/5,$

third: $b_\omega(x):\ b_i = 100(-1)^i,$ $\qquad c_\omega(x) = \begin{cases} c_4 = 50, \\ c_i = 0, \quad i \neq 4. \end{cases}$

We illustrate the babaviour of the weak solutions in the following figures.

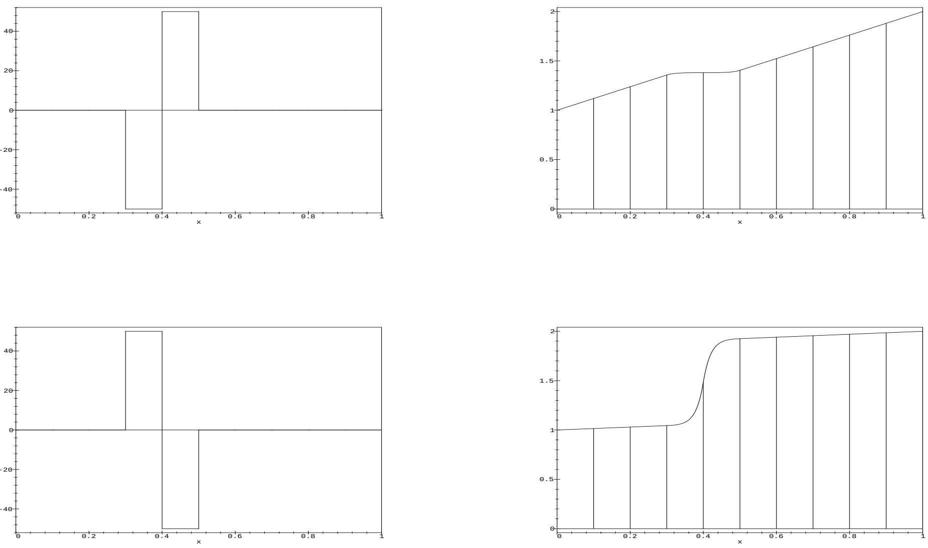
Example 1: $b_\omega(x) \equiv 0$

$c_\omega(x)$ $u(x)$



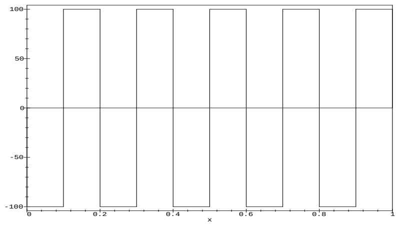
Example 2: $c_\omega(x) \equiv 0$

$b_\omega(x)$ $u(x)$

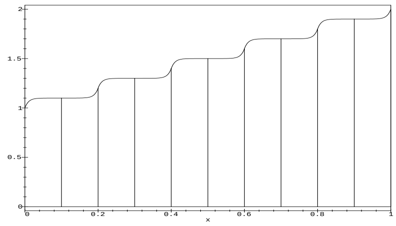


Example 3: $\quad b_\omega(x) \quad$ with $\quad b_i = 100(-1)^i, \quad i = 1, \ldots, 10,$

$c_\omega(x) \equiv 0,$
$b_\omega(x): \; b_i = 100(-1)^i,$



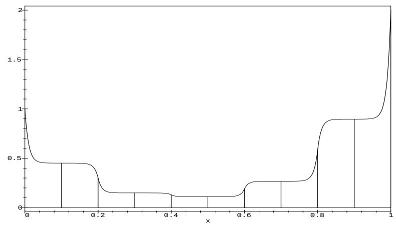
$u(x)$



$c_\omega(x) \equiv \frac{1}{5},$

$b_\omega(x): \; b_i = 100(-1)^i,$

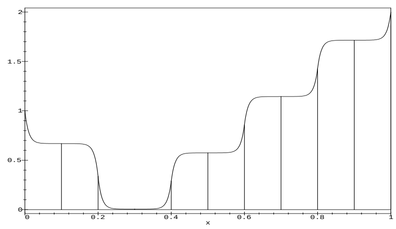
$u(x)$



$$c_\omega(x) = \begin{cases} c_4 = 50, & \\ c_i = 0, & i \neq 4, \end{cases}$$

$b_\omega(x): \; b_i = 100(-1)^i,$

$u(x)$



Example 2 shows that the weak solution $u(x)$ exhibits a plateau at the grid point x_i if $b_i < 0$, $b_{i+1} > 0$ and $|b_i| = b_{i+1} \gg 1$.

Conversely, the weak solution $u(x)$ has an interior layer around x_i if $b_i > 0$, $b_{i+1} < 0$ with $b_i = |b_{i+1}| \gg 1$.

In the first situation of Example 3 the just described situations alter from one grid point to the next. The weak solution shows this typical behaviour also for certain $c_\omega \not\equiv 0$. An illustration is given in the last two situations of Example 3.

Appendix 2

Proof of Theorem 1 in Section 3

From (2.25) we see that $\kappa_i > 0$ implies $p_i > 0,\ q_i > 0,\ \rho_i > 0$. Furthermore, define the auxiliary variable

$$\gamma_i = \frac{e^{(\lambda_1^i+\lambda_2^i)h_i}}{p_i^2} = \frac{e^{b_i h_i}}{p_i^2} > 0. \tag{3.3}$$

Consider first

$$a_\omega(\phi_{i-1}, \phi_i) = \int_{x_{i-1}}^{x_i} \left(\phi'_{i-1}\phi'_i + b_i\phi'_{i-1}\phi_i + c_i\phi_{i-1}\phi_i\right) d\xi. \tag{3.4}$$

For $\kappa_i = 0$ ($\Leftrightarrow b_i = c_i = 0$) we immediately get $a_\omega(\phi_{i-1}, \phi_i) = -\frac{1}{h_i}$. On the other hand, if $\kappa_i > 0$ after several steps of elementary integrations we find

$$\int_{x_{i-1}}^{x_i} \phi'_{i-1}\phi'_i \, d\xi = \frac{\gamma_i(\lambda_1^i - \lambda_2^i)}{\lambda_1^i + \lambda_2^i}\left(\lambda_2^i \sinh(\lambda_2^i h_i) - \lambda_1^i \sinh(\lambda_1^i h_i)\right),$$

$$\int_{x_{i-1}}^{x_i} \phi'_{i-1}\phi_i \, d\xi = \gamma_i \left(\sinh(\lambda_1^i h_i) + \sinh(\lambda_2^i h_i) + \frac{\lambda_2^i e^{-\lambda_1^i h_i} + \lambda_1^i e^{-\lambda_2^i h_i} - \lambda_2^i e^{\lambda_2^i h_i} - \lambda_1^i e^{\lambda_1^i h_i}}{\lambda_1^i + \lambda_2^i}\right),$$

$$\int_{x_{i-1}}^{x_i} \phi_{i-1}\phi_i \, d\xi = \frac{\gamma_i(\lambda_1^i - \lambda_2^i)}{\lambda_1^i\lambda_2^i(\lambda_1^i + \lambda_2^i)}\left(\lambda_2^i \sinh(\lambda_1^i h_i) - \lambda_1^i \sinh(\lambda_2^i h_i)\right),$$

which simplifies to

$$a_\omega(\phi_{i-1}, \phi_i) = -\rho_i \, p_i < 0. \tag{3.5}$$

Hence, $a_\omega(\phi_{i-1}, \phi_i) = a_{i,i-1}$ for $\forall \kappa_i \geq 0$, see (2.26), (2.27).

It will be useful to split $a_\omega(\phi_i, \phi_i)$ into the sum as follows

$$a_\omega(\phi_i, \phi_i) = a_\omega^-(\phi_i, \phi_i) + a_\omega^+(\phi_i, \phi_i), \tag{3.6}$$

where

$$\begin{aligned} a_\omega^-(\phi_i, \phi_i) &= \int_{x_{i-1}}^{x_i} \left(\phi'_i\phi'_i, + b_i\phi'_i\phi_i + c_i\phi_i\phi_i\right) d\xi, \\ a_\omega^+(\phi_i, \phi_i) &= \int_{x_i}^{x_{i+1}} \left(\phi'_i\phi'_i + b_{i+1}\phi'_i\phi_i + c_{i+1}\phi_i\phi_i\right) d\xi. \end{aligned} \tag{3.7}$$

If $\kappa_i = 0$ then we get $a_\omega^-(\phi_i, \phi_i) = \frac{1}{h_i}$. Otherwise, for $\kappa_i > 0$ we derive from

$$\int\limits_{x_{i-1}}^{x_i} \phi_i' \phi_i' \, d\xi = \gamma_i \left(\frac{2\lambda_1^i \lambda_2^i}{\lambda_1^i + \lambda_2^i} + \frac{(\lambda_1^i - \lambda_2^i)^2 e^{(\lambda_1^i + \lambda_2^i) h_i}}{2(\lambda_1^i + \lambda_2^i)} - \frac{\lambda_2^i e^{(\lambda_1^i - \lambda_2^i) h_i}}{2} - \frac{\lambda_1^i e^{-(\lambda_1^i - \lambda_2^i) h_i}}{2} \right),$$

$$\int\limits_{x_{i-1}}^{x_i} \phi_i' \phi_i \, d\xi = \gamma_i (\cosh((\lambda_1^i - \lambda_2^i) h_i) - 1),$$

$$\int\limits_{x_{i-1}}^{x_i} \phi_i \phi_i \, d\xi = \gamma_i \left(\frac{2}{\lambda_1^i + \lambda_2^i} + \frac{(\lambda_1^i - \lambda_2^i)^2 e^{(\lambda_1^i + \lambda_2^i) h_i}}{2\lambda_1^i \lambda_2^i (\lambda_1^i + \lambda_2^i)} - \frac{e^{(\lambda_1^i - \lambda_2^i) h_i}}{2\lambda_2^i} - \frac{e^{-(\lambda_1^i - \lambda_2^i) h_i}}{2\lambda_1^i} \right),$$

that

$$a_\omega^-(\phi_i, \phi_i) = \tfrac{\lambda_1^i - \lambda_2^i}{2} \coth\left(\tfrac{(\lambda_1^i - \lambda_2^i) h_i}{2} \right) + \tfrac{b_i}{2} > 0. \tag{3.8}$$

Thus $a_\omega^-(\phi_i, \phi_i) = a_{ii}^-$ for $\forall \kappa_i \geq 0$, see (2.26), (2.27).

Next, if $\kappa_{i+1} = 0$ ($\Leftrightarrow b_{i+1} = c_{i+1} = 0$) we find $a_\omega^+(\phi_i, \phi_i) = \frac{1}{h_{i+1}}$ and for $\kappa_{i+1} > 0$ we derive

$$a_\omega^+(\phi_i, \phi_i) = \tfrac{\lambda_1^{i+1} - \lambda_2^{i+1}}{2} \coth\left(\tfrac{(\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1}}{2} \right) - \tfrac{b_{i+1}}{2} > 0, \tag{3.9}$$

because of

$$\begin{aligned}
\int\limits_{x_i}^{x_{i+1}} \phi_i' \phi_i' \, d\xi = \; & \gamma_{i+1} \Big(-\tfrac{2\lambda_1^{i+1} \lambda_2^{i+1}}{\lambda_1^{i+1} + \lambda_2^{i+1}} - \tfrac{(\lambda_1^{i+1} - \lambda_2^{i+1})^2 e^{-(\lambda_1^{i+1} + \lambda_2^{i+1}) h_{i+1}}}{2(\lambda_1^{i+1} + \lambda_2^{i+1})} \\
& + \tfrac{\lambda_1^{i+1} e^{(\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1}}}{2} + \tfrac{\lambda_2^{i+1} e^{-(\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1}}}{2} \Big), \\
\int\limits_{x_i}^{x_{i+1}} \phi_i' \phi_i \, d\xi = \; & \gamma_{i+1} (1 - \cosh((\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1})), \\
\int\limits_{x_i}^{x_{i+1}} \phi_i \phi_i \, d\xi = \; & \gamma_{i+1} \Big(-\tfrac{2}{\lambda_1^{i+1} + \lambda_2^{i+1}} - \tfrac{(\lambda_1^{i+1} - \lambda_2^{i+1})^2 e^{-(\lambda_1^{i+1} + \lambda_2^{i+1}) h_{i+1}}}{2\lambda_1^{i+1} \lambda_2^{i+1} (\lambda_1^{i+1} + \lambda_2^{i+1})} \\
& + \tfrac{e^{(\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1}}}{2\lambda_1^{i+1}} + \tfrac{e^{-(\lambda_1^{i+1} - \lambda_2^{i+1}) h_{i+1}}}{2\lambda_2^{i+1}} \Big).
\end{aligned}$$

This shows that $a_\omega^+(\phi_i, \phi_i) = a_{ii}^+$ for $\forall \kappa_i \geq 0$, see (2.28), (2.29).

As a last one consider

$$a_\omega(\phi_{i+1}, \phi_i) = \int\limits_{x_i}^{x_{i+1}} (\phi_{i+1}' \phi_i' + b_{i+1} \phi_{i+1}' \phi_i + c_{i+1} \phi_{i+1} \phi_i) \, d\xi. \tag{3.10}$$

If $\kappa_{i+1} = 0$ then $a_\omega(\phi_{i+1}, \phi_i) = -\frac{1}{h_{i+1}}$ and if $\kappa_{i+1} > 0$ we find by

$$\int\limits_{x_i}^{x_{i+1}} \phi'_{i+1}\phi'_i \, d\xi = \frac{\gamma_{i+1}(\lambda_1^{i+1}-\lambda_2^{i+1})}{\lambda_1^{i+1}+\lambda_2^{i+1}} \left(\lambda_2^{i+1}\sinh(\lambda_2^{i+1}h_{i+1}) - \lambda_1^{i+1}\sinh(\lambda_1^{i+1}h_{i+1})\right),$$

$$\int\limits_{x_i}^{x_{i+1}} \phi'_{i+1}\phi_i \, d\xi = \gamma_{i+1}(\sinh(\lambda_1^{i+1}h_{i+1}) + \sinh(\lambda_2^{i+1}h_{i+1})$$

$$+ \frac{\lambda_1^{i+1}e^{-\lambda_1^{i+1}h_{i+1}} + \lambda_2^{i+1}e^{-\lambda_2^{i+1}h_{i+1}} - \lambda_1^{i+1}e^{\lambda_2^{i+1}h_{i+1}} - \lambda_2^{i+1}e^{\lambda_1^{i+1}h_{i+1}}}{\lambda_1^{i+1}+\lambda_2^{i+1}}),$$

$$\int\limits_{x_i}^{x_{i+1}} \phi_{i+1}\phi_i \, d\xi = \frac{\gamma_{i+1}(\lambda_1^{i+1}-\lambda_2^{i+1})}{\lambda_1^{i+1}\lambda_2^{i+1}(\lambda_1^{i+1}+\lambda_2^{i+1})} \left(\lambda_2^{i+1}\sinh(\lambda_1^{i+1}h_{i+1}) - \lambda_1^{i+1}\sinh(\lambda_2^{i+1}h_{i+1})\right),$$

that

$$a_\omega(\phi_{i+1}, \phi_i) = -\rho_{i+1}\; q_{i+1} < 0.$$

Thus, $a_\omega(\phi_{i+1}, \phi_i) = a_{i,i+1}$ for $\forall \kappa_{i+1} \geq 0$, see (2.28), (2.29) and the proof is complete. □

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Author: Dr. Günther Windisch,
Fakultät für Mathematik, TU Chemnitz-Zwickau,
D-09107 Chemnitz,
e-mail: windisch@mathematik.tu-chemnitz.de.

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