Technische Universität Chemnitz
Sonderforschungsbereich 393
Numerische Simulation auf massiv parallelen Rechnern

G. Gattica H. Harbrecht R. Schneider

Least squares methods for the coupling of FEM and BEM

Preprint SFB393/01-32

This research was partially supported by Conicyt-Chile through the FONDAP Program in Applied Mathematics, by the Ministery of Education of Chile through the MECESUP project UCO9907, and by the Dirección de Investigación of the Universidad de Concepción through the Advanced Research Groups Program.

Preprint-Reihe des Chemnitzer SFB 393

SFB393/01-32 **November 2001**

Contents

Author's addresses:

G. Gattica
Universidad de Concepción
Departamento de Ingeniería Matemática
Casilla 160-C, Concepción, Chile

H. Harbrecht & R. Schneider
TU Chemnitz
Fakultät für Mathematik
D-09107 Chemnitz

`http://www.tu-chemnitz.de/sfb393/`

Abstract

In the present paper we propose least squares formulations for the numerical solution of exterior boundary value problems. The partial differential equation is a first order system in a bounded subdomain, and the unbounded subdomain is treated by means of boundary integral equations. The first order system is derived from a strongly elliptic second order system. The analysis of the present least squares formulations is reduced to the analysis of the Galerkin method for the coupling of finite element and boundary element methods (FEM and BEM) of the second order problem. The least squares approach requires no stability condition. But it requires the computation of negative as well as of half integer Sobolev norms. The arising linear systems can be preconditioned to have condition numbers ~ 1. The present methods benefit strongly from the use of biorthogonal wavelets on the coupling boundary and the computation of corresponding equivalent norms in Sobolev spaces. Our approach leads to a very efficient discretization of the least squares formulations.

Mathematics Subject Classifications (1991): 65J15, 65N30, 65N38, 65R20

1 Introduction

The combined use of finite elements (FEM) and boundary elements (BEM), also called coupling of FEM and BEM, is already known as a very powerful tool to solve a large class of transmission problems in physics and engineering sciences (see, e.g. [13], [22], [27], [31], [37], and the references therein). In addition, the interest in using mixed finite element methods instead of the usual FEM has been increasing during the last few years. Indeed, the combination of mixed finite elements with either boundary integral equations or Dirichlet-to-Neumann mappings has been recently used to solve several interior and exterior boundary value problems appearing in potential theory and elasticity (see, e.g. [2], [8], [20], [23], [25] and [34]).

The reasons for this new interest arise mainly from structural mechanics, where the use of mixed finite element methods allows to compute stresses more accurately than displacements, whereas the utilization of boundary elements or Dirichlet-to-Neumann mappings is more appropriate for linear homogeneous materials in bounded and unbounded domains. In the framework of dual-mixed methods, the recent papers [8] and [34], dealing with an exterior problem from potential theory and the linear elasticity problem, respectively, are the first ones on the subject that consider the $H(\mathrm{div};\Omega)$ spaces in the finite element domain, and the two boundary integral equations approach from [13] and [27] in the boundary element region. Now, the method from [8] and [34] was extended in [20], [25] and [2], where a suitable combination of dual-mixed FEM with either BEM or Dirichlet-to-Neumann mappings, was applied to some nonlinear transmission problems. However, this extension has not been completely successful since the derivation of explicit finite element subspaces satisfying the corresponding discrete inf-sup conditions is still an open problem. As a first attempt to overcome this difficulty, we examined in [3] the use of a primal-mixed finite element method. More recently, we obtained quite satisfactory results, at both the continuous and discrete levels, by applying what we called a dual-dual mixed variational formulation (see [4], [21] and [24]). This latter approach requires an extension of the usual Babuska-Brezzi theory to a special class of nonlinear variational problems with constraints, which was derived with full details in [19].

On the other hand, a possibility that has not been yet fully investigated, is the utilization of least squares methods. As it is well known, this approach avoids the necessity of inf-sup conditions, and hence it becomes attractive to use it jointly with mixed finite element formulations. One of the main methods, introduced in [1], uses the general theory of elliptic boundary value problems of Agmon-Douglis-Nirenberg and reduces the system to the minimization of a least-squares functional that consists of a weighted sum of the residuals occurring in the equations and the boundary conditions. This is a generalization of both the method of Jespersen [32] and the method of Wendland [39]. Another approach, mostly used for second order elliptic problems written as first order systems, introduces a least-squares functional and studies the resulting minimization problem by proving that the hypotheses of the Lax-Milgram lemma are satisfied on appropriate spaces (see, e.g. [9] and [35]). More recently, a least-squares functional involving a discrete inner product related to the inner product in the Sobolev space of order -1, was introduced in [5], and an approach more closely coupled to the Galerkin method was studied by the same authors in [6].

Following the approach of [5], [6], the design of the least squares method requires the use of some negative and half integer Sobolev norms, such as the norms of $H^{-1}(\Omega)$ and $H^{-1/2}(\Gamma)$, which seem to be difficult to compute in practice. However, due to recent results in multilevel preconditioning [7] and multiscale methods or wavelet approximations (see [14], [16], [36]), these norms are computable in suitable finite dimensional subspaces. Moreover, in the framework of multiscale methods or biorthogonal wavelets, these computations are fairly simple and can be carried out within optimal complexity. We like to mention that these approaches gives rise to positive definite system matrices that can be easily preconditioned. In particular, using multilevel methods one can reduce the condition number to $\mathcal{O}(1)$.

Now, we remark that the first main approach for least squares formulations requires the flux to be in $H(\mathrm{div};\Omega)$ and minimizes the equilibrium equation in the $L^2(\Omega)$-norm (see [9]), whereas the present ones are somehow sharper performing the minimization in the $H^{-1}(\Omega)$-norm. For the computation of the latter Sobolev norm in the bounded subdomain Ω, there are yet mainly two possibilities, the use of wavelet bases (see [15], [17]) or alternatively, the utilization of suitable preconditioners (see [5], [6]), which are applicable to standard multigrid finite element discretizations. For the computation of the Sobolev norms along the boundary we recommend wavelet bases. It is worth mentioning that the negative and half integer Sobolev norms can be computed only on finite dimensional test spaces. This means that we must prove also the stability of the discrete formulations, since this is not automatically guaranteed by the continuous formulation. Our present proofs are completely based on the theory for the Galerkin scheme of the second order problem. From these results we conclude the stability of the present methods. Only in the case of the functional $\mathbf{J}_4$ (see Section 3 below) we have to enlarge the test spaces slightly. An important feature of the present approaches is that, for the least squares discretizations of the boundary integral operators we only need the coefficients of the Galerkin matrices of the layer potentials and not of compositions of layer potentials (see Section 7). Finally, it is worth mentioning that in the framework of multiscale methods these matrices are sparse (see, e.g. [36], [38]) and hence, preconditioning becomes a simple task.

Consequently, the purpose of the present work is to examine the use of least squares formulations for the coupling of mixed-FEM and BEM, as applied to linear exterior boundary value problems. This must be considered as the first step toward the future extension to nonlinear exterior transmission problems. The rest of the paper is organized as follows. In Section 2 we describe the exterior second order model problem and apply the boundary integral equation method to reduce it to an equivalent non-local boundary value problem in a bounded annular domain. Then, after setting the flux as a new unknown, the non-local problem is rewritten as a first order system, which yields the underlying equations for the discretization. Various continuous least squares formulations, induced by this first order system, are introduced in Section 3. Although existence and uniqueness for the least squares minimization problems can be easily deduced from the mapping properties of the underlying operators, we provide explicit proofs by using coercivity estimates of the usual variational formulation for the coupling procedure, since the method of these proofs can be used for the validation of the corresponding results for the discrete least squares formulations in Section 5. Next, in Section 4 we define the finite dimensional subspaces. The discrete least squares formulations and the corresponding error analysis are studied in Section 5. Among the various approaches discussed in this paper the one using $\mathbf{J}_3$ (see Section 3 below) seems to be the most canonical one. It avoids completely any restriction concerning $H(\mathrm{div};\Omega)$ spaces, it is easy to implement and it requires no kind of stabilization. In Section 6 we give a brief description of the equivalence of norms based on wavelet bases, and indicate the utilization of these functions for the present least square approach. In addition, we remark how to use the wavelet bases provided by [17] for the treatment of three dimensional problems. In the last section we consider a numerical example and demonstrate how to set up the discrete matrices related to the minimization of $\mathbf{J}_3$.

2 The exterior boundary value problem

Let G be a bounded and simply connected domain in $I\!R^2$ with Lipschitz-continuous boundary ∂G, and let Γ_D and Γ_N be two disjoint subsets of ∂G such that $|\Gamma_D| \neq 0$ and $\partial G = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. In addition, let Ω be the annular domain bounded by ∂G and a second Lipschitz-continuous curve Γ whose interior region contains G. We denote $\Omega_e := I\!R^2 - (\overline{G} \cup \overline{\Omega})$. Then, given $f \in L^2(\Omega)$ and a matrix valued function $\mathbf{a}(\cdot) := (a_{ij}(\cdot))_{2\times 2}$, we consider the exterior boundary value problem: *Find* $u \in H^1_{loc}(I\!R^2 - \overline{G})$ *such that*

$$
\begin{aligned}
u = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad (\mathbf{a}\nabla u)\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_N\,,\\
-\operatorname{div}(\mathbf{a}\nabla u) = f \quad \text{in} \quad \Omega\,,\\
\lim_{\substack{x\to x_0\\ x\in\Omega}} u(x) = \lim_{\substack{x\to x_0\\ x\in\Omega_e}} u(x) \quad \forall\, x_0\in\Gamma\,,\\
\lim_{\substack{x\to x_0\\ x\in\Omega}} \mathbf{a}(x)\nabla u(x)\cdot\mathbf{n}(x_0) = \lim_{\substack{x\to x_0\\ x\in\Omega_e}} \nabla u(x)\cdot\mathbf{n}(x_0) \quad \forall\, x_0\in\Gamma\,,\\
-\Delta u = 0 \quad \text{in} \quad \Omega_e \quad , \quad u(x) = O(1) \quad \text{as} \quad \|x\|\to+\infty\,,
\end{aligned}
\tag{1}
$$

where $\mathbf{n}$ (resp. $\mathbf{n}(x_0)$) denotes the unit outward normal to $\partial\Omega$ (to $x_0 \in \partial\Omega$). Here, we assume that $a_{ij} \in L^\infty(\Omega)$ and that there exists $\alpha > 0$ such that

$$
\alpha\,\|z\|^2 \le z^T\,\mathbf{a}(x)\,z \qquad \forall\, z\in I\!R^2 \quad \text{and for almost all } x\in\Omega\,. \tag{2}
$$

We observe that the fourth and fifth equations of (1) constitute the usual transmission conditions along the interface Γ.

In what follows, we use the boundary integral equation method in the region Ω_e and reduce the problem (1) to a nonlocal boundary value problem on the bounded domain Ω. To this end, we let

$$
E(x,y) := -\frac{1}{2\pi}\log\|x-y\|
$$

be the fundamental solution of the Laplacian, and recall that the Green representation formula in Ω_e becomes

$$
u(x) = \int_\Gamma \left\{ \frac{\partial}{\partial\mathbf{n}(y)} E(x,y)\,u(y) - E(x,y)\,\frac{\partial u}{\partial\mathbf{n}}(y) \right\} ds_y - \lambda \quad \forall\, x\in\Omega_e,
$$

where λ is an unknown constant.

Then, according to the well known jump conditions of the layer potentials, and using the transmission conditions from (1), we obtain the following integral equations

$$
\begin{aligned}
0 &= \left(\frac{1}{2}\mathbf{I} - \mathbf{K}\right) u + \mathbf{V}\sigma + \lambda \quad \text{on } \Gamma\,,\\
\sigma &= -\,\mathbf{W}u + \left(\frac{1}{2}\mathbf{I} - \mathbf{K}'\right)\sigma \quad \text{on } \Gamma\,,
\end{aligned}
\tag{3}
$$

where we have introduced the new unknown $\sigma := (\mathbf{a}\,\nabla u)\cdot\mathbf{n}$ on Γ, and $\mathbf{V}$, $\mathbf{K}$, $\mathbf{K}'$ and $\mathbf{W}$ are the boundary integral operators of the simple, double, adjoint of the double and hyper-singular layer potentials, respectively.

Now, the condition at infinity of u implies that σ satisfies

$$\int_\Gamma \sigma\, ds = \int_\Gamma (\mathbf{a}\,\nabla u)\cdot\mathbf{n}\, ds = 0\,,$$

which means that $\sigma \in H_0^{-1/2}(\Gamma)$, where $H_0^{-1/2}(\Gamma) := \{\tau \in H^{-1/2}(\Gamma) : \langle \tau, 1\rangle = 0\}$ and, hereafter, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product.

In this way, the original exterior boundary value problem (1) reduces to the following non-local boundary value problem in Ω: *Find* $(u,\sigma,\lambda) \in H^1(\Omega)\times H_0^{-1/2}(\Gamma)\times I\!R$ *such that*

$$\begin{aligned}
u = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad (\mathbf{a}\,\nabla u)\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_N\,,\\
-\operatorname{div}(\mathbf{a}\,\nabla u) = f \quad \text{in} \quad \Omega\,,\\
\sigma = (\mathbf{a}\nabla u)\cdot\mathbf{n} \quad \text{on} \quad \Gamma\,,\\
\sigma = -\,\mathbf{W}u + \left(\frac{1}{2}\mathbf{I} - \mathbf{K}'\right)\sigma \quad \text{on} \quad \Gamma\,,\\
0 = \left(\frac{1}{2}\mathbf{I} - \mathbf{K}\right)u + \mathbf{V}\sigma + \lambda \quad \text{on} \quad \Gamma\,.
\end{aligned} \tag{4}$$

We now introduce the flux $\boldsymbol{\theta} := \mathbf{a}\,\nabla u$. Since $\sigma \in H_0^{-1/2}(\Gamma)$ and $\boldsymbol{\theta}\cdot\mathbf{n} = \sigma$ on Γ, we note that the unknown $\boldsymbol{\theta}$ must belong to $H_0(\operatorname{div};\Omega)$, where

$$H_0(\operatorname{div};\Omega) := \left\{ \boldsymbol{\zeta} \in H(\operatorname{div};\Omega) : \quad \boldsymbol{\zeta}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_N \quad \text{and} \quad \langle \boldsymbol{\zeta}\cdot\mathbf{n}, 1\rangle = 0 \right\}.$$

As usual, $H(\operatorname{div};\Omega)$ is the space of functions $\boldsymbol{\zeta} \in [L^2(\Omega)]^2$ such that $\operatorname{div}\boldsymbol{\zeta} \in L^2(\Omega)$. Provided with the inner product

$$(\boldsymbol{\theta},\boldsymbol{\zeta})_{H(\operatorname{div};\Omega)} := (\boldsymbol{\theta},\boldsymbol{\zeta})_{[L^2(\Omega)]^2} + (\operatorname{div}\boldsymbol{\theta}, \operatorname{div}\boldsymbol{\zeta})_{L^2(\Omega)}\,,$$

$H(\operatorname{div};\Omega)$ is a Hilbert space. Here, $(\cdot,\cdot)_{[L^2(\Omega)]^2}$ and $(\cdot,\cdot)_{L^2(\Omega)}$ denote the inner products of the spaces indicated. Moreover, for all $\boldsymbol{\zeta} \in H(\operatorname{div};\Omega)$ and $\boldsymbol{\zeta}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ there holds $\|\boldsymbol{\zeta}\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma)} \le \|\boldsymbol{\zeta}\|_{H(\operatorname{div};\Omega)}$ (see [26] for the proof of these results).

Consequently, our problem (4) can be rewritten as the following equivalent first order system: *Find* $(\boldsymbol{\theta}, u, \sigma, \lambda) \in H_0(\mathrm{div};\Omega) \times H^1(\Omega) \times H_0^{-1/2}(\Gamma) \times I\!R$ *such that*

$$
\begin{aligned}
u \,=\, 0 \quad \text{on} \quad \Gamma_D\,,\\
\boldsymbol{\theta} \,-\, \mathbf{a}\nabla u \,=\, 0 \quad \text{and} \quad -\,\mathrm{div}\,\boldsymbol{\theta} \,=\, f \quad \text{in} \quad \Omega\,,\\
\sigma \,=\, \boldsymbol{\theta}\cdot\mathbf{n} \quad \text{on} \quad \Gamma\,,\\
\sigma = -\,\mathbf{W}u \,+\, \left(\frac{1}{2}\mathbf{I} - \mathbf{K}'\right)\sigma \quad \text{on} \quad \Gamma\,,\\
0 = \left(\frac{1}{2}\mathbf{I} \,-\, \mathbf{K}\right)u + \mathbf{V}\sigma + \lambda \quad \text{on} \quad \Gamma\,.
\end{aligned}
\tag{5}
$$

This system is the starting point for the least squares formulations that we propose below in Section 3.

Before ending the present section, we recall that the boundary integral operators used above are formally defined by

$$
(\mathbf{V}\tau)(x) := \int_\Gamma E(x,y)\,\tau(y)\,ds_y \quad \forall\,\tau \in H^{-1/2}(\Gamma)\,, \quad \forall\,x \in \Gamma\,,
$$

$$
(\mathbf{K}\mu)(x) := \int_\Gamma \frac{\partial}{\partial\mathbf{n}(y)}E(x,y)\,\mu(y)\,ds_y \quad \forall\,\mu \in H^{1/2}(\Gamma)\,, \quad \forall\,x \in \Gamma\,,
$$

$$
(\mathbf{K}'\tau)(x) := \int_\Gamma \frac{\partial}{\partial\mathbf{n}(x)}E(x,y)\,\tau(y)\,ds_y \quad \forall\,\tau \in H^{-1/2}(\Gamma)\,, \quad \forall\,x \in \Gamma\,,
$$

$$
(\mathbf{W}\mu)(x) := -\,\frac{\partial}{\partial\mathbf{n}(x)}\int_\Gamma \frac{\partial}{\partial\mathbf{n}(y)}E(x,y)\,\mu(y)\,ds_y \quad \forall\,\mu \in H^{1/2}(\Gamma)\,, \quad \forall\,x \in \Gamma\,.
$$

Moreover, their main mapping properties are collected in the following lemma.

Lemma 2.1 *Let* Γ *be a Lipschitz boundary. The operators*

$$
\mathbf{V} : H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma), \qquad \mathbf{K} : H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma)
$$

$$
\mathbf{K}' : H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma), \quad \mathbf{W} : H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma),
$$

are continuous for all $s \in [-1/2,\, 1/2]$*. Furthermore, there exist positive constants* α_1, α_2 *such that*

$$
\langle \tau, \mathbf{V}\tau\rangle \;\geq\; \alpha_1\,\|\tau\|^2_{H^{-1/2}(\Gamma)} \quad \forall\,\tau \in H_0^{-1/2}(\Gamma)\,,
$$

and

$$
\langle \mathbf{W}\mu, \mu\rangle \;\geq\; \alpha_2\,\|\mu\|^2_{H^{1/2}(\Gamma)} \quad \forall\,\mu \in H_0^{1/2}(\Gamma)\,,
$$

where

$$
H_0^{1/2}(\Gamma) := \{\mu \in H^{1/2}(\Gamma) : \quad \langle 1, \mu\rangle = 0\}\,.
$$

Proof. See [12]. □

3 The continuous least squares formulations

According to the system (5), and taking into account the least squares formulations already described in Section 1, we consider here four different approaches.

First, we introduce the operator $\mathbf{P}_0 : H^{1/2}(\Gamma) \to H_0^{1/2}(\Gamma)$, where

$$\mathbf{P}_0 \, \mu \; := \; \mu \; - \; \frac{1}{|\Gamma|} \, \langle 1, \mu \rangle \quad \forall \, \mu \in H^{1/2}(\Gamma) \, . \tag{6}$$

Note that $\mathbf{P}_0 \, \mu \equiv 0$ for all constant μ on Γ, and that there exists $C > 0$, depending only on Γ, such that

$$\| \mathbf{P}_0 \, \mu \|_{H^{1/2}(\Gamma)} \; \leq \; C \, \| \mu \|_{H^{1/2}(\Gamma)} \quad \forall \, \mu \in H^{1/2}(\Gamma) \, . \tag{7}$$

Then, we define the space

$$H^1_{\Gamma_D}(\Omega) := \{ \, v \in H^1(\Omega) : \quad v = 0 \quad \text{on} \quad \Gamma_D \} \, ,$$

and consider the following minimization problem: *Find* $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_1 := \; H_0(\text{div}; \Omega) \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ *such that*

$$\mathbf{J}_1(\boldsymbol{\theta}, u, \sigma) \; = \; \min_{(\boldsymbol{\zeta}, v, \tau) \in \mathbf{X}_1} \mathbf{J}_1(\boldsymbol{\zeta}, v, \tau) \, , \tag{8}$$

where $\mathbf{J}_1$ is the quadratic functional defined by

$$\begin{aligned} \mathbf{J}_1(\boldsymbol{\zeta}, v, \tau) \; &:= \; \| \mathbf{a} \nabla v - \boldsymbol{\zeta} \|^2_{[L^2(\Omega)]^2} \; + \; \| \text{div} \, \boldsymbol{\zeta} + f \|^2_{L^2(\Omega)} \\ + \; \Big\| \mathbf{W} v + \boldsymbol{\zeta} \cdot \mathbf{n} - \left(\tfrac{1}{2} \mathbf{I} - \mathbf{K}' \right) \tau \Big\|^2_{H^{-1/2}(\Gamma)} \; &+ \; \Big\| \mathbf{P}_0 \left[\left(\tfrac{1}{2} \mathbf{I} - \mathbf{K} \right) v + \mathbf{V} \tau \right] \Big\|^2_{H^{1/2}(\Gamma)} \, . \end{aligned} \tag{9}$$

In the sequel, let $H^{-1}(\Omega)$ denote the dual of $H^1_{\Gamma_D}(\Omega)$. Then, since the fourth equation from (5) must be understood at least in the distributional sense, it suffices to assume that the data f belongs to $H^{-1}(\Omega)$ and that the unknown $\boldsymbol{\theta}$ is sought in the space

$$H \; := \; \left\{ \, \boldsymbol{\zeta} \in [L^2(\Omega)]^2 : \quad \boldsymbol{\zeta} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_N \quad \text{and} \quad \boldsymbol{\zeta} \cdot \mathbf{n} \in H_0^{-1/2}(\Gamma) \, \right\} ,$$

which is provided with the norm of $[L^2(\Omega)]^2$.

The above remark leads us to the following sharper minimization problem: *Find* $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_2 \; := \; H \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ *such that*

$$\mathbf{J}_2(\boldsymbol{\theta}, u, \sigma) \; = \; \min_{(\boldsymbol{\zeta}, v, \tau) \in \mathbf{X}_2} \mathbf{J}_2(\boldsymbol{\zeta}, v, \tau) \, , \tag{10}$$

where $\mathbf{J}_2$ is the quadratic functional defined by

$$\begin{aligned} \mathbf{J}_2(\boldsymbol{\zeta}, v, \tau) \; &:= \; \| \mathbf{a} \nabla v - \boldsymbol{\zeta} \|^2_{[L^2(\Omega)]^2} \; + \; \| \text{div} \, \boldsymbol{\zeta} + f \|^2_{H^{-1}(\Omega)} \\ + \; \Big\| \mathbf{W} v + \boldsymbol{\zeta} \cdot \mathbf{n} - \left(\tfrac{1}{2} \mathbf{I} - \mathbf{K}' \right) \tau \Big\|^2_{H^{-1/2}(\Gamma)} \; &+ \; \Big\| \mathbf{P}_0 \left[\left(\tfrac{1}{2} \mathbf{I} - \mathbf{K} \right) v + \mathbf{V} \tau \right] \Big\|^2_{H^{1/2}(\Gamma)} \, . \end{aligned} \tag{11}$$

We remark that the only differences between (8)-(9) and (10)-(11) lies on the norm that measures the error arising from the equilibrium equation $(\operatorname{div} \boldsymbol{\zeta} + f) = 0$, and on the space in which the unknown $\boldsymbol{\theta}$ lives. In any case, it is easy to see that the minimum of both $\mathbf{J}_1$ and $\mathbf{J}_2$ is attained for any solution $(\boldsymbol{\theta}, u, \sigma)$ of problem (5). Also, it is important to mention that, instead of the first term in the definitions of $\mathbf{J}_1$ and $\mathbf{J}_2$, one may use the weighted norm $\|\mathbf{a}^{-1/2}(\mathbf{a}\nabla v - \boldsymbol{\zeta})\|^2_{[L^2(\Omega)]^2}$, which leads to a better conditioning of the corresponding discrete problems (see [15] for details).

The use of norms are motivated by the proper functional analytical setting $\mathcal{L} : \mathbf{X} \to \mathbf{X}'$. The paper [15] provides a general framework for least squares methods based on variational formulations. In contrast to Galerkin methods the least square methods are stable iff $\mathcal{L}$ is normally solvable, i.e. $\operatorname{Im}\mathcal{L} \subset \mathbf{X}'$ is a closed subset of $\mathbf{X}'$. However, the previous formulations do not fit directly into the framework of [15]. Nevertheless, there is a slight modification of the functional $\mathbf{J}_2$ fitting into this setting which can be derived from the variational formulation of the second order problem. This realization facilitates the implementation, see Section 7. Taking the equation $-\operatorname{div} \boldsymbol{\theta} = f$ in its weak form we can apply Green's Theorem

$$-(\operatorname{div} \boldsymbol{\theta}, v)_{L^2(\Omega)} = (\boldsymbol{\theta}, \nabla v)_{[L^2(\Omega)]^2} - \langle \boldsymbol{\theta} \cdot \mathbf{n}, v \rangle \quad \forall\, v \in H^1_{\Gamma_D}(\Omega)\,.$$

Here we are considering the Hilbert space $\mathbf{X}_3 := [L^2(\Omega)]^2 \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ and $\mathbf{X}_3'$ is the dual space of $\mathbf{X}_3$ with respect to the canonical L^2-inner product. This yields the following least squares minimization problem: *Find* $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_3$ *such that*

$$\mathbf{J}_3(\boldsymbol{\theta}, u, \sigma) = \min_{(\boldsymbol{\zeta}, v, \tau) \in \mathbf{X}_3} \mathbf{J}_3(\boldsymbol{\zeta}, v, \tau)\,, \tag{12}$$

where $\mathbf{J}_3$ is the quadratic functional defined by

$$\begin{aligned} \mathbf{J}_3(\boldsymbol{\zeta}, v, \tau) &:= \|\mathbf{a}\nabla v - \boldsymbol{\zeta}\|^2_{[L^2(\Omega)]^2} + \left\|\mathbf{P}_0\left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right)v + \mathbf{V}\tau\right]\right\|^2_{H^{1/2}(\Gamma)} \\ &+ \left\|\operatorname{div} \boldsymbol{\zeta} + f - \delta_\Gamma \otimes \left(\mathbf{W}v + \boldsymbol{\zeta}\cdot\mathbf{n} - \left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}'\right)\tau\right)\right\|^2_{H^{-1}(\Omega)} \end{aligned} \tag{13}$$

Here $\delta_\Gamma \otimes \tau$ is a distribution in $H^{-1}(\Omega)$ which is supported on the interface boundary Γ. Though this minimization problem looks unusual it is relatively simple to implement. One advantage of this formulation is that the flux can be chosen simply in $[L^2(\Omega)]^2$.

There is another more simplified version which is obtained by inserting the transmission condition $\boldsymbol{\theta} \cdot \mathbf{n} = \sigma$ directly into the above formulation. Then, the trace norms $\|\cdot\|_{H^{1/2}(\Gamma)}$ and $\|\cdot\|_{H^{-1/2}(\Gamma)}$ in (11) are redundant and we can derive a simpler minimization problem, that is: *Find* $(\boldsymbol{\theta}, u) \in \mathbf{X}_4 := H \times H^1_{\Gamma_D}(\Omega)$ *such that*

$$\mathbf{J}_4(\boldsymbol{\theta}, u) = \min_{(\boldsymbol{\zeta}, v) \in \mathbf{X}_4} \mathbf{J}_4(\boldsymbol{\zeta}, v)\,, \tag{14}$$

where $\mathbf{J}_4$ is the quadratic functional defined by

$$\begin{aligned} \mathbf{J}_4(\boldsymbol{\zeta}, v) &:= \|\mathbf{a}\nabla v - \boldsymbol{\zeta}\|^2_{[L^2(\Omega)]^2} + \|\operatorname{div} \boldsymbol{\zeta} + f\|^2_{H^{-1}(\Omega)} \\ &+ \left\|\mathbf{P}_0\left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right)v + \mathbf{V}(\boldsymbol{\zeta}\cdot\mathbf{n})\right]\right\|^2_{H^{1/2}(\Gamma)}\,. \end{aligned} \tag{15}$$

Since $H^{-1}(\Omega)$, $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are Hilbert spaces the norms are defined by the corresponding inner products $(\cdot,\cdot)_{H^{-1}(\Omega)}$, $\langle\cdot,\cdot\rangle_{H^{-1/2}(\Gamma)}$ and $\langle\cdot,\cdot\rangle_{H^{1/2}(\Gamma)}$. For example, the quadratic functional $\mathbf{J}_4$ can be rewritten by

$$\begin{aligned}\mathbf{J}_4(\boldsymbol{\zeta}, v) \;:=\; & (\mathbf{a}\nabla v - \boldsymbol{\zeta}, \mathbf{a}\nabla v - \boldsymbol{\zeta})_{[L^2(\Omega)]^2} \;+\; (\operatorname{div}\boldsymbol{\zeta} + f, \operatorname{div}\boldsymbol{\zeta} + f)_{H^{-1}(\Omega)} \\ & + \langle \mathbf{P}_0\left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right) v + \mathbf{V}(\boldsymbol{\zeta}\cdot\mathbf{n})\right], \mathbf{P}_0\left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right) v + \mathbf{V}(\boldsymbol{\zeta}\cdot\mathbf{n})\right]\rangle_{H^{1/2}(\Gamma)}\,.\end{aligned}$$

In what follows, we develop the necessary tools to study the solvability and discrete approximations of our least squares formulations. However, the second, third and fourth formulation are sharper than the first one. In fact, the resulting convergence rate is higher and the system matrices can be preconditioned quite well. In the third and forth formulation the $H^{-1/2}$-norm is avoided. Moreover, in the third formulation, the flux is computed by means of Green's Theorem, see Section 7. This means that we neither need $\boldsymbol{\zeta}\cdot\mathbf{n}$ nor the assumption $\boldsymbol{\zeta}\cdot\mathbf{n} \in H^{-1/2}(\Gamma)$ explicitly. In our opinion, it is the most favourable approach. The fourth formulation looks most simple and it avoids the computation of the hyper-singular operator $\mathbf{W}$. However, its discretization requires some kind of stabilization which will be discussed below in Section 5. Therefore, throughout the rest of the paper, we will just concentrate on the problems (10)-(11), (14)-(15) and (14)-(15). Since the computation of the $L^2(\Omega)$-inner product offers no difficulties, the corresponding extension to (8)-(9) will be straightforward.

Now, following the general setting from [15], we find that (8), (10) and (12) are equivalent to

$$\mathbf{J}_i(\boldsymbol{\theta}, u, \sigma) \;=\; \min_{(\boldsymbol{\zeta}, v, \tau)\in\mathbf{X}_i} \mathbf{J}_i(\boldsymbol{\zeta}, v, \tau)\,, \tag{16}$$

with

$$\mathbf{J}_i(\boldsymbol{\zeta}, v, \tau) \;=\; \frac{1}{2}\, B_i\big((\boldsymbol{\zeta}, v, \tau), (\boldsymbol{\zeta}, v, \tau)\big) \;-\; \mathbf{G}_i(\boldsymbol{\zeta}, v, \tau) \;+\; \textit{const.}\,, \tag{17}$$

$i = 1, 2, 3$, with corresponding bilinear forms $B_i : \mathbf{X}_i \times \mathbf{X}_i \to I\!R$, and linear functionals $\mathbf{G}_i : \mathbf{X}_i \to I\!R$. An analogous setting holds for (15). Then, the minimzation problems are equivalent to the following linear equations: *Find* $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_i$ *such that*

$$B_i\big((\boldsymbol{\theta}, u, \sigma), (\boldsymbol{\zeta}, v, \tau)\big) \;=\; \mathbf{G}_i(\boldsymbol{\zeta}, v, \tau) \tag{18}$$

for all $(\boldsymbol{\zeta}, v, \tau) \in \mathbf{X}_i$. This equation is solved approximatively on a finite dimensional subspace in $\mathbf{X}_i$. Therein, the major difficulty is the computation of the underlying bilinear and linear forms. However, this can be done only approximatively, which means B_i is replaced by some discrete bilinear form B_i^h.

4 Coercivity estimates

It is easy to prove, using the mapping properties of the boundary integral operators (cf. Lemma 2.1) and (7), that B_2, B_3 and B_4 are symmetric and bounded in the corresponding energy norms. In addition, $\mathbf{G}_2$, $\mathbf{G}_3$ and $\mathbf{G}_4$ are also bounded. Therefore, in order to conclude the unique solvability of our least squares formulations (10)-(11), (12)-(13) and (14)-(15), it remains to show that B_2, B_3 and B_4 are strongly coercive in $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$, respectively. Usually, coercivity estimates for least squares formulations are valid under much weaker conditions than for Galerkin formulations since only the normal solvability of the operator is required. Since the Sobolev norms cannot be computed exactly (see below) we need to apply a more sophisticated tool for the investigation of the present discrete least squares methods. For this purpose, we have to state some previous results concerning the Galerkin scheme of the original second order non-local boundary value problem (4).

First, proceeding in the usual way (see, e.g. [13], [22], [27]), we find that the weak formulation of (4) reduces to: *Find* $(u,\sigma) \in \mathbf{H} := H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ *such that*

$$A\big((u,\sigma),(v,\tau)\big) \;=\; \mathbf{F}(v,\tau) \quad \forall\,(v,\tau) \in \mathbf{H}\,, \tag{19}$$

where $A : \mathbf{H} \times \mathbf{H} \to I\!R$ is the bounded bilinear form defined by

$$\begin{aligned} A\big((u,\sigma),(v,\tau)\big) \;:=\; & (\mathbf{a}\,\nabla u, \nabla v)_{[L^2(\Omega)]^2} \,+\, \langle \mathbf{W}u, v\rangle \,-\, \langle \big(\tfrac{1}{2}\mathbf{I} - \mathbf{K}'\big)\,\sigma, v\rangle \\ & +\, \langle \tau, \mathbf{V}\sigma\rangle \,+\, \langle \tau, \big(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\big)\,u\rangle \end{aligned} \tag{20}$$

for all (u,σ), $(v,\tau) \in \mathbf{H}$, and $\mathbf{F} \in \mathbf{H}'$ is given by

$$\mathbf{F}(v,\tau) \;:=\; \int_\Omega f\,v\,dx \quad \forall\,(v,\tau) \in \mathbf{H}\,. \tag{21}$$

The product space $\mathbf{H}$ is provided with the corresponding norm, that is

$$\|(v,\tau)\|_{\mathbf{H}} \;:=\; \left\{\, \|v\|^2_{H^1(\Omega)} \,+\, \|\tau\|^2_{H^{-1/2}(\Gamma)} \,\right\}^{1/2}.$$

In the sequel, given two expressions a and b, the relation $a \lesssim b$ means that a is bounded by some constant times b uniformly in all parameters upon which a and b may depend. An analogue definition holds for the relation $a \gtrsim b$. Also, $a \sim b$ means that $a \lesssim b$ and $a \gtrsim b$.

Lemma 4.1 *The bilinear form A is strongly coercive in $\mathbf{H}$, that is*

$$A\big((v,\tau),(v,\tau)\big) \;\gtrsim\; \|(v,\tau)\|^2_{\mathbf{H}} \quad \forall\,(v,\tau) \in \mathbf{H}\,.$$

Proof. Using that $\mathbf{K}'$ is the adjoint of $\mathbf{K}$, we obtain from (20) that

$$A\big((v,\tau),(v,\tau)\big) \;=\; (\mathbf{a}\,\nabla v, \nabla v)_{[L^2(\Omega)]^2} \;+\; \langle \mathbf{W} v, v\rangle \;+\; \langle \tau, \mathbf{V}\tau\rangle\,.$$

Since $|\Gamma_D| \neq 0$, Poincaré's inequality yields the equivalence between the norm and the semi-norm of $H^1(\Omega)$ in the subspace $H^1_{\Gamma_D}(\Omega)$, which, together with (2), implies that

$$(\mathbf{a}\,\nabla v, \nabla v)_{[L^2(\Omega)]^2} \;\gtrsim\; \|v\|^2_{H^1(\Omega)} \quad \forall\, v \in H^1_{\Gamma_D}(\Omega)\,.$$

Then, the above inequality and the coerciveness properties of $\mathbf{V}$ and $\mathbf{W}$ given in Lemma 2.1 complete the proof. □

For the sake of completeness, we also provide the following consequence of the previous lemma.

Theorem 4.1 *There exists a unique solution $(u,\sigma) \in \mathbf{H}$ of the variational formulation* (19)*. Moreover, this solution satisfies the a-priori estimate $\|(u,\sigma)\|_{\mathbf{H}} \;\lesssim\; \|\mathbf{F}\|_{\mathbf{H}'}$.*

Proof. It is a straightforward application of the Lax-Milgram Lemma. □

The following lemma reveals a well known fact about boundary integral operators.

Lemma 4.2 *For $u \in H_0^{1/2}(\Gamma)$ and $\sigma \in H_0^{-1/2}(\Gamma)$ holds*

$$\big\|\mathbf{W}u + (\tfrac{1}{2}\mathbf{I} + \mathbf{K}')\sigma\big\|_{H^{-1/2}(\Gamma)} \;\sim\; \big\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u + \mathbf{V}\sigma\right]\big\|_{H^{1/2}(\Gamma)}\,. \tag{22}$$

Proof. It follows easily from the equality $\mathbf{W} = -(\frac{1}{2}\mathbf{I} + \mathbf{K}')\mathbf{V}^{-1}(\frac{1}{2}\mathbf{I} - \mathbf{K})$ together with the mapping properties of the double layer potential operators in the spaces $H^{\pm 1/2}(\Gamma)$. □

Theorem 4.2 *For all functions $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_2 := H \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ the following a-priori estimate is valid*

$$\begin{aligned} \|u\|_{H^1(\Omega)} + \|\boldsymbol{\theta}\|_{[L^2(\Omega)]^2} + \|\sigma\|_{H^{-1/2}(\Gamma)} \;&\lesssim\; \|\mathbf{a}\,\nabla u - \boldsymbol{\theta}\|_{[L^2(\Omega)]^2} + \|\mathrm{div}\,\boldsymbol{\theta}\|_{H^{-1}(\Omega)} \\ &+ \big\|\mathbf{W}u + \boldsymbol{\theta}\cdot\mathbf{n} - (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\sigma\big\|_{H^{-1/2}(\Gamma)} + \big\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u + \mathbf{V}\sigma\right]\big\|_{H^{1/2}(\Gamma)}\,. \end{aligned} \tag{23}$$

Moreover, for all $(\boldsymbol{\theta}, u, \sigma) \in \mathbf{X}_3 := [L^2(\Omega)]^2 \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$ there holds

$$\begin{aligned} \|u\|_{H^1(\Omega)} \;+\; \|\boldsymbol{\theta}\|_{[L^2(\Omega)]^2} \;+\; \|\sigma\|_{H^{-1/2}(\Gamma)} \;&\lesssim\; \|\mathbf{a}\,\nabla u - \boldsymbol{\theta}\|_{[L^2(\Omega)]^2} \\ &+ \big\|\mathrm{div}\,\boldsymbol{\theta} - \delta_\Gamma \otimes \left[\mathbf{W}u + \boldsymbol{\theta}\cdot\mathbf{n} - (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\sigma\right]\big\|_{H^{-1}(\Omega)} \\ &+ \big\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u + \mathbf{V}\sigma\right]\big\|_{H^{1/2}(\Gamma)}\,. \end{aligned} \tag{24}$$

In addtion, for any $(\boldsymbol{\theta}, u) \in \mathbf{X}_4 := H \times H^1_{\Gamma_D}(\Omega)$ there holds the a-priori estimate

$$\begin{aligned} \|u\|_{H^1(\Omega)} \;+\; \|\boldsymbol{\theta}\|_{[L^2(\Omega)]^2} \;+\; \|\boldsymbol{\theta}\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma)} \;&\lesssim\; \|\mathbf{a}\,\nabla u - \boldsymbol{\theta}\|_{[L^2(\Omega)]^2} \\ &+ \|\mathrm{div}\,\boldsymbol{\theta}\|_{H^{-1}(\Omega)} \;+\; \big\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u + \mathbf{V}(\boldsymbol{\theta}\cdot\mathbf{n})\right]\big\|_{H^{1/2}(\Gamma)}\,. \end{aligned} \tag{25}$$

Proof. We provide a particular proof of this result because we need this argumentation below to prove the main Theorem 5.1. In virtue of Theorem 4.1 we estimate

$$\|u\|_{H^1(\Omega)} + \|\sigma\|_{H^{-1/2}(\Gamma)} \lesssim \sup_{\delta\in H_0^{-1/2}(\Gamma)} \frac{1}{\|\delta\|_{H^{-1/2}(\Gamma)}} \left\{\langle \delta, (\tfrac{1}{2}\mathbf{I}-\mathbf{K})u + \mathbf{V}\sigma\rangle\right\}$$
$$+ \sup_{v\in H^1_{\Gamma_D}(\Omega)} \frac{1}{\|v\|_{H^1(\Omega)}} \left\{(\mathbf{a}\nabla u, \nabla v)_{[L^2(\Omega)]^2} + \langle \mathbf{W}u - (\tfrac{1}{2}\mathbf{I}-\mathbf{K}')\sigma, v\rangle\right\}$$
$$\lesssim \sup_{\delta\in H_0^{-1/2}(\Gamma)} \frac{1}{\|\delta\|_{H^{-1/2}(\Gamma)}} \left\{\langle \delta, (\tfrac{1}{2}\mathbf{I}-\mathbf{K})u + \mathbf{V}\sigma\rangle\right\}$$
$$+ \sup_{v\in H^1_{\Gamma_D}(\Omega)} \frac{1}{\|v\|_{H^1(\Omega)}}\left\{(\mathbf{a}\nabla u - \boldsymbol{\theta}, \nabla v)_{[L^2(\Omega)]^2} + (\boldsymbol{\theta}, \nabla v)_{[L^2(\Omega)]^2} + \langle \mathbf{W}u - (\tfrac{1}{2}\mathbf{I}-\mathbf{K}')\sigma, v\rangle\right\}.$$

Next, we apply the divergence theorem and use that $\boldsymbol{\theta}\cdot\mathbf{n} = 0$ on Γ_N, whence

$$\|u\|_{H^1(\Omega)} + \|\sigma\|_{H^{-1/2}(\Gamma)} \lesssim \sup_{\delta\in H_0^{-1/2}(\Gamma)} \frac{1}{\|\delta\|_{H^{-1/2}(\Gamma)}} \left\{\langle \delta, (\tfrac{1}{2}\mathbf{I}-\mathbf{K})u + \mathbf{V}\sigma\rangle\right\}$$
$$+ \sup_{v\in H^1_{\Gamma_D}(\Omega)} \frac{1}{\|v\|_{H^1(\Omega)}}\left\{(\mathbf{a}\nabla u - \boldsymbol{\theta}, \nabla v)_{[L^2(\Omega)]^2} - (\operatorname{div}\boldsymbol{\theta}, v)_{[L^2(\Omega)]^2}\right.$$
$$\left. + \langle \mathbf{W}u + \boldsymbol{\theta}\cdot\mathbf{n} - (\tfrac{1}{2}\mathbf{I}-\mathbf{K}')\sigma, v\rangle\right\},$$

which implies both estimates, (23) and (24), immediately. We remark that we have used the trace theorem $\|v\|_{H^{1/2}(\Gamma)} \lesssim \|v\|_{H^1(\Omega)}$ and the fact that $v|_{\Gamma_D} = 0$ for all $v \in H^1_{\Gamma_D}(\Omega)$. To prove (25) we choose $\sigma = \boldsymbol{\theta}\cdot\mathbf{n}$ in (23) and apply the result of Lemma 4.2. □

5 Finite element approximations

For the definition of the Ritz and Galerkin methods for (16)- (17), we consider finite dimensional subspaces $\mathbf{X}_i^h := X_h^i \times V_h \times S_h$ of $\mathbf{X}_i$ and assume the following.

- *Approximation property of* V_h. There exists $d := d_V > 1$ such that for all $s < \min\{\frac{3}{2}, d\}$ and for all $u \in H^d(\Omega)$

$$\inf_{v_h\in V_h} \|u - v_h\|_{H^s(\Omega)} \lesssim h^{d-s}\|u\|_{H^d(\Omega)}.$$

- *Inverse property* of V_h. For all $v_h \in V_h$ and for all $t < s < \frac{3}{2}$ there holds

$$\|v_h\|_{H^s(\Omega)} \lesssim h^{t-s}\|v_h\|_{H^t(\Omega)}.$$

Similar properties are also assumed for X_h^i and S_h with constants d_X and d_S, respectively.

Typical candidates for these spaces are finite element spaces V_h, subordinated to a triangulation $\mathcal{T}_h = \{\tau_k\}$ of Ω consisting of triangles or quadrilaterals τ_k with diameter h_k. The above properties are valid for shape regular quasi-uniform triangulations. The results for (16)-(17) remain valid also for non-uniform triangulations. The results with respect to $\mathbf{J}_4$ seem to be also true on non-uniform grids (perhaps the proof becomes rather technical).

Here d denotes polynomial degree on each triangle. Since $V_h \subset H^1_{\Gamma_D}(\Omega)$, the functions $v_h \in V_h$ are supposed continuous on Ω. For a consistent discretization it is sufficient to choose $d_X = d_V - 1 = d - 1$. The spaces S_h are defined analogously on the boundary and they should be exact at least of order $d_S = d - 1$.

The $H^{-1}(\Omega)$-norm can be computed by introducing the operator $\mathbf{T}_h : H^{-1}(\Omega) \to V_h$, where for each $f \in H^{-1}(\Omega)$ the function $w_h := \mathbf{T}_h f$ is the unique function in V_h satisfying

$$(\nabla w_h, \nabla v_h)_{[L^2(\Omega)]^2} \;=\; (f, v_h)_{L^2(\Omega)} \quad \forall\, v_h \in V_h \,. \tag{26}$$

The computation of the operator $\mathbf{T}_h$ requires the solution of a Neumann problem which is relatively expensive. For an efficient computation it is much more feasible to use a symmetric preconditioner $\mathbf{B}_h : V_h^* \to V_h$ instead of $\mathbf{T}_h$ satisfying

$$\|\mathbf{B}_h f_h\|_{H^1_{\Gamma_D}(\Omega)} \;\sim\; \|f_h\|_{H^{-1}(\Omega)} \qquad \forall\, f_h \in V_h^* \,, \tag{27}$$

or equivalently

$$(\mathbf{T}_h f_h, f_h)_{L^2(\Omega)} \;\sim\; (\mathbf{B}_h f_h, f_h)_{L^2(\Omega)} \quad \forall\, f_h \in V_h^* \,, \tag{28}$$

where $V_h^* \subset H^{-1}(\Omega)$ is a suitable finite dimensional subspace. Such preconditioners are available from multigrid or multilevel algorithms [5], [6], in which case one can choose $V_h^* = V_h$, as well as from wavelet bases [15], where the space V_h^* is generated by the dual wavelet basis.

Now, in order to compute the inner products $\langle \cdot, \cdot \rangle_{H^{\pm 1/2}(\Gamma)}$, one can use (cf. Lemma 2.1) that $\langle \lambda, \mu \rangle_{H^{1/2}(\Gamma)} \sim \langle \mathbf{W}\mu, \lambda \rangle$ for all $\lambda, \mu \in H_0^{1/2}(\Gamma)$ and that $\langle \sigma, \tau \rangle_{H^{-1/2}(\Gamma)} \sim \langle \sigma, \mathbf{V}\tau \rangle$ for all $\sigma, \tau \in H_0^{-1/2}(\Gamma)$, which, however, are not accessible for numerical computations. Again we have to consider only $H^{\pm 1/2}(\Gamma)$-norms and $\langle \cdot, \cdot \rangle_{H^{\pm 1/2}(\Gamma)}$-inner products on finite dimensional subspaces. However, one can apply a preconditioner $\mathbf{D}_h$ for $\mathbf{W}$ in the same way as described above, see e.g. [30] and [36]. It is computable on a finite dimensional subspace $\tilde{V}_h(\Gamma)$ of $H_0^{1/2}(\Gamma)$ and satisfies

$$\langle \mathbf{W}\lambda_h, \lambda_h \rangle \;\sim\; \langle \mathbf{D}_h \lambda_h, \lambda_h \rangle \quad \forall\, \lambda_h \in \tilde{V}_h(\Gamma) \,. \tag{29}$$

Similarly, we introduce an operator $\mathbf{C}_h$ as a preconditioner for $\mathbf{V}$ satisfying

$$\langle \sigma_h, \mathbf{V}\sigma_h \rangle \;\sim\; \langle \sigma_h, \mathbf{C}_h \sigma_h \rangle \quad \forall\, \sigma_h \in \tilde{S}_h \,, \tag{30}$$

where $\tilde{S}_h$ is a finite dimensional subspace of $H_0^{-1/2}(\Gamma)$. In the case one is dealing only with traditional boundary elements, we simply have $\tilde{V}_h = V_h|_\Gamma$ and $\tilde{S}_h = S_h$.

For wavelet preconditioner we refer to the subsequent section. It is worth to mention that the dual basis functions are never used explicitly.

Since these operators are symmetric and coercive, we define for notation's convenience the square roots $\mathbf{B}_h^{1/2}$ by $(\mathbf{B}_h^{1/2})^*\mathbf{B}_h^{1/2} = \mathbf{B}_h$ and similarly we set up $\mathbf{C}_h^{1/2}$ and $\mathbf{D}_h^{1/2}$. In addition, we define $V_h(\Gamma) := V_h|_\Gamma \cap H_0^{1/2}(\Gamma)$, and let $P_h^* : H^1_{\Gamma_D}(\Omega) \to V_h^*$, $\mathcal{Q}_h : H^{1/2}(\Gamma) \to V_h(\Gamma)$, and $Q_h : H^{-1/2}(\Gamma) \to S_h$ be bounded projectors with *adjoint* operators $P_h^* : H^{-1}(\Omega) \to V_h^*$, $\mathcal{Q}_h^* : H^{-1/2}(\Gamma) \to \tilde{S}_h$ and $Q_h^* : H^{1/2}(\Gamma) \to \tilde{V}_h(\Gamma)$, respectively. Then, according to (27), (29) and (30), we deduce that

$$\begin{aligned}
(\mathbf{B}_h P_h^* f, P_h^* f)_{L^2(\Omega)} &\sim \|P_h^* f\|^2_{V_h^*} && \forall\, f \in H^{-1}(\Omega)\,, \\
\langle \mathbf{D}_h Q_h^* \lambda, Q_h^* \lambda\rangle &\sim \|Q_h^* \lambda\|^2_{H^{1/2}(\Gamma)} && \forall\, \lambda \in H_0^{1/2}(\Gamma)\,, \\
\langle \mathcal{Q}_h^* \sigma, \mathbf{C}_h \mathcal{Q}_h^* \sigma\rangle &\sim \|\mathcal{Q}_h^* \sigma\|^2_{H^{-1/2}(\Gamma)} && \forall\, \sigma \in H_0^{-1/2}(\Gamma)\,.
\end{aligned} \tag{31}$$

The above means that we will use truncated bilinear forms instead of the original ones for the computation of the Galerkin solutions. Certainly, this truncation may influence the stability of the methods. Hence, we prove next that stability is not violated by this procedure.

Theorem 5.1 *For arbitrary functions $(\boldsymbol{\theta}_h, u_h, \sigma_h) \in \mathbf{X}_2^h$, the following a-priori estimate hold*

$$\begin{aligned}
\|u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} + \|\sigma_h\|_{H^{-1/2}(\Gamma)} &\lesssim \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} \\
+ \|\mathbf{B}_h^{1/2} P_h^* \operatorname{div} \boldsymbol{\theta}_h\|_{L^2(\Omega)} &+ \big\|\mathbf{C}_h^{1/2}\mathcal{Q}_h^*\big[\mathbf{W}u_h + \boldsymbol{\theta}_h\cdot\mathbf{n} - (\tfrac12\mathbf{I} - \mathbf{K}')\sigma_h\big]\big\|_{L^2(\Gamma)} \\
&+ \big\|\mathbf{D}_h^{1/2} Q_h^* \mathbf{P}_0\big[(\tfrac12\mathbf{I} - \mathbf{K})u_h + \mathbf{V}\sigma_h\big]\big\|_{L^2(\Gamma)}\,,
\end{aligned} \tag{32}$$

and for $(\boldsymbol{\theta}_h, u_h, \sigma_h) \in \mathbf{X}_3^h$ we find

$$\begin{aligned}
\|u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} + \|\sigma_h\|_{H^{-1/2}(\Gamma)} &\lesssim \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} \\
+ \big\|\mathbf{B}_h^{1/2} P_h^*\big(\operatorname{div} \boldsymbol{\theta}_h - \delta_\Gamma \otimes \big[\mathbf{W}u_h &+ \boldsymbol{\theta}_h\cdot\mathbf{n} - (\tfrac12\mathbf{I} - \mathbf{K}')\sigma_h\big]\big)\big\|_{L^2(\Omega)} \\
&+ \big\|\mathbf{D}_h^{1/2} Q_h^* \mathbf{P}_0\big[(\tfrac12\mathbf{I} - \mathbf{K})u_h + \mathbf{V}\sigma_h\big]\big\|_{L^2(\Gamma)}\,.
\end{aligned} \tag{33}$$

Proof. We estimate the expression in the same fashion as in the proof of Theorem 4.2. First we observe that the stability of the Galerkin scheme implies the following estimate

$$\|u_h\|_{H^1(\Omega)} + \|\sigma_h\|_{H^{-1/2}(\Gamma)} \lesssim \sup_{\delta_h\in S_h} \frac{1}{\|\delta_h\|_{H^{-1/2}(\Gamma)}} \big\{\langle \delta_h, (\tfrac12\mathbf{I}-\mathbf{K})u_h + \mathbf{V}\sigma_h\rangle\big\}$$

$$+ \sup_{v_h\in V_h} \frac{1}{\|v_h\|_{H^1(\Omega)}} \big\{(\mathbf{a}\nabla u_h, \nabla v_h)_{[L^2(\Omega)]^2} + \langle \mathbf{W}u_h - (\tfrac12\mathbf{I}-\mathbf{K}')\sigma_h, v_h\rangle\big\}$$

$$\lesssim \ \big\|Q_h^*\,\mathbf{P}_0\left[(\tfrac12\mathbf{I}-\mathbf{K})u_h + \mathbf{V}\sigma_h\right]\big\|_{H^{1/2}(\Gamma)} + \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}$$

$$+ \big\|P_h^*\left(\operatorname{div}\boldsymbol{\theta}_h - \delta_\Gamma\otimes\left[\mathbf{W}u_h + \boldsymbol{\theta}_h\cdot\mathbf{n} - (\tfrac12\mathbf{I}-\mathbf{K}')\sigma_h\right]\right)\big\|_{H^{-1}(\Omega)}$$

$$\lesssim \ \big\|\mathbf{D}_h^{1/2}Q_h^*\,\mathbf{P}_0\big[(\tfrac12\mathbf{I}-\mathbf{K})u_h + \mathbf{V}\sigma_h\big]\big\|_{L^2(\Gamma)} + \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}$$

$$+ \big\|\mathbf{B}_h^{1/2}P_h^*\big(\operatorname{div}\boldsymbol{\theta}_h - \delta_\Gamma\otimes\left[\mathbf{W}u_h + \boldsymbol{\theta}_h\cdot\mathbf{n} - (\tfrac12\mathbf{I}-\mathbf{K}')\sigma_h\right]\big)\big\|_{L^2(\Omega)},$$

where we have used the properties of the operators $\mathbf{B}_h$ and $\mathbf{D}_h$. From this estimate the assertion (33) follows immediately. Similarly one can prove the estimate (32). □

This suggests that one has to solve the following linear problem

$$B_i^h\big((\boldsymbol{\theta}_h, u_h, \sigma_h), (\boldsymbol{\zeta}_h, v_h, \tau_h)\big) \ = \ \mathbf{G}_i^h(\boldsymbol{\zeta}_h, v_h, \tau_h), \quad i = 2, 3, \tag{34}$$

with the truncated bilinear forms $B_i^h : \mathbf{X}_i^h \times \mathbf{X}_i^h \to I\!R$ and the functionals $\mathbf{G}_i^h : \mathbf{X}_i^h \to I\!R$. The computation of the bilinear functionals B_1, B_2^h and B_4^h requires the computation of $\operatorname{div}\boldsymbol{\zeta}$ which is possible e.g. if $\boldsymbol{\zeta}\in H(\operatorname{div};\Omega)$ or $X_h \subset H$, despite the fact that the energy space using B_2^h or B_4^h is $[L^2(\Omega)]^2$. The differentiation of $\boldsymbol{\zeta}$ can be avoided by the help of Green's Theorem. This is used in the third formulation using B_3^h which requires only that $\boldsymbol{\zeta}\in[L^2(\Omega)]^2$, i.e. $X_h\subset[L^2(\Omega)]^2$.

It turns out that, for the fourth formulation, the truncation must be performed on a probably finer grid to preserve the stability. Here, the bilinear form $B_4^h : \mathbf{X}_4^h\times\mathbf{X}_4^h \to I\!R$ is defined by

$$B_4^h\big((\boldsymbol{\theta}_h, u_h), (\boldsymbol{\zeta}_h, v_h)\big) \ := \ (\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h, \mathbf{a}\nabla v_h - \boldsymbol{\zeta}_h)_{[L^2(\Omega)]^2}$$

$$+ (\mathbf{B}_h P_h^*\operatorname{div}\boldsymbol{\theta}_h, P_h^*\operatorname{div}\boldsymbol{\zeta}_h)_{L^2(\Omega)}$$

$$+ \langle \mathbf{D}_{h'}Q_{h'}^*\,\mathbf{P}_0\left[\big(\tfrac12\mathbf{I}-\mathbf{K}\big)\,u_h + \mathbf{V}(\boldsymbol{\theta}_h\cdot\mathbf{n})\right], Q_{h'}^*\,\mathbf{P}_0\left[\big(\tfrac12\mathbf{I}-\mathbf{K}\big)\,v_h + \mathbf{V}(\boldsymbol{\zeta}_h\cdot\mathbf{n})\right]\rangle_{L^2(\Gamma)},$$

where the positive parameter h' has to be chosen such that $h' \lesssim h$. In fact, we have the following result.

Theorem 5.2 *Assume that $\boldsymbol{\theta}_h\cdot\mathbf{n}\in S_h$ for all $\boldsymbol{\theta}_h\in X_h$. Then there exists a mesh size $h' \lesssim h$ such that for $u_h\in V_h$ and $\boldsymbol{\theta}_h\in X_h$ there holds the a-priori estimate*

$$\|u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2} + \|\boldsymbol{\theta}_h\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma)} \lesssim \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega)]^2}$$

$$+ \big\|\mathbf{B}_h^{1/2}P_h^*\operatorname{div}\boldsymbol{\theta}_h\big\|_{L^2(\Omega)} + \big\|\mathbf{D}_{h'}^{1/2}Q_{h'}^*\,\mathbf{P}_0\big[(\tfrac12\mathbf{I}-\mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h\cdot\mathbf{n})\big]\big\|_{L^2(\Gamma)}.$$

Proof. Given $(\boldsymbol{\theta}_h, u_h) \in X_h \times V_h$ we take $\sigma_h := \boldsymbol{\theta}_h \cdot \mathbf{n}$ in the estimate (32), and then apply Lemma 4.2 to obtain

$$
\begin{aligned}
\|u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h\|_{[L^2(\Omega]^2} + \|\boldsymbol{\theta}_h \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} &\lesssim \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega]^2} \\
+ \left\|\mathbf{B}_h^{1/2} P_h^* \operatorname{div} \boldsymbol{\theta}_h\right\|_{L^2(\Omega)} &+ \left\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right]\right\|_{H^{1/2}(\Gamma)} \\
&\lesssim \|\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h\|_{[L^2(\Omega]^2} + \|\mathbf{B}_h^{1/2} P_h^* \operatorname{div} \boldsymbol{\theta}_h\|_{L^2(\Omega)} \\
&+ \left\|Q_{h'}^* \mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right]\right\|_{H^{1/2}(\Gamma)} \\
&+ \left\|(I - Q_{h'}^*)\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right]\right\|_{H^{1/2}(\Gamma)} .
\end{aligned} \tag{35}
$$

Next, using the approximation and inverse properties of the subspaces involved, we get

$$
\begin{aligned}
&\left\|(I - Q_{h'}^*)\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right]\right\|_{H^{1/2}(\Gamma)} \\
&\lesssim (h')^\alpha \left\|\mathbf{P}_0\left[(\tfrac{1}{2}\mathbf{I} - \mathbf{K})u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right]\right\|_{H^{1/2+\alpha}(\Gamma)} \\
&\lesssim (h')^\alpha \left\{\|u_h\|_{H^{1/2+\alpha}(\Gamma)} + \|\boldsymbol{\theta}_h \cdot \mathbf{n}\|_{H^{-1/2+\alpha}(\Gamma)}\right\} \\
&\lesssim (h')^\alpha h^{-\alpha} \left\{\|u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta}_h \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)}\right\} .
\end{aligned} \tag{36}
$$

Therefore, replacing (36) back into (35), choosing $h' \lesssim h$ and using (31), we conclude the proof. □

The error analysis of both methods then is a standard application of the well known Second Strang Lemma.

Theorem 5.3 *The bilinear forms B_i^h, $i = 2, 3$, satisfy $B_i^h\big((\boldsymbol{\theta}_h, u_h, \sigma_h), (\boldsymbol{\theta}_h, u_h, \sigma_h)\big) \sim \|(\boldsymbol{\theta}_h, u_h, \sigma_h)\|_{\mathbf{X}_i}^2$ and the following convergence estimate holds in both cases*

$$
\|u - u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{L^2(\Omega)} + \|\sigma - \sigma_h\|_{H^{-1/2}(\Gamma)} \lesssim h^{d-1}\|u\|_{H^d(\Omega)} .
$$

In addition, there exists $h' \lesssim h$ such that the bilinear form B_4^h satisfies

$$
B_4^h\big((\boldsymbol{\theta}_h, u_h), (\boldsymbol{\theta}_h, u_h)\big) \gtrsim \|u_h\|_{H^1(\Omega)}^2 + \|\boldsymbol{\theta}_h\|_{L^2(\Omega)}^2 + \|\boldsymbol{\theta}_h \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} ,
$$

and the following convergence estimate holds

$$
\|u - u_h\|_{H^1(\Omega)} + \|\boldsymbol{\theta} - \boldsymbol{\theta}_h\|_{L^2(\Omega)} + \|\boldsymbol{\theta} \cdot \mathbf{n} - \boldsymbol{\theta}_h \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \lesssim h^{d-1}\|u\|_{H^d(\Omega)} .
$$

6 Wavelet bases and related matrices

In the framework of the present least squares methods we would like to recommend the use of wavelet bases at least for the discretization of the boundary integral operators. Wavelet bases facilitate the computation of the Sobolev norms. In fact, one can exploit several features simultaneously, namely the computation of the half integer Sobolev norms [15], the preconditioning [14], [36], together with a sparse discretization by matrix compression [16], [36], [38], and the use of wavelet bases for an adaptive approximation [11]. The matrix compression accelerates computation with the boundary element matrices enormously. In fact, it reduces the quadratic complexity dealing with full matrices of size N to order N or $N \log^a N$, cf. [36]. This might be not a major concern for two dimensional problems, since the finite element part already has N^2 unknowns. But for three-dimensional problems the complexity of the boundary element part would dominate that of the finite element part. Therefore, fast methods for boundary integral equations become necessary when dealing with very large systems of integral equations [28].

Wavelet bases and particular wavelet bases for boundary integral equations is by now a well-studied notion. There are many excellent accounts about wavelets in general, and for boundary integral equations we refer the reader to the survey paper [14] and the references therein. Here we focus only on those aspects which are important for the present purpose. Particularly, more information about wavelet least squares methods are contained in [15].

In general, a multiresolution analysis consists of a nested family of finite dimensional subspaces

$$S_0 \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots$$

where e.g. $\bigcup_{j\geq 0} S_j$ is supposed dense in $L^2(\Gamma)$. For example, we may consider $S_j = S_h$ with $h \sim 2^{-j}$ and where $\bigcup_{j\geq 0} S_j$ is dense in $H^{-1/2}(\Gamma)$.

Each space S_j is defined by a single-scale basis, i.e. $S_j = \operatorname{span}\{\varphi_k^j : k \in \Delta_j\}$, where Δ_j denotes a suitable index set with cardinality $\#\Delta_j \sim 2^{nj}$. These basis functions might be classical piecewise constant or piecewise linear basis functions for boundary element methods. The wavelets $\Psi^j = \{\psi_k^j : k \in \nabla_j = \Delta_{j+1} \setminus \Delta_j\}$ are the bases of complementary spaces $W_j = \operatorname{span}\{\psi_k^j : k \in \nabla_j\}$ of S_j in S_{j+1}, i.e.

$$S_{j+1} = S_j \oplus W_j, \qquad S_j \cap W_j = \{0\}.$$

In the sequel we adhere the following short hand notation. We write $\psi_k^{-1} := \varphi_k^0$ and $\nabla_{-1} := \Delta_0$. By Ψ_j we denote the (column-) vector $\Psi_j = (\psi_k^l)_{k\in\nabla_l, -1\leq l<j}$. For a given vector $\mathbf{v} \in I\!R^{\#\Delta_j}$ we write simply

$$\Psi_j^T \mathbf{v} = \mathbf{v}^T \Psi_j = \sum_{l=-1}^{j-1} \sum_{k\in\nabla_l} v_{l,k} \psi_k^l .$$

It is supposed that the collection Ψ_j builds a uniformly stable basis of S_{j+1} and a Riesz-basis in L^2. This property is guaranteed if there exists a biorthogonal, or dual, collection $\tilde{\Psi} = \{\tilde{\psi}_k^l : k \in \nabla_l,\ l \geq -1\}$ generating spaces $\tilde{S}_0 \subset \ldots \subset \tilde{S}_j \subset \ldots$ such that $\langle \tilde{\psi}_k^j, \psi_l^i \rangle = \delta_{k,l}\delta_{i,j}$. In this case, every $v \in L^2(\Gamma)$ has the representations

$$v = \langle v, \tilde{\Psi} \rangle^T \Psi , \qquad v = \langle v, \Psi \rangle^T \tilde{\Psi} . \tag{37}$$

Then, the projectors Q_j and Q_j^* are given by

$$Q_j v \;=\; \langle v, \tilde{\Psi}_j \rangle^T \Psi_j, \qquad Q_j^* v \;=\; \langle v, \Psi_j \rangle^T \tilde{\Psi}_j \,.$$

In addition, the wavelets are supposed local on the corresponding scale. We refer to [14], [17] and [36] for further details.

Let $\gamma := \sup\{s \in I\!R : S_j \subset H^s(\Gamma)\}$ and $\tilde{\gamma}$ be defined analogously. Then, for a given function v the following norm equivalences hold

$$\|v\|^2_{H^s(\Gamma)} \sim \sum_{l \geq -1} 2^{-2ls}\, \|\langle v, \tilde{\Psi}^l \rangle\|^2, \qquad \|v\|^2_{H^{-s}(\Gamma)} \sim \sum_{l \geq -1} 2^{2ls}\, \|\langle v, \Psi^l \rangle\|^2, \tag{38}$$

where $-\tilde{\gamma} < s < \gamma$. It is important to remark that one does not need the dual basis for the computation of the norm.

To describe the application of these norm equivalences, let us take a single operator and consider $h' \leq h$ for example. Let $\Phi_{j'}$ be a wavelet basis for the traces of $V_{h'}$ on the boundary Γ. Then, we define the matrix $\mathbf{V}_{h',h} := \langle \mathbf{P}_0(\mathbf{V}\Psi_j), \Phi_{j'} \rangle$ where $h' = 2^{-j'}$ and $h = 2^{-j}$ which is nothing but a part of the Galerkin matrix for the operator $\mathbf{V}$ together with the diagonal matrix $\mathbf{D}^{-2s}_{h',h'} = \mathrm{diag}(2^{-2ls})$. For instance, we compute the $H^{1/2}(\Gamma)$-norm by setting $s = 1/2$ and obtain

$$\|Q_j \mathbf{P}_0(\mathbf{V}\Psi_j)\|^2_{H^{1/2}(\Gamma)} \sim \mathbf{c}_j^T \mathbf{V}^T_{h',h} \mathbf{D}^{-1}_{h',h'} \mathbf{V}_{h',h} \mathbf{c}_j \,.$$

This means that the preconditioner defined in the previous section is of the following form $\mathbf{D}_h u = \Psi_j \mathbf{D}^{-1}_{h,h} \langle \Psi_j, u \rangle$. Similarly, the other parts of the system matrices are derived. For the combination of finite element spaces and the use of the BPX preconditioner for the computation of the $H^{-1}(\Omega)$-inner products and wavelet bases on the boundary we need to apply the wavelet transform (we refer to [28] and [29] for further details). We like to remark that the size of the matrix $\mathbf{V}^T_{h',h}\mathbf{D}^{-1}_{h',h'}\mathbf{V}_{h',h}$ is already $\sim 2^{jn} \times 2^{jn}$ and can be sparsified by wavelet matrix compression.

Remark. Wavelets on surfaces are defined e.g. in [17] and [18]. The first construction in [17] seems to be simpler than the final one in [18]. Since in [17] the duality is based on a modified inner product $< \cdot, \cdot >$ defined via the local parametrizations, a comment about the use of this construction is required for the correct utilization of these bases computing $H^{1/2}(\Gamma)$-inner products according to (38). Instead of using the inner products $\langle f, \psi^j_k \rangle$ one has to use the modified inner product $< \cdot, \cdot >$, whereas for the computation of the $H^{-1/2}(\Gamma)$ inner products one has to use the canonical inner product $\langle \cdot, \cdot \rangle$.

Remark. A major restriction of the present approach is that the traces along the boundary Γ of the spaces V_h must also admit a multiresolution analysis. This restriction can be removed by introducing an additional unknown $\mu \in H^{1/2}(\Gamma)$ for the traces of u along Γ like in [10]. Here μ will be discretized by wavelet bases. This means that the coupling is defined by a slightly weaker condition, see [10]. The generalization of the present method to this case is rather straightforward.

7 Numerical Computations

In this section, we show how to compute the corresponding system matrices and right hand sides for the minimization of the functional $\mathbf{J}_3$ and present some numerical results. The energy space of $\mathbf{J}_3$ is $\mathbf{X}_3 = [L^2(\Omega)]^2 \times H^1_{\Gamma_D}(\Omega) \times H_0^{-1/2}(\Gamma)$. For a conforming discretization this requires only $\boldsymbol{\zeta}_h \in X_h \subset [L^2(\Omega)]^2$, which, for instance, allows the functions in X_h to be discontinuous. In our tests we use both, piecewise constant functions and continuous piecewise linear functions, subordinated to the triangulation $\mathcal{T}_h$. The trial functions $u_h \in V_h \subset H^1_{\Gamma_D}(\Omega)$ are chosen piecewise linear and continuous and $\sigma_h \in S_h \subset H^{-1/2}(\Gamma)$ consists of piecewise constant functions.

It is worthwhile to describe the present realization of that least squares method more detailed. Abbreviating

$$g(\boldsymbol{\theta}_h, u_h, \sigma_h) := \operatorname{div} \boldsymbol{\theta}_h - \delta_\Gamma \otimes (\boldsymbol{\theta}_h \cdot \mathbf{n}) - \left(\delta_\Gamma \otimes \left[\mathbf{W} u_h - (\tfrac{1}{2}\mathbf{I} - \mathbf{K}')\sigma_h\right]\right)$$

the discrete bilinear form $B_3^h : \mathbf{X}_3^h \times \mathbf{X}_3^h \to I\!R$ is defined by

$$\begin{aligned} B_3^h\big((\boldsymbol{\theta}_h, u_h, \sigma_h), (\boldsymbol{\zeta}_h, v_h, \tau_h)\big) \;:=\; & (\mathbf{a}\nabla u_h - \boldsymbol{\theta}_h, \mathbf{a}\nabla v_h - \boldsymbol{\zeta}_h)_{[L^2(\Omega)]^2} \\ & + \big(\mathbf{B}_h P_h^* \, g(\boldsymbol{\theta}_h, u_h, \sigma_h), P_h^* \, g(\boldsymbol{\zeta}_h, v_h, \tau_h)\big)_{L^2(\Omega)} \\ + \langle \mathbf{D}_h Q_h^* \, \mathbf{P}_0 \left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right) u_h + \mathbf{V}(\boldsymbol{\theta}_h \cdot \mathbf{n})\right], & \, Q_h^* \, \mathbf{P}_0 \left[\left(\tfrac{1}{2}\mathbf{I} - \mathbf{K}\right) v_h + \mathbf{V}(\boldsymbol{\zeta}_h \cdot \mathbf{n})\right]\rangle_{L^2(\Gamma)} \,, \end{aligned}$$

and the linear functional $\mathbf{G}_3^h : \mathbf{X}_3^h \to I\!R$ is given by

$$\mathbf{G}_3^h(\boldsymbol{\zeta}_h, v_h, \tau_h) \;:=\; \big(P_h^* \, f, P_h^* \, g(\boldsymbol{\zeta}_h, v_h, \tau_h)\big)_{L^2(\Omega)} \,.$$

Let us denote by Φ_h the vector of basis functions $\phi_k^h \in V_h$, Θ_h consists of the basis functions in X_h and Λ_h indicates the vector of basis functions in S_h. To built up the system matrix and the right hand side for the corresponding least squares method we require the following matrices and vectors

$$\mathbf{A}_h \;:=\; (\mathbf{a}\nabla\Phi_h, \nabla\Phi_h)_{[L^2(\Omega)]^2} \,, \qquad \mathbf{F}_h \;:=\; (\nabla\Phi_h, \Theta_h)_{[L^2(\Omega)]^2} \,,$$

$$\mathbf{G}_h \;:=\; (\Theta_h, \Theta_h)_{[L^2(\Omega)]^2} \,, \qquad \mathbf{f}_h \;:=\; (f, \Phi_h)_{[L^2(\Omega)]^2} \,,$$

together with the matrices of basis functions belonging to the interface boundary

$$\mathbf{V}_h \;:=\; \langle \mathbf{V}\Lambda_h, \Lambda_h \rangle \,, \qquad \mathbf{K}_h \;:=\; \langle \mathbf{K}\Phi_h, \Lambda_h \rangle \,,$$

$$\mathbf{W}_h \;:=\; \langle \mathbf{W}\Phi_h, \Phi_h \rangle \,, \qquad \mathbf{I}_h \;:=\; \langle \Phi_h, \Lambda_h \rangle \,.$$

We use the matrices $\mathbf{C}_h$ to define the inner product in $H^{-1/2}(\Gamma)$ and $\mathbf{B}_h$ for the computation of the inner product in $H^{-1}(\Omega)$. We choose $\mathbf{B}_h$ as a BPX preconditioner [7] and $\mathbf{C}_h := (\text{diag}\mathbf{V}_h)^{-1}$, where $\mathbf{V}_h = \langle \mathbf{V}\Psi_h, \Psi_h \rangle$, is given with respect to a wavelet basis Ψ_h of S_h. Then, the corresponding linear system for the present least squares method can be written in the following form

$$\left\{ \begin{bmatrix} \mathbf{G}_h & -\mathbf{F}_h^T & \mathbf{0} \\ -\mathbf{F}_h & \mathbf{A}_h & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{F}_h^T \\ \mathbf{W}_h \\ (\mathbf{K}_h - \frac{1}{2}\mathbf{I}_h) \end{bmatrix} \mathbf{B}_h \left[\mathbf{F}_h \quad \mathbf{W}_h \quad (\mathbf{K}_h - \tfrac{1}{2}\mathbf{I}_h)^T \right] \right.$$

$$\left. + \begin{bmatrix} \mathbf{0} \\ (\frac{1}{2}\mathbf{I}_h - \mathbf{K}_h)^T \\ \mathbf{V}_h \end{bmatrix} \mathbf{C}_h \left[\mathbf{0} \quad (\tfrac{1}{2}\mathbf{I}_h - \mathbf{K}_h) \quad \mathbf{V}_h \right] \right\} \begin{bmatrix} \boldsymbol{\theta}_h \\ \mathbf{u}_h \\ \boldsymbol{\sigma}_h \end{bmatrix} = - \begin{bmatrix} \mathbf{F}_h \\ \mathbf{W}_h \\ (\mathbf{K}_h - \frac{1}{2}\mathbf{I}_h) \end{bmatrix} \mathbf{B}_h \mathbf{f}_h .$$

This system is preconditioned by the operator $\operatorname{diag}(\mathbf{Id}, \mathbf{B}_h, \mathbf{C}_h)$. We remark that $P_h^*\big(\operatorname{div} \boldsymbol{\zeta}_h - \delta_\Gamma \otimes (\boldsymbol{\zeta}_h \cdot \mathbf{n})\big)$ is computed from the inner products

$$\big(\operatorname{div} \boldsymbol{\zeta}_h - \delta_\Gamma \otimes (\boldsymbol{\zeta} \cdot \mathbf{n}), v_h\big)_{L^2(\Omega)} = -(\boldsymbol{\zeta}, \nabla v_h)_{[L^2(\Omega)]^2} \quad \forall\, v_h \in V_h \,.$$

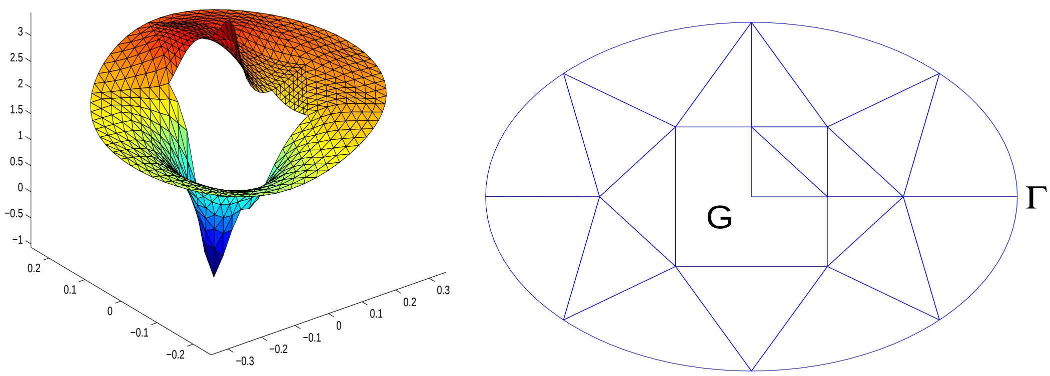

Figure 1: The solution u and the initial triangulation of Ω.

For the numerical tests we choose G as the annulus outside the two dimensional L-shape $\left[-\frac{1}{10}, \frac{1}{10}\right]^2 \setminus \left[0, \frac{1}{10}\right]^2$ and inside an ellipse. Similiar to [29] we consider a problem for which an analytical solution is known. We split

$$u(x,y) = u_1(x,y) + u_2(x,y) \in C^2\Big(I\!R \setminus \Big[\begin{smallmatrix} -1/20 \\ 0 \end{smallmatrix}\Big]\Big)$$

with the harmonical function

$$u_1(x,y) = \frac{1}{100} \cdot \frac{(x + \frac{1}{20}) + y}{(x + \frac{1}{20})^2 + y^2} \in C^\infty\Big(I\!R \setminus \Big[\begin{smallmatrix} -1/20 \\ 0 \end{smallmatrix}\Big]\Big)$$

and the nonharmonical function $u_2 \in C^2(I\!R)$ defined by

$$u_2(x,y) = 2 + \begin{cases} \left(\frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} - 1\right)^3, & \text{if } \frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} \le 1, \\ 0, & \text{if } \frac{x^2}{0.3^2} + \frac{y^2}{0.2^2} > 1. \end{cases}$$

The function $f := -\triangle u_2 \in C^1(I\!R)$ is supported in the ellipse with semiaxis 0.3 and 0.2. Thus, setting $g := u|_{\partial G}$ we obtain a boundary value problem with nonhomogeneous Dirichlet data at the boundary $\Gamma_D = \partial G$. The interface boundary Γ is chosen as the boundary of the ellipse with semiaxis 0.35 and 0.25. The solution u and the initial triangulation using curved triangles is shown in Figure 1.

We depict in Figure 2 the errors with respect to the energy norm using piecewise constant and continuous piecewise linear functions, respectively, for the approximation of the flux $\boldsymbol{\theta}_h$. In Figure 3 one finds the corresponding errors with respect to the L^2-norms. Note that we use double logarithmical scales.

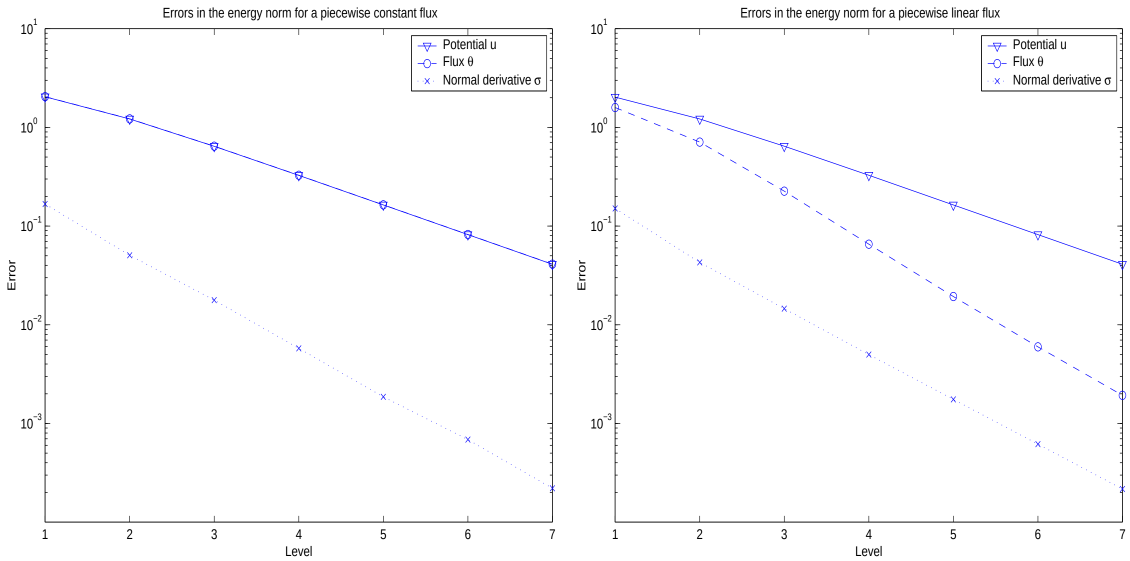

Figure 2: Error in the energy norm

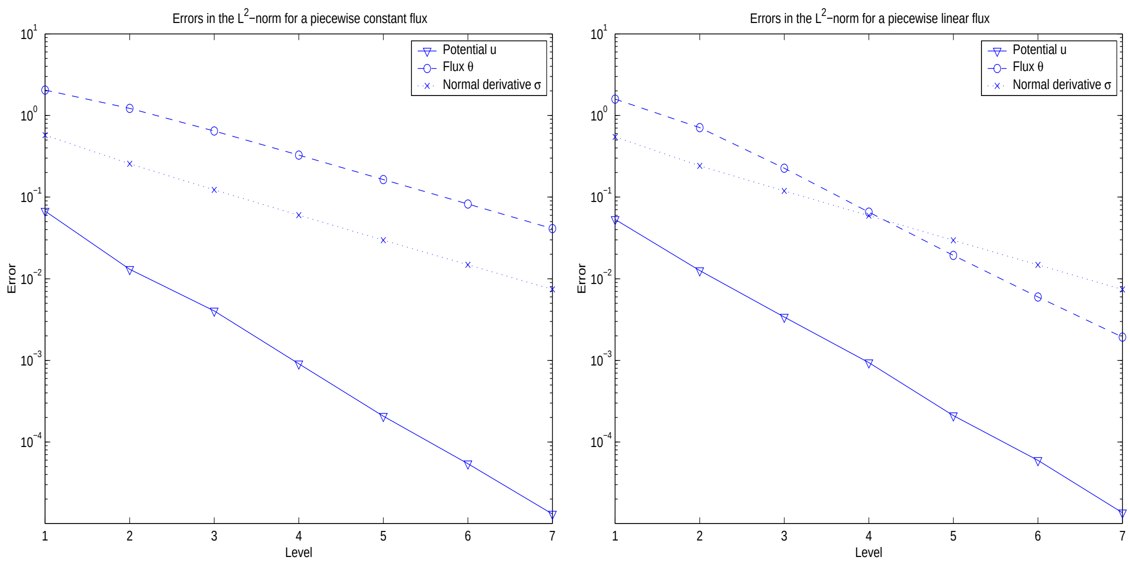

Figure 3: Error in L^2-norms

In the previous sections, we have already proved convergence estimates with respect to the energy norm. The numerical experiments confirm the claimed convergence rate $\mathcal{O}(h)$. This does not include L^2-estimates for the potential u. But it can be observed that the potential u converges in L^2 with the order h^2 which is optimal for piecewise linear functions. The measured convergence rate for flux in L^2 is h^1 for the piecewise constant approximation. With respect to continuous piecewise linear functions it seems to be between $h^{3/2}$ and h^2. However, we have not proved these types of convergence rates. But we mention that the application of the Aubin Nitzsche trick is limited due to the concave vertices of Ω. Obviously, we observe a better approximation of the flux when using piecewise linear functions.

It is confirmed by our experience that the expenses for the boundary integral part is small compared to the finite element part if the integral equations are treated by fast methods. Therefore, the efficiency of the present algorithm is comparable to the efficiency of corresponding finite element least squares methods for interior boundary value problems. Wavelet methods are proved to be an efficient tool for the treatment of boundary integral operators in the coupling. Our approach requires the same boundary element matrices as the FEM-BEM coupling for the second order system.

References

[1] Aziz, A.K., Kellog, R.B. and Stephens, A.B.: Least-squares methods for elliptic systems. *Mathematics of Computation* **44**, pp. 53-70, (1985).

[2] Barrenechea, G.R., Gatica, G.N. and Hsiao, G.C.: Weak solvability of interior transmission problems via mixed finite elements and Dirichlet-to-Neumann mappings. *Journal of Computational and Applied Mathematics* **100**, 2, pp. 145-160, (1998).

[3] Barrenechea, G.R., Gatica, G.N. and Thomas, J.-M.: Primal mixed formulations for the coupling of FEM and BEM. Part I: linear problems. *Numerical Functional Analysis and Optimization* **19**, 1&2, pp. 7-32, (1998).

[4] Barrientos, M.A., Gatica, G.N. and Stephan, E.P.: A mixed finite element method for nonlinear elasticity: two-fold saddle point approach and a-posteriori error estimate. To appear in *Numerische Mathematik*.

[5] Bramble, J., Lazarov, R.D. and Pasciak, J.E.: A least-squares approach based on a discrete minus one inner product for first order systems. *Mathematics of Computation* **66**, 219, pp. 935-955, (1997).

[6] Bramble, J., Lazarov, R.D. and Pasciak, J.E.: Least-squares for second order elliptic problems. *Computer Methods in Applied Mechanics and Engineering* **152**, pp. 195-210, (1998).

[7] Bramble, J., Pasciak, J.E. and Xu, J.: Parallel multilevel preconditioners. *Mathematics of Computation* **55**, pp. 1-22, (1990).

[8] Brink, U., Carstensen, C. and Stein, E.: Symmetric coupling of boundary elements and Raviart-Thomas type mixed finite elements in elastostatics. *Numerische Mathematik* **75**, pp. 153-174, (1996).

[9] Cai, Z., Lazarov, R., Manteuffel, T. and McCormick, S.: First-order system least-squares for second-order partial differential equations: Part I. SIAM *Journal on Numerical Analysis* **31**, pp. 1785-1799, (1994).

[10] Carstensen, C., Funken, S.A.: Coupling of nonconforming finite elements and boundary elements I: a-priori estimates. *Computing*, **62**, pp. 229-241, (1999)

[11] Cohen, A., Dahmen, W. and de Vore, R.: Adaptive wavelet schemes for elliptic operator equations-convergence rates. *Mathematics of Computation*, **70**, pp. 27–75, (2001).

[12] Costabel, M.: Boundary integral operators on Lipschitz domains: Elementary results. SIAM *Journal on Mathematical Analysis* **19**, pp. 613-621, (1988).

[13] Costabel, M. and Stephan, E.P.: Coupling of finite and boundary element methods for an elasto-plastic interface problem. SIAM *Journal on Numerical Analysis* **27**, 5, pp. 1212-1226, (1990).

[14] Dahmen, W.: Stability of multiscale transformations. *Journal of Fourier Analysis and Applications* **4**, pp. 341-361, (1996).

[15] Dahmen, W., Kunoth, A. and Schneider, R.: Wavelet least square methods for boundary value problems. To appear in SIAM *Journal on Numerical Analysis.*

[16] Dahmen, W., Prössdorf, S. and Schneider, R.: Multiscale methods for pseudo-differential equations on smooth manifolds. In: *Proceedings of the International Conference on Wavelets: Theory, Algorithms, and Applications*, C.K. Chui, L. Montefusco, L. Puccio (eds.), Academic Press, 1994, pp. 385-424.

[17] Dahmen, W. and Schneider, R.: Composite wavelet bases for operator equations. *Mathematics of Computation* **68**, pp. 1533–1567, (1999).

[18] Dahmen, W. and Schneider, R.: Wavelets on manifolds I. Construction and domain decomposition. SIAM *Journal on Mathematical Analysis* **31**, pp. 184–230, (1999).

[19] Gatica, G.N.: Solvability and Galerkin approximations of a class of nonlinear operator equations. *Technical Report* 99-03, Departamento de Ingeniería Matemática, Universidad de Concepción, (1999).

[20] Gatica, G.N.: Combination of mixed finite element and Dirichlet to Neumann methods in nonlinear plane elasticity. *Applied Mathematics Letters* **10**, pp. 29-35, (1997).

[21] Gatica, G.N. and Heuer, N.: A dual-dual formulation for the coupling of mixed-FEM and BEM in hyperelasticity. SIAM *Journal on Numerical Analysis* **38**, 2, pp. 380-400, (2000).

[22] Gatica, G.N. and Hsiao, G.C.: *Boundary-Field Equation Methods for a Class of Nonlinear Problems.* Pitman Research Notes in Mathematics Series, vol. 331, Longman, 1995.

[23] Gatica, G.N. and Meddahi, S.: An a-posteriori error estimate for the coupling of BEM and mixed-FEM. *Numerical Functional Analysis and Optimization* **20**, pp. 449-472, (1999).

[24] Gatica, G.N. and Meddahi, S.: A dual-dual mixed formulation for nonlinear exterior transmission problems. *Mathematics of Computation* **70**, 236, pp. 1461-1480, (2001).

[25] Gatica, G.N. and Wendland, W.L.: Coupling of mixed finite elements and boundary elements for a hyperelastic interface problem. SIAM *Journal on Numerical Analysis* **34**, 6, pp. 2335-2356, (1997).

[26] Girault, V. and Raviart, P.-A.: *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms.* Springer Verlag, 1986.

[27] Han, H.: A new class of variational formulations for the coupling of finite and boundary element methods. *Journal of Computational Mathematics* **8**, pp. 223–232, (1990).

[28] Harbrecht, H., Perez, C., Paiva, F. and Schneider, R.: Biorthogonal wavelet approximation for the coupling of FEM-BEM. *Technical Report* 99-24, Departamento de Ingeniería Matemática, Universidad de Concepción, (1999). To appear in *Numerische Mathematik*

[29] Harbrecht, H., Perez, C., Paiva, F. and Schneider, R.: Multiscale preconditioning for the coupling of FEM-BEM. *Technical Report* 00-04, Departamento de Ingeniería Matemática, Universidad de Concepción, (2000).

[30] Heuer, N., Stephan, E.P. and Tran,T.: Multilevel additive Schwarz method for the *h-p* version of the Galerkin boundary element method. *Mathematics of Computation* **67**, 501–518, (1998).

[31] Hsiao, G.C.: The coupling of boundary element and finite element methods. A plenary lecture at the GAMM-Tagung 1989. *Zeitschrift fur Angewandte Mathematik und Mechanics* (ZAMM) **70**, 6, pp. 493-503, (1990).

[32] Jespersen, D.C.: A least-square decomposition method for solving elliptic systems. *Mathematics of Computation* **31**, pp. 873-880, (1977).

[33] Kress, R.: *Linear Integral Equations.* Springer Verlag, 1989.

[34] Meddahi, S., Valdés, J., Menéndez, O. and Pérez, P.: On the coupling of boundary integral and mixed finite element methods. *Journal of Computational and Applied Mathematics* **69**, pp. 113-124, (1996).

[35] Pehlivanov, A.I., Carey, G.F. and Lazarov, R.D.: Least-squares mixed finite elements for second order elliptic problems. SIAM *Journal on Numerical Analysis* **31**, pp. 1368-1377, (1994).

[36] Schneider, R.: *Multiskalen und Wavelet-Matrixkompression: Analysisbasierte Methoden zur Lösung Großer Vollbesetzter Gleichungssysteme. Advances in Numerical Mathematics*, Teubner Stuttgart, (1998) (in German).

[37] Stephan, E.P.: Coupling of finite elements and boundary elements for some nonlinear interface problems. *Computer Methods in Applied Mechanics and Engineering* **101**, pp. 61-72, (1992).

[38] von Petersdorff, T., Schneider, R. and Schwab, C.: Multiwavelets for second kind integral equations. SIAM *Journal on Numerical Analysis* **34**, pp. 2212-2227, (1997).

[39] Wendland, W.L.: *Elliptic Systems in the Plane.* Pitman, London, 1979.