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Adaptive Wavelet Galerkin BEM

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Abstract

The wavelet Galerkin scheme for the fast solution of boundary integral equations produces approximate solutions within discretization error accuracy offered by the underlying Galerkin method at a computational expense that stays proportional to the number of unknowns. In this paper we present an adaptive version of the scheme which preserves the super-convergence of the Galerkin method.

Key Words: Boundary integral equations; biorthogonal wavelet bases; Galerkin scheme; adaptive methods.

1 Introduction

As shown by Dahmen, Harbrecht and Schneider [3, 8, 9], the fully discrete wavelet Galerkin scheme for boundary integral equations scales linearly with the number of unknowns without compromising the accuracy of the underlying Galerkin scheme.

In view of nonsmooth geometries or singularities of the solution, a modern method should put into practice adaptivity. The most adaptive methods realize the adaptivity in the energy norm such that the super-convergence of the underlying Galerkin method is generally not realized. Wavelet bases offer the possibility to measure a wide range of Sobolev norms. In particular, adaptive schemes working with the optimal negative norm can be performed. Based on our actual approach we present these developments and provide numerical results which demonstrate the potential of our algorithm.

2 Problem formulation and preliminaries

We are interested in solving a boundary integral equation

$$Au = f \quad \text{on } \Gamma, \tag{1}$$

where $\Gamma \in \mathbb{R}^3$ is a boundary manifold and the kernel of the boundary integral operator

$$A : H^q(\Gamma) \to H^{-q}(\Gamma), \qquad u \mapsto Au(x) = \int_\Gamma k(x,y)u(y)d\Gamma_y,$$

satisfies estimates of the type

$$|\partial_x^\alpha \partial_y^\beta k(x,y)| \lesssim \|x-y\|^{-(2+2q+|\alpha|+|\beta|)}, \tag{2}$$

We partition the manifold Γ into a finite number of *patches*

$$\Gamma = \bigcup_{i=1}^M \Gamma_i, \qquad \Gamma_i = \gamma_i(\square), \qquad i = 1, 2, \ldots, M,$$

where each $\gamma_i : \square \to \Gamma_i$ defines a diffeomorphism of $\square := [0,1]^n$ onto Γ_i. The intersection $\Gamma_i \cap \Gamma_{i'}$, $i \neq i'$, of the patches Γ_i and $\Gamma_{i'}$ is supposed to be either empty or a common edge or vertex.

A mesh of level j on Γ is induced by dyadic subdivisions of depth j of the unit square into 4^j squares $C_{j,k} = [2^{-j}k_1, 2^{-j}(k_1+1)] \times [2^{-j}k_2, 2^{-j}(k_2+1)]$ with $0 \leq k_1, k_2 < 2^j$. This yields $4^j M$ *elements* $\Gamma_{i,j,k} := \gamma_i(C_{j,k}) \subseteq \Gamma_i$, $i = 1, 2, \ldots, M$, cf. Figure 3.

The nested trial spaces $V_j \subseteq V_{j+1}$ that we employ in the Galerkin scheme are spanned the traditional piecewise constant or bilinear ansatz functions defined on the given partition, where the latter ones are supposed to be patchwise continuous, but along the interfaces of the patches double nodes or continuity might be considered.

Instead of these *single-scale bases* we discretize Eq (1) by *biorthogonal spline wavelets* $\{\psi_{j,k}\}$ constructed in several papers [1, 2, 4, 8]. Notice that j indicates the level whereas k describes the locality with respect to the scale j. Besides compacts supports, i.e. $\operatorname{diam}\operatorname{supp}\psi_{j,k} \sim 2^{-j}$, such wavelets provide vanishing moments

$$|\langle v, \psi_{j,k}\rangle| \lesssim 2^{-j(\widetilde{d}+1)} |v|_{W^{\widetilde{d},\infty}(\operatorname{supp}\psi_{j,k})}.$$

A plot of such biorthogonal spline wavelets can be found in Figure 1. If the number of vanishing moments is sufficiently large, that is $d < \widetilde{d} + 2q$, the associated system matrix becomes quasi-sparse due to the decay property of the kernels, cf. Eq (2). Neglecting the nonrelevant matrix coefficients is called matrix compression. The compressed linear system of equations can be computed and solved within linear complexity, see Dahmen, Harbrecht and Schneider [3, 8, 9] for the details.

3 Adaptivity

This section is concerned with finding a sequence of spaces

$$V_0 = \widehat{V}_0 \subseteq \widehat{V}_1 \subseteq \widehat{V}_2 \subseteq \cdots \subseteq \widehat{V}_J \subseteq V_J,$$

where $\widehat{V}_j \subseteq V_j$, such that the Galerkin solution with respect to $\widehat{V}_j$ provides the same accuracy as the Galerkin solution with respect to V_j.

To define a wavelet basis well on $\widehat{V}_j$, we have to ensure that the support of a small wavelet does not intersect large elements. We call a mesh *1-graded*, if the levels of adjacent elements differ at most by one. Likewise, a graded mesh is *2-graded*, if the levels of all neighbours of each element differ at most by one, and so on, cf. Figure 2. An exact definition of m-gradedness can be found in Dahmen et al. [5]. The gradedness ensures that we find a tree structured (with respect to the supports of the associated wavelets) index set Λ_j such that $\widehat{V}_j = \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda_j\}$ with $|\Lambda_j| = \dim \widehat{V}_j$. Moreover, completing Λ_j by the sons of all leaves, we obtain the index set $\Lambda_{j,\boxplus}$, which generates the trial space $\widehat{V}_{j,\boxplus} = \operatorname{span}\{\psi_\lambda : \lambda \in \Lambda_{j,\boxplus}\}$ that arises from uniform subdivision of $\widehat{V}_j$. We mention that the mesh has to be patchwise 1-graded in the case of the piecewise constant wavelets presented in Figure 1. The piecewise bilinear wavelets require 2-gradedness, which has to be extended to global 2-gradedness if we consider them globally continuous.

The traditional formulation of adaptive algorithms is based on the energy norm. To our experience this restricts in general the super-convergence of the Galerkin scheme. The highest order of convergence of the boundary element method is achieved with respect to the norm in $H^{2q-d}(\Gamma)$, see Wendland [10]. Since the number of vanishing moments is chosen such that $d < \widetilde{d} + 2q$, we can estimate this norm by the estimates

$$\Big\| \sum_{j,k} v_{j,k}\psi_{j,k} \Big\|_t^2 \lesssim \sum_{j,k} 2^{2jt}|v_{j,k}|^2 \lesssim \begin{cases} \big\| \sum_{j,k} v_{j,k}\psi_{j,k} \big\|_{t'}^2, & t \in (-\widetilde{d}, -\widetilde{\gamma}],\ t < t', \\ \big\| \sum_{j,k} v_{j,k}\psi_{j,k} \big\|_t^2, & t \in (-\widetilde{\gamma}, \gamma), \end{cases}$$

where γ and $\widetilde{\gamma}$ denote the regularity of the primal and dual wavelets, respectively.

Assumption. *Let $\widehat{V}_j$ denote an arbitrary m-graded trial space and let $\widehat{V}_{j,\boxplus}$ be the trial space that arises from uniform subdivision of $\widehat{V}_j$. For a fixed $t \in [2q-d, \gamma)$ we assume that that there exists a constant $\theta < 1$ such that the solutions u_j with respect to $\widehat{V}_j$ and $u_{j,\boxplus}$ with respect to $\widehat{V}_{j,\boxplus}$ satisfy*

$$\|u - u_{j,\boxplus}\|_t \le \theta \|u - u_j\|_t. \tag{3}$$

Theorem. *Assume that (3) holds. If the Galerkin solution u_{j+1} with respect to the trial space $\widehat{V}_j \subseteq \widehat{V}_{j+1} \subseteq \widehat{V}_{j,\boxplus}$ satisfies*

$$\|u_{j,\boxplus} - u_{j+1}\|_t \le \epsilon \|u_{j,\boxplus} - u_j\|_t, \tag{4}$$

then it holds

$$\|u - u_{j+1}\|_t \le [\theta(1+\epsilon) + \epsilon] \|u - u_j\|_t,$$

i.e. the solution u_{j+1} is more accurate than u_j if $\epsilon < (1-\theta)/(1+\theta)$.

Proof. The proof follows immediately from Eqs (3), (4) and the triangle inequality.

Up to now we can compute the Galerkin solutions u_j and $u_{j,\boxplus}$. Our problem reads now: find the smallest index set $\Lambda_j \subseteq \Lambda_{j+1} \subseteq \Lambda_{j,\boxplus}$, such that the Galerkin solution u_{j+1} with respect to $\psi_{\Lambda_{j+1}}$ satisfies Eq (4). At present we choose the canonical strategy and compute elementwise error portions by bunching the wavelets which correspond to the subdivision of an element of $\widehat{V}_j$. This procedure is simple to implement and corresponds completely to that when using hierarchical error estimators, see Mund et al. [6, 7]. After sorting these error portions by their modulus, we increase the index set Λ_j successively by activating the wavelets corresponding to the largest error portions until Eq (4) is satisfied. Possibly the so constructed index set Λ_{j+1} has to be extended to ensure the m-gradedness of the new trial space $\widehat{V}_{j+1}$.

We are now in the position to formulate our adaptive algorithm, which is based on a nested iteration.

initialization: $\widehat{V}_0 := V_0$
for $j := 1$ **to** $J-1$ **do begin**
  compute the compressed system matrix for $\widehat{V}_{j-1,\boxplus}$
  compute the solutions $u_j - 1$ **and** $u_{j-1,\boxplus}$
  determine $\widehat{V}_j$ **such that** (4) **holds**
end
compute the compressed system matrix for $\widehat{V}_{J-1,\boxplus}$
compute the final solution $u_J := u_{J-1,\boxplus}$

4 Numerical results

For a given function $f \in H^{1/2}(\Gamma)$ we consider an interior Dirichlet problem, i.e., we seek $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \Gamma. \tag{5}$$

We consider the crankshaft presented in Figure 3 as domain Ω. Choosing the harmonical function $u(x) = \langle a, x - b\rangle \|x - b\|^{-3}$, where $a = (4, 2, 1)$ and $b = (0, 0, 1.5) \notin \Omega$, and setting $f = u|_\Gamma$, the Dirichlet problem has the unique solution u.

We use the indirect formulation involving the *single layer operator*

$$V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \qquad (V\rho)(x) := \frac{1}{4\pi}\int_\Gamma \frac{1}{\|x-y\|}\rho(y)d\Gamma_y,$$

which gives a Fredholm integral equation of the first kind

$$V\rho = f \quad \text{on } \Gamma.$$

We discretize this boundary integral equation by piecewise constant wavelets with three vanishing moments as well as patchwise continuous piecewise bilinear wavelets with four vanishing moments. The approximate potentials $\mathbf{u}_J = [(V\rho)(x_i)]$ are calculated in many points x_i distributed inside the crankshaft. The exact potential is denoted by $\mathbf{u} = [u(x_i)]$. The computations are performed by a standard personal computer with 1 Gigabyte main memory.

First, in Table 1 we compare the adaptive scheme with the nonadaptive one in the case of the piecewise constants wavelets. The setting for the adaptive scheme is $t = -2$ and $\epsilon = 1/3$. We tabulate the maximum norm of the absolute error of u_J. The optimal order of convergence is cubic, but it cannot be expected due to concave angles between the patches. Notice that level 6 is no more computable with the nonadaptive scheme. Both schemes provide a nearly identical accuracy, but the adaptive one does it with essentially less unknowns and, hence, less memory requirement and cpu-time (measured in seconds). The adaptive mesh on level 4 is presented in Figure 4.

In Table 2 we consider the piecewise bilinear wavelets. The setting for the adaptive scheme is $t = -3$ and $\epsilon = 1/6$. We loose one computable level but increase the accuracy due to the higher order of the ansatz functions. The absolute error of the adaptive scheme is 1.70e-4 on the level 5, which corresponds to a relative error of 3.11e-5 due to $\|\mathbf{u}\|_\infty = 5.488$.

5 Conclusion

In this paper we presented an easily performable adaptive algorithm based on the fully discrete wavelet Galerkin scheme for boundary integral equations. We demonstrated by numerical results that our method solves a given boundary integral equations highly efficient.

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Figures and Tables

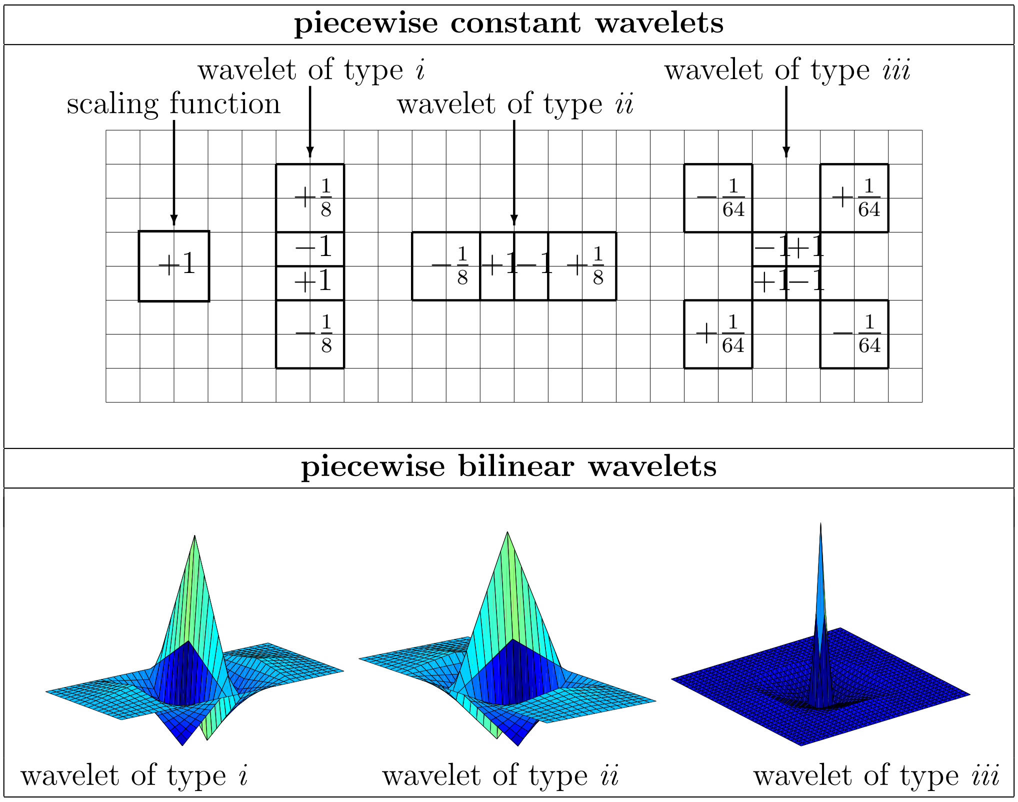

Figure 1: (Interior) piecewise constant/bilinear wavelets with three/four vanishing moments.

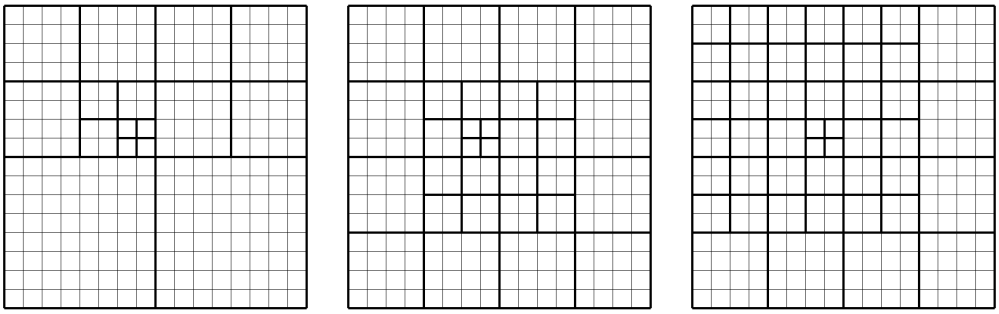

Figure 2: A nongraded mesh (left) and corresponding graded (mid) and 2-graded (right) meshes.

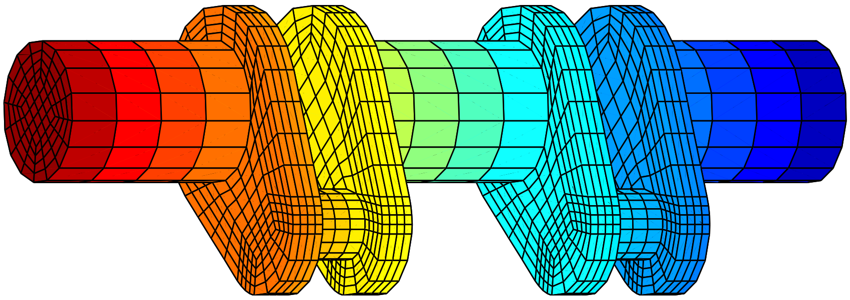

Figure 3: The mesh of a crankshaft parametrized by 172 patches after two subdivision steps.

unknowns		adaptive scheme			nonadaptive scheme	
J	$\dim V_J$	$\dim \widehat{V}_J / \dim V_J$	$\|\mathbf{u}-\mathbf{u}_J\|_\infty$	cpu-time	$\|\mathbf{u}-\mathbf{u}_J\|_\infty$	cpu-time
1	568	100	20.1	3	20.1	2
2	2272	99	1.03	27	1.01	16
3	9088	26	2.45e-1	95	2.40e-1	52
4	36352	7.2	1.82e-2	330	2.80e-2	1539
5	145408	3.0	5.27e-3	1230	5.24e-3	12125
6	581632	1.4	2.10e-3	4244	—	—

Table 1: Numerical results for the crankshaft in the case of piecewise constant wavelets.

unknowns		adaptive scheme			nonadaptive scheme	
J	$\dim V_J$	$\dim \widehat{V}_J / \dim V_J$	$\Vert\mathbf{u}-\mathbf{u}_J\Vert_\infty$	cpu-time	$\Vert\mathbf{u}-\mathbf{u}_J\Vert_\infty$	cpu-time
1	1278	100	2.95	21	2.95	21
2	3550	100	1.33	87	1.33	87
3	11502	33	6.82e-2	838	6.66e-2	1164
4	41038	9.7	3.18e-3	2978	1.26e-3	11573
5	154638	3.5	1.70e-4	12696	—	—

Table 2: Numerical results for the crankshaft in the case of piecewise bilinear wavelets.

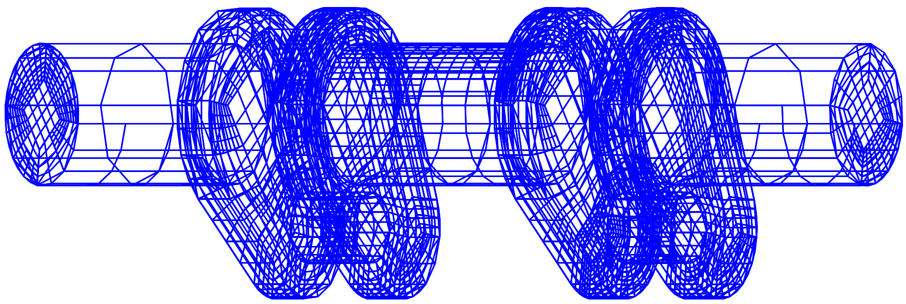

Figure 4: The adaptive mesh of the crankshaft on the level 4.

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