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Nitsche mortar finite element method for transmission problems with singularities

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Nitsche mortar finite element method for the transmission problem with singularities

B. Heinrich S. Nicaise

Abstract

The paper deals with Nitsche type finite element method (FEM) for treating non-matching meshes at the interface of some domain decomposition. This method is applied to some transmission (or interface) problems of the plane with Dirichlet boundary conditions presenting some corner singularities. In a natural way, the interface of the non-matching grids is taken as the interface of the problem. Properties of the finite element scheme and error estimates are given. For appropriate graded meshes, optimal convergence rates are obtained as for the classical FEM with regular solutions. Some numerical tests illustrate the approach and confirm the theoretical results.

Key words. finite element method, non-matching meshes, mortar finite elements, corner singularities, transmission problems, Nitsche type mortaring

AMS subject classification. 65N30, 65N55

1 Introduction

Domain decomposition methods are widely used for an efficient numerical treatment of boundary value problems (BVPs). They allow to work in parallel: for the generation of the meshes in subdomains, the calculation of the corresponding parts of the stiffness matrix and of the right-hand side, and the resolution of the system of the finite element equations.

Triangulations which do not match at the interface of the subdomains are of particular interest. Indeed such non-matching meshes arise, for example, if the meshes in different subdomains are generated independently, or if an adaptive remeshing in some subdomain is of primary interest. This is often caused by a large jump of the data (material properties or right-hand sides) of the BVP in different subdomains or by a complicated geometry of the domain, the first property also leads to solutions with singular or anisotropic behaviour. Non-matching meshes also appear if different discretization approaches are used in different subdomains.

There are several ways to manage non-matching meshes. In order to satisfy the continuities on the interface (e.g. continuity of the solution and of its conormal derivative) we may use iterative procedures (e.g. Schwarz's method) or direct methods like the Lagrange multiplier technique.

In the Lagrange multiplier mortar technique, see e.g. [5, 6, 9, 29, 30] and the literature quoted in these papers, new unknowns (the Lagrange multipliers) are introduced. Therefore the stability of the problem has to be ensured by checking an inf-sup condition (for the actual mortar method) or using a stabilization technique.

Another approach which is of particular interest here is related to the classical Nitsche method [20] of treating essential boundary conditions. This approach has been worked out more generally in [27, 24] and transferred to interior continuity conditions by Stenberg [25] (Nitsche type mortaring), cf. also [1]. As shown in [4, 10], the Nitsche type mortaring can be interpreted as a stabilized variant of the mortar method based on a saddle point problem. Compared with the classical mortar method, the Nitsche type mortaring method has several advantages. Namely the saddle point problem, the inf–sup–condition as well as the calculation of the Lagrange multipliers are circumvented. The method employs only a single variational equation which is, compared with the usual equations (without any mortaring), slightly modified by an interface term. This allows to apply existing software tools by slight modifications. Moreover, the Nitsche type method yields symmetric and positive definite discretization matrices if the BVP has these properties. Although the approach involves a stabilizing parameter γ, but is not a penalty method since it is consistent with the solution of the BVP. Note further that the parameter γ can be easily estimated (see below).

Basic aspects of the Nitsche type mortaring and error estimates for regular solutions $u \in H^k(\Omega)$ ($k \geq 2$) on quasi-uniform meshes may be found in [25, 4]. The extension of this method to the Laplace equation with non-regular solutions and locally refined meshes, which are not quasi-uniform, is given in [16].

Here we consider transmission problems with Dirichlet boundary conditions and presenting corner singularities. Naturally we take as interface of our non-matching meshes the interface of the problem. For the appropriate treatment of corner singularities we use meshes locally refined around the singular corner, the grading parameter being related to the importance of the singularity. For meshes with and without grading, basic inequalities, stability and boundedness of the bilinear form as well as error estimates in a discrete H^1-norm are proved using new trace estimates in weighted Sobolev spaces. Adapting the standard Aubin-Nitsche trick we further show that the rate of convergence in L_2 is twice of that in the H^1-norm. For an appropriate choice of the grading parameter, the rate of convergence is optimal as in the case of regular solutions on quasi-uniform meshes. Finally, some numerical experiments are given which confirm the theoretical rates of convergence.

2 Some preliminaries

In the following, for a domain X and a real number s, we denote by $H^s(X)$, the usual Sobolev space, with the corresponding norm $\| \, . \, \|_{s,X} := \| \, . \, \|_{H^s(X)}$. As usual if $s = 0$, we write $H^0 = L_2$. In the sequel constants C or c occuring in the inequalities are generic constants. We sometimes write $a \cong b$ if $ca \leq b \leq Ca$.

Let us fix a bounded polygonal domain Ω of $\mathbb{R}^2$, with a Lipschitz-boundary $\partial\Omega$ consisting of straight line segments. We suppose that Ω is decomposed into two non-overlapping subdomains (only for the sake of simplicity) Ω_1 and Ω_2 with an interface Γ satisfying

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma.$$

We assume that the boundaries $\partial\Omega_i$ of Ω_i ($i = 1, 2$) are also Lipschitz-continuous and formed by open straight line segments Γ_j such that

$$\Gamma \;=\; \bigcup_{j=1}^{J} \overline{\Gamma}_j.$$

We distinguish two important types of interfaces Γ:

case 1: the intersection $\Gamma \cap \partial\Omega$ consists of two points P_1, P_2 ($P_1 \neq P_2$) being the endpoints of Γ, like in Figure 1,

case 2: $\Gamma \cap \partial\Omega = \emptyset$, i.e., Γ does not touch the boundary $\partial\Omega$, like in Figure 2.

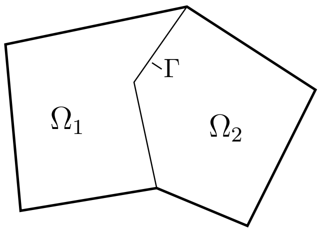

Figure 1:

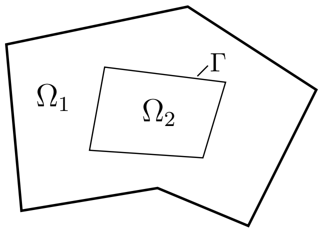

Figure 2:

We further fix two positive constants p_1, p_2 and define the function p on Ω as follows:

$$p|_{\Omega_i} = p_i, \ i = 1, 2.$$

For the sake of shorthness, if a function v is defined in Ω, we shall denote by v^i its restriction to Ω_i, $i = 1, 2$.

We now consider the transmission problem with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\text{ div } (p\nabla u) &= f && \text{in } \Omega\,, \\ u &= 0 && \text{on } \partial\Omega\,, \end{aligned} \tag{2.1}$$

where f is supposed to be in $L_2(\Omega)$. The variational formulation of (2.1) is as follows. Find $u \in H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ such that

$$a(u, v) \ = \ f(v) \qquad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

$$\text{with } \ a(u, v) \ := \ \int_\Omega p\, (\nabla u, \nabla v)\; dx\,, \qquad f(v) \ := \ \int_\Omega f v\, dx.$$

For the error analysis of our method we need to describe the regularity of the weak solution u of (2.2) whose existence and uniqueness directly follows from Lax-Milgram's lemma with the a priori estimate $\|u\|_{1,\Omega} \leq C\,\|f\|_{0,\Omega}$. For $p_1 = p_2$ the standard regularity theory of (2.2) yields $u \in H^2(\Omega)$ and $\|u\|_{2,\Omega} \leq C\,\|f\|_{0,\Omega}$ if Ω is convex. On the contrary u has a singular behaviour near the corner points of Ω_i [13, 19], that may be described as follows (see [13, 19] for the details): Assume that $\partial\Omega$ has convex corners except the corners belonging to Γ and that $1 \notin \Lambda_j$, for all $j \in \{0, \cdots, J\}$, then u admits the representation

$$u \ = \ w + \sum_{j=0}^{J} \eta_j \sum_{\lambda_{j,k} \in \Lambda_j} a_j\, r_j^{\lambda_{j,k}}\, \phi_{j,k}(\varphi_j), \tag{2.3}$$

where w is the so-called regular part which belongs to $PH^2(\Omega) := \{w \in H^1(\Omega) : w_i \in H^2(\Omega_i), i = 1, 2\}$. Here, (r_j, φ_j) denote local polar coordinates of center $P_j = \bar{\Gamma}_j \cap \bar{\Gamma}_{j+1}$. The set Λ_j is the set of singular exponents $\lambda_{j,k}$ at P_j in $]0, 1[$, which are characterized as the zeroes of an analytic function (which has only real zeroes), $\phi_{j,k}$ is the associated eigenvalue which is piecewise smooth (in φ_j) and η_j is a locally acting (smooth) cut-off function around the vertex P_j. Moreover we have the estimate

$$\sum_{i=1,2} \|w_i\|_{2,\Omega_i} + \sum_{j=0}^{J} \sum_{\lambda_{j,k}\in\Lambda_j} |a_{j,k}| \leq C||f||_{0,\Omega}. \tag{2.4}$$

The above decomposition (2.3) implies that our solution belongs to some weighted Sobolev spaces that we recall hereafter: For $l = 1$ or 2 and $\alpha \geq 0$, let us denote by

$$PH^{l,\alpha}(\Omega) = \{v \in L_2(\Omega) : v^i \in H^{l-1}(\Omega_i), r^\alpha D^\beta v^i \in L_2(\Omega_i), \forall|\beta| = l, i = 1,2\},$$

where r is the distance to the points P_j, $j = 0, \cdots, J$. It is clearly a Hilbert space for the norm

$$||v||_{l,\alpha} = \left(\sum_{i=1,2} ||v^i||^2_{l-1,\Omega_i} + \sum_{|\beta|=l} ||r^\alpha D^\beta v_i||^2_{0,\Omega_i}\right)^{1/2}.$$

We can now state the

Theorem 2.1. *Let $u \in H^1_0(\Omega)$ be the unique solution of (2.2) with $f \in L_2(\Omega)$. If $1 \notin \Lambda_j$, for all $j \in \{0, \cdots, J\}$, then*

$$u \in PH^{2,\alpha}(\Omega),$$

for all $\alpha \in [0, 1)$ satisfying

$$\alpha > 1 - \lambda_{j,k}, \forall\lambda_{j,k} \in \Lambda_j, j = 0, \cdots, J.$$

Moreover one has

$$||u||_{2,\alpha} \leq C||f||_{0,\Omega}. \tag{2.5}$$

Proof. It suffices to check that each singular function $\eta_j r_j^{\lambda_{j,k}} \phi_{j,k}(\varphi_j)$ belongs to $PH^{2,\alpha}(\Omega)$ for $\alpha > 1 - \lambda_{j,k}$.

The estimate (2.5) follows directly from (2.4) and the above regularity of the singular function. □

In the context of our domain decomposition method the solution of the problem (2.1) is equivalent seen as the solution of the following interface problem: Find (u^1, u^2) such that

$$\begin{aligned} -p_i\Delta u^i &= f^i \quad \text{in } \Omega_i\,, && i = 1,2\,,\\ u^i &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega\,, && i = 1,2\,,\\ u^1 &= u^2 \quad \text{on } \Gamma\,,\\ p_1\frac{\partial u^1}{\partial n_1} + p_2\frac{\partial u^2}{\partial n_2} &= 0 \quad \text{on } \Gamma\,, \end{aligned} \tag{2.6}$$

where n_i $(i = 1, 2)$ denotes the outward normal vector along $\partial\Omega_i \cap \Gamma$. If we introduce the spaces V^i $(i = 1, 2)$ by

$$V^i := \left\{v^i \in H^1(\Omega_i) : v^i\big|_{\partial\Omega\cap\partial\Omega_i} = 0\right\}, \tag{2.7}$$

(note that in the case 2 $V^i = H^1(\Omega_i)$ for $\partial\Omega \cap \partial\Omega_i = \emptyset$) and the space $V := V^1 \times V^2$, then the problem (2.6) has to be interpretated in the weak form (see e.g. [2]): $u = (u^1, u^2) \in V$ with $u^i \in H^1(\Delta, \Omega_i) := \{v \in H^1(\Omega_i) : \Delta v \in L_2(\Omega_i)\}$ $(i = 1, 2)$, satisfies $-p_i \Delta u^i = f^i$ in $L_2(\Omega_i)$, the continuity condition $u_1 = u_2$ on Γ is required in the sense of $H_*^{\frac{1}{2}}(\Gamma)$ while the last condition is in the sense of $H_*^{-\frac{1}{2}}(\Gamma)$ (the dual space of $H_*^{\frac{1}{2}}(\Gamma)$). Here we define $H_*^{\frac{1}{2}}(\Gamma)$ (also written $H_{00}^{\frac{1}{2}}(\Gamma)$) as the trace space of $H_0^1(\Omega)$ provided with the quotient norm, see e.g. [9, 14]. So in case 2, we have $H_*^{\frac{1}{2}}(\Gamma) = H^{\frac{1}{2}}(\Gamma)$. By $\langle . , . \rangle_\Gamma$ we shall denote the duality pairing of $H_*^{-\frac{1}{2}}(\Gamma)$ and $H_*^{\frac{1}{2}}(\Gamma)$.

3 Non-matching finite element discretization

Let us fix a positive constant γ (to be specified subsequently) and some real parameters α_1, α_2 satisfying

$$0 \le \alpha_i \le 1 \quad (i = 1, 2), \qquad \alpha_1 + \alpha_2 = 1.$$

Let $\mathcal{T}_h^i$ be a triangulation of $\overline{\Omega}_i$ $(i = 1, 2)$ made of triangles. The triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent to each other. Moreover, no compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of these triangulations, with $0 < h \le h_0$ and sufficiently small h_0. Take e.g. $h = \max\{h_T : T \in \mathcal{T}_h^1 \cup \mathcal{T}_h^2\}$, where $h_T := \operatorname{diam} T$ is the diameter of the triangle T.

Assumption 3.1.

(i) For $i = 1, 2$, it holds
$$\overline{\Omega}_i = \bigcup_{T \in \mathcal{T}_h^i} T. \tag{3.1}$$

(ii) Two arbitrary triangles $T, T' \in \mathcal{T}_h^i$ $(T \neq T', i = 1, 2)$ are either disjoint or have a common vertex, or a common edge.

(iii) The mesh in $\overline{\Omega}_i$ $(i = 1, 2)$ is shape regular, *i.e., for the diameter h_T of T and the diameter ϱ_T of the largest inscribed sphere of T, we have*

$$\frac{h_T}{\varrho_T} \le C \text{ for any } T \in \mathcal{T}_h^i, \tag{3.2}$$

where C is independent of T and h.

Clearly, the condition (3.2) implies that the angle θ at any vertex and the length h_F of any side F of the triangle T satisfy the inequalities

$$0 < \theta_0 \le \theta \le \pi - \theta_0, \qquad \varepsilon_1 h_T \le h_F \le h_T, \quad (0 < \varepsilon_1 < 1),$$

with constants θ_0 and ε_1 being independent of h and T. Note that (3.2) does not imply the quasi-uniformity of the triangulation $\mathcal{T}_h^i$ $(i = 1, 2)$.

Consider further some triangulation $\mathcal{E}_h$ of Γ by intervals E $(E = \overline{E})$, i.e. $\Gamma = \bigcup_{E \in \mathcal{E}_h} E$, and denote by h_E the diameter of E. A natural choice for the triangulation $\mathcal{E}_h$ of Γ is $\mathcal{E}_h := \mathcal{E}_h^1$ if $\alpha_1 = 1$ or $\mathcal{E}_h := \mathcal{E}_h^2$ if $\alpha_2 = 1$, where

$$\mathcal{E}_h^i = \{E : E = \partial T \cap \Gamma, \text{ if } E \text{ is a segment}, T \in \mathcal{T}_h^i\}, \quad \text{for } i = 1, 2, \tag{3.3}$$

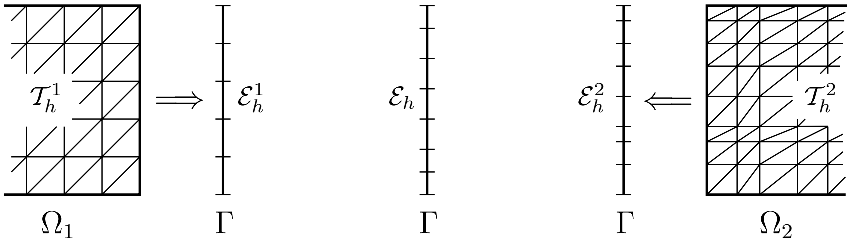

Figure 3:

cf. Figure 3.

We finally require the asymptotic behaviour of the triangulations $\mathcal{T}_h^1, \mathcal{T}_h^2$ and of $\mathcal{E}_h$ to be consistent on Γ in the sense of the following assumption.

Assumption 3.2. *For $T \in \mathcal{T}_h^i$ $(i = 1, 2)$ and $E \in \mathcal{E}_h$ with $\partial T \cap E \neq \emptyset$, there are positive constants C_1 and C_2 independent of h_T, h_E and h $(0 < h \leq h_0)$ such that the condition*

$$C_1 h_T \;\leq\; h_E \;\leq\; C_2 h_T \tag{3.4}$$

is satisfied.

Relation (3.4) guarantees that the diameter h_T of the triangle T touching the interface Γ at E is asymptotically equivalent to the diameter h_E of the segment E, but the equivalence between h_T and h_E is required only locally.

For $i = 1, 2$ we introduce the finite element space V_h^i (finite subspace of V^i from (2.7)) of functions v^i on Ω_i by

$$V_h^i \;:=\; \left\{ v^i \in H^1(\Omega_i) : v^i\big|_T \in \mathbb{P}_1(T) \;\; \forall\, T \in \mathcal{T}_h^i,\; v^i\big|_{\partial\Omega \cap \partial\Omega_i} = 0 \right\}, \tag{3.5}$$

where $\mathbb{P}_1(T)$ denotes the set of all polynomials on T with degree ≤ 1. The finite element space V_h of vectorial functions v_h with components v_h^i on Ω_i is given by

$$V_h \;:=\; V_h^1 \times V_h^2 \;=\; \left\{ v_h = (v_h^1, v_h^2) : v_h^1 \in V_h^1,\, v_h^2 \in V_h^2 \right\}. \tag{3.6}$$

In general, $v_h \in V_h$ is not continuous across Γ.

Following [25] we now introduce the bilinear form $\mathcal{B}_h(.\,,.)$ on $V_h \times V_h$ and the linear form $\mathcal{F}_h(.)$ on V_h as follows:

$$\begin{aligned}
\mathcal{B}_h(u_h, v_h) \;&:=\; \sum_{i=1}^{2} \left(p_i \nabla u_h^i, \nabla v_h^i\right)_{\Omega_i} - \left\langle \alpha_1 p_1 \frac{\partial u_h^1}{\partial n_1} - \alpha_2 p_2 \frac{\partial u_h^2}{\partial n_2}, v_h^1 - v_h^2 \right\rangle_\Gamma \\
&-\; \left\langle \alpha_1 p_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 p_2 \frac{\partial v_h^2}{\partial n_2}, u_h^1 - u_h^2 \right\rangle_\Gamma + \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \left\langle u_h^1 - u_h^2, v_h^1 - v_h^2 \right\rangle_E, \qquad (3.7) \\
\mathcal{F}_h(v_h) \;&:=\; \sum_{i=1}^{2} \left(f^i, v_h^i\right)_{\Omega_i} \qquad \text{for } u_h, v_h \in V_h.
\end{aligned}$$

Here, $(.\,,.)_{\Omega_i}$ denotes the $L_2(\Omega_i)$-inner product, $\langle .\,,. \rangle_\Gamma$ the $H_*^{-\frac{1}{2}}(\Gamma) \times H_*^{\frac{1}{2}}(\Gamma)$-duality pairing and $\langle .\,,. \rangle_E$ the $L_2(E)$-inner product.

The finite element approximation u_h of the solution u of problem (2.2) on the non-matching triangulation $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ is now defined by $u_h = \left(u_h^1, u_h^2\right) \in V_h = V_h^1 \times V_h^2$ solution of

$$\mathcal{B}_h(u_h, v_h) \;=\; \mathcal{F}_h(v_h) \qquad \forall\, v_h \in V_h. \tag{3.8}$$

4 Properties of the discretization

First we show that the solution u of the problem (2.1) satisfies the variational equation (3.8), i.e., u is consistent with the approach (3.8).

Theorem 4.1. *Let u be the solution of (2.1). Then $u = \left(u^1, u^2\right)$ satisfies*

$$\mathcal{B}_h(u, v_h) \;=\; \mathcal{F}_h(v_h) \qquad \forall\, v_h \in V_h. \tag{4.1}$$

Proof. First of all, note that $\mathcal{B}_h(u, v_h)$ is meaningful for any $v_h \in V_h$ since Theorem 2.1 and Hardy's inequalities [14, p.28] imply that $r^{\alpha-1/2}\frac{\partial u^i}{\partial n_i}$ belongs to $L_2(\Gamma)$, therefore the duality pairing

$$\left\langle \alpha_1 p_1 \frac{\partial u^1}{\partial n_1} - \alpha_2 p_2 \frac{\partial u^2}{\partial n_2}, v_h^1 - v_h^2 \right\rangle_\Gamma$$

is well defined.

Fix any $v_h \in V_h$. Since $u^1\big|_\Gamma = u^2\big|_\Gamma$ and $p_1 \frac{\partial u^1}{\partial n_1}\Big|_\Gamma = -\, p_2 \frac{\partial u^2}{\partial n_2}\Big|_\Gamma$, cf. (2.6), we get

$$\mathcal{B}_h(u, v_h) \;=\; \sum_{i=1}^{2} \left(p_i \nabla u^i, \nabla v_h^i\right)_{\Omega_i} - \left\langle p_1 \frac{\partial u^1}{\partial n_1}, v_h^1 \right\rangle_\Gamma - \left\langle p_2 \frac{\partial u^2}{\partial n_2}, v_h^2 \right\rangle_\Gamma .$$

As $u_i \in H^1(\Delta, \Omega_i)$, we may apply half Green's formula on the domains Ω_i (see Theorem 1.5.3.11 of [14]), therefore the above relation becomes

$$\mathcal{B}_h(u, v_h) \;=\; -\sum_{i=1}^{2} p_i \left(\Delta u^i, v_h^i\right)_{\Omega_i} \;=\; \sum_{i=1}^{2} \left(f^i, v_h^i\right)_{\Omega_i} \;=\; \mathcal{F}_h(v_h).$$

This proves the assertion. ◻

Note that due to (4.1) and (3.8) we also have the $\mathcal{B}_h$-orthogonality of the error $u - u_h$ on V_h, i.e.,

$$\mathcal{B}_h(u - u_h, v_h) \;=\; 0 \qquad \forall\, v_h \in V_h. \tag{4.2}$$

For further results on stability and convergence of the method, we recall the following "weighted discrete trace theorem", which describes also an inverse inequality and is proved in [16].

Lemma 4.2. *Let Assumption 3.1 and 3.2 be satisfied. Then, for any $v_h \in V_h$ the inequality*

$$\sum_{E \in \mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \;\le\; C_I^{(i)} \sum_{F \in \mathcal{E}_h^i} \left\| \nabla v_h^i \right\|_{0,T_F}^2 \tag{4.3}$$

holds, for $i = 1, 2$, where $F \in \mathcal{E}_h^i$ is the face of a triangle $T_F \in \mathcal{T}_h^i$ touching Γ by F ($T_F \cap \Gamma = F$). The constants $C_I^{(i)}$ ($i = 1, 2$) do not depend on h, h_T, h_E.

Note that extending the norms on the right-hand side of (4.3) to the whole of Ω_i implies

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \;\leq\; C_I^{(i)} \left\|\nabla v_h^i\right\|_{0,\Omega_i}^2, \quad i=1,2. \tag{4.4}$$

For inequalities on quasi-uniform meshes related with (4.4) see [27, 25, 4].

To derive the V_h-ellipticity and V_h-boundedness of the discrete bilinear form $\mathcal{B}_h(.,.)$, we introduce the following discrete norm $\|\,.\,\|_{1,h}$:

$$\|v_h\|_{1,h}^2 \;:=\; \sum_{i=1}^{2} \left\|\nabla v_h^i\right\|_{0,\Omega_i}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|v_h^1 - v_h^2\right\|_{0,E}^2 \tag{4.5}$$

cf. [25] and [9, 4] for uniform weights. Then we can prove the following theorem.

Theorem 4.3. *Let Assumptions 3.1 and 3.2 for $\mathcal{T}_h^i$ $(i=1,2)$ and for $\mathcal{E}_h$ be satisfied. Choose the constant γ in (3.7) independently of h and such that $\gamma > C_I^{(i)} p_i$ for $\alpha_i = 1$ $(i=1$ or $i=2)$, and $\gamma > 2C_I^{(i)}\alpha_i^2 p_i$ for $0<\alpha_i<1$ $(i=1,2)$, with $C_I^{(i)}$ from (4.3). Then it holds*

$$\mathcal{B}_h(v_h,v_h) \;\geq\; \mu_1 \|v_h\|_{1,h}^2 \qquad \forall\, v_h \in V_h, \tag{4.6}$$

with a constant $\mu_1 > 0$ independent of h.

Proof. From the definition (3.7) of $\mathcal{B}_h(.,.)$, we have the identity

$$\begin{aligned}\mathcal{B}_h(v_h,v_h) \quad &= \sum_{i=1}^{2} p_i \left\|\nabla v_h^i\right\|_{0,\Omega_i}^2 - 2\sum_{E\in\mathcal{E}_h} \left\langle \alpha_1 p_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 p_2 \frac{\partial v_h^2}{\partial n_2}, v_h^1 - v_h^2 \right\rangle_E \\ &\quad + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|v_h^1 - v_h^2\right\|_{0,E}^2 .\end{aligned}$$

Using Cauchy-Schwarz's inequality and Young's inequality $(2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$, for $\varepsilon > 0)$ we get

$$\begin{aligned}\mathcal{B}_h(v_h,v_h) \;\geq\; &\sum_{i=1}^{2} p_i \left\|\nabla v_h^i\right\|_{0,\Omega_i}^2 - \frac{1}{\varepsilon}\sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 p_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 p_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \\ &- \varepsilon \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|v_h^1 - v_h^2\right\|_{0,E}^2 + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|v_h^1 - v_h^2\right\|_{0,E}^2 .\end{aligned}$$

The application of inequality (4.3) yields (4.6), with $\mu_1 > 0$, if ε is chosen as follows. For $\alpha_i = 1$ $(i=1$ or $i=2)$, take $C_I^{(i)} p_i < \varepsilon < \gamma$, and for $0<\alpha_i<1$ $(i=1,2)$: $2C_I^{(i)}\alpha_i^2 p_i < \varepsilon < \gamma$. □

Beside the V_h-ellipticity of $\mathcal{B}_h(.,.)$ we also prove the V_h-boundedness.

Theorem 4.4. *Let Assumption 3.1 and 3.2 be satisfied. Then there is a constant $\mu_2 > 0$ such that*

$$\left|\mathcal{B}_h\left(w_h, v_h\right)\right| \;\leq\; \mu_2 \|w_h\|_{1,h} \|v_h\|_{1,h} \qquad \forall w_h, v_h \in V_h. \tag{4.7}$$

Proof. The proof is a direct consequence of Cauchy-Schwarz's inequality and of the estimate (4.3). □

The above results together with Lax-Milgram's lemma yield

Corollary 4.5. *Under the assumptions of Theorem 4.3, problem (3.8) has a unique solution $u_h \in V_h$.*

5 Error estimates with weighted norms

Let u be the solution of (2.1) and u_h from (3.8) its finite element approximation. We shall study the error $u - u_h$ in the norm $\| \, . \, \|_{1,h}$ given in (4.5). An estimate of the error $||u - u_h||_{1,h}$ for regular solutions u is given in [24, 4, 16]. Error estimate for corner singularities with singular exponents $> 1/2$ is also given in [16] using the special form of them. Here since singular exponents may be less than $1/2$ (see [19, 12]), we make use of the regularity in term of weighted Sobolev spaces.

Since the influence of corner singularities is local, without loss of generality, we may suppose that only one corner P_j is singular in the sense that only the set Λ_j is not empty. We further denote by λ the smallest eigenvalue in Λ_j and write $P = P_j$ for shortness. For basic approaches of treating corner singularities by finite element methods see e.g. [3, 7, 14, 21, 23, 26]. We now suppose that our meshes are refined near P_j according to the following rule:

Assumption 5.1. *There exists $\mu \in (0, 1]$ and $R > 0$ such that the triangulations $\mathcal{T}_h^i, i = 1, 2$ satisfy the assumptions 3.1 and 3.2 and are graded around the singular vertex P in the following way:*

$$\begin{aligned} \varrho_1 h^{\frac{1}{\mu}} \;&\leq\; h_T \;\leq\; \varrho_1^{-1} h^{\frac{1}{\mu}} && \text{for } T \in \mathcal{T}_h^i : r_T = 0, \\ \varrho_2 h r_T^{1-\mu} \;&\leq\; h_T \;\leq\; \varrho_2^{-1} h r_T^{1-\mu} && \text{for } T \in \mathcal{T}_h^i : 0 < r_T < R, \\ \varrho_3 h \;&\leq\; h_T \;\leq\; \varrho_3^{-1} h && \text{for } T \in \mathcal{T}_h^i : r_T \geq R, \end{aligned} \tag{5.1}$$

where r_T is the distance from T to P, i.e., $r_T := \inf_{x \in T} |x - P|$, with some constants ϱ_k, $0 < \varrho_k \leq 1$ $(k = 1, 2, 3)$ independent of h.

The value $\mu = 1$ yields quasi-uniform meshes in the whole domain Ω, i.e., $\frac{\max_{T \in \mathcal{T}_h^i} h_T}{\min_{T \in \mathcal{T}_h^i} \varrho_T} \leq C$. In [3, 21, 23] graded meshes of the above type are described. In [18] a mesh generator is given which automatically generates a mesh of type (5.1).

We first give a trace inequality in weighted Sobolev spaces motivated by the fact that the solution u of (2.1) satifies

$$\nabla u \in PH^{1,\alpha}(\Omega)^2,$$

for any $\alpha \geq 0$ from Theorem 2.1.

Theorem 5.2. *Let $T \in \mathcal{T}_h^i, i = 1, 2$ be a triangle containing the singular point P as vertex. Then for any $\alpha > 1/2$ and any edge E of T containing P, we have*

$$\left\| r^{\alpha - 1/2} v \right\|_{0,E}^2 \;\leq\; c ||r^{\alpha-1} v||_{0,T} \left(h_T^{-1} ||r^\alpha v||_{0,T} + ||r^\alpha \nabla v||_{0,T} \right), \tag{5.2}$$

for all $v \in H^{1,\alpha}(T) = \{v \in L_2(T) : r^\alpha \nabla v \in L_2(T)\}$, where r is the distance to P.

Proof. Let us assume that (5.2) holds on the reference element $\hat{T}$, namely

$$\left\| \hat{r}^{\alpha - 1/2} \hat{v} \right\|_{0,\hat{E}}^2 \;\leq\; c ||\hat{r}^{\alpha-1} \hat{v}||_{0,\hat{T}} \left(||\hat{r}^\alpha \hat{v}||_{0,\hat{T}} + ||\hat{r}^\alpha \nabla \hat{v}||_{0,\hat{T}} \right), \tag{5.3}$$

where $\hat{r}$ is the distance to the origin. Then (5.2) holds for any element T by the change of variables

$$x = B_T \hat{x} + b_T,$$

where $B_T \in \mathbb{R}^{2\times 2}$ and $b_T \in \mathbb{R}^2$ such that $||B_T|| \cong h_T$ and $P = b_T$, which imply $r \cong h_T \hat{r}$. Therefore by the regularity of T, we have

$$\begin{aligned}
\left\| r^{\alpha-1/2} v \right\|_{0,E}^2 &\leq c h_T^{2\alpha} \left\| \hat{r}^{\alpha-1/2} \hat{v} \right\|_{0,\hat{E}}^2 \\
&\leq c h_T^{2\alpha} ||\hat{r}^{\alpha-1} \hat{v}\|_{0,\hat{T}} \left(||\hat{r}^{\alpha}\hat{v}\|_{0,\hat{T}} + ||\hat{r}^{\alpha}\nabla \hat{v}\|_{0,\hat{T}} \right) \\
&\leq c h_T^{2\alpha} h_T^{-\alpha} ||r^{\alpha-1} v\|_{0,T} \left(h_T^{-1-\alpha} ||r^{\alpha} v\|_{0,T} + h_T^{-\alpha} ||r^{\alpha}\nabla v\|_{0,T} \right),
\end{aligned}$$

which yields (5.2).

It then remains to prove (5.3). For that purpose, we write

$$\hat{v} = v_1 + v_2 \text{ on } \hat{E},$$

where $v_1 = \lambda_1 \hat{v}$, $v_2 = \lambda_2 \hat{v}$ and λ_1 (resp. λ_2) is the barycentric coordinate associated with $(0,0)$ (resp. $(0,1)$) (since without loss of generality we may assume that the extremities of $\hat{E}$ are $(0,0)$ and $(0,1)$).

For v_1 we remark that it satisfies

$$v_1(x, 1-x) = 0 \qquad \forall x \in [0,1],$$

therefore for any $x \in (0,1)$ we may write

$$\begin{aligned}
x^{2\alpha-1} |v_1(x,0)|^2 &= x^{2\alpha-1} (|v_1(x,0)|^2 - |v_1(x,1-x)|^2) \\
&= -x^{2\alpha-1} \int_0^{1-x} \partial_y |v_1(x,y)|^2 \, dy \\
&= -x^{2\alpha-1} \int_0^{1-x} 2 v_1(x,y) \partial_y v_1(x,y) \, dy.
\end{aligned}$$

Integrating this estimate on $x \in (0,1)$, we obtain

$$\int_0^1 x^{2\alpha-1} |v_1(x,0)|^2 \, dx = -2 \int_{\hat{T}} x^{2\alpha-1} v_1(x,y) \partial_y v_1(x,y) \, dxdy.$$

Since $x \leq \hat{r}$, Cauchy-Schwarz's inequality yields

$$\left\| \hat{r}^{\alpha-1/2} v_1 \right\|_{0,\hat{E}}^2 \leq 2 ||\hat{r}^{\alpha-1} v_1\|_{0,\hat{T}} ||\hat{r}^{\alpha} \nabla v_1\|_{0,\hat{T}}. \tag{5.4}$$

We show the same estimate for v_2, namely we may write

$$\begin{aligned}
x^{2\alpha-1} |v_2(x,0)|^2 &= x^{2\alpha-1} (|v_2(x,0)|^2 - |v_2(0,x)|^2) \\
&= x^{2\alpha-1} \int_0^1 \partial_t |v_2(tx,(1-t)x)|^2 \, dt \\
&= 2x^{2\alpha-1} \int_0^1 v_2(tx,(1-t)x) \{(\partial_x v_2)(tx,(1-t)x) - (\partial_y v_2)(tx,(1-t)x)\} \, xdt.
\end{aligned}$$

Integrating this estimate on $x \in (0,1)$, we obtain

$$\begin{aligned}
\int_0^1 x^{2\alpha-1} |v_2(x,0)|^2 \, dx &= 2 \int_0^1 \int_0^1 x^{2\alpha-1} v_2(tx,(1-t)x) \{(\partial_x v_2)(tx,(1-t)x) \\
&- (\partial_y v_2)(tx,(1-t)x)\} \, xdxdt.
\end{aligned}$$

By the change of variables $x_1 = tx$ et $y_1 = (1-t)x$ and remarking that

$$x_1^2 + y_1^2 \geq x^2/2,$$

the previous inequality becomes

$$\int_0^1 x^{2\alpha-1}|v_2(x,0)|^2\,dx \leq 4\int_{\hat{T}} \hat{r}^{2\alpha-1}v_2(x,y)(\partial_x v_2(x,y) - \partial_y v_2(x,y))\,dxdy.$$

By Cauchy-Schwarz's inequality we arrive at

$$\left\|\hat{r}^{\alpha-1/2}v_2\right\|_{0,\hat{E}}^2 \quad\leq\quad 8||\hat{r}^{\alpha-1}v_2||_{0,\hat{T}}||\hat{r}^{\alpha}\nabla v_2||_{0,\hat{T}}. \tag{5.5}$$

As $v_i = \lambda_i\hat{v}$, we have

$$\begin{aligned} ||\hat{r}^{\alpha-1}v_i||_{0,\hat{T}} &\leq ||\hat{r}^{\alpha-1}\hat{v}||_{0,\hat{T}}, & (5.6)\\ ||\hat{r}^{\alpha}\nabla v_i||_{0,\hat{T}} &\leq 2||\hat{r}^{\alpha}\hat{v}||_{0,\hat{T}} + ||\hat{r}^{\alpha}\nabla\hat{v}||_{0,\hat{T}}. & (5.7)\end{aligned}$$

The estimates (5.4) to (5.7) lead to (5.3). □

The above proof fails in the case $\alpha \leq 1/2$, therefore we change the left-hand side of (5.2) into the L_2-norm.

Corollary 5.3. *Let $T \in \mathcal{T}_h^i, i = 1,2$ be a triangle containing the singular point P as vertex. Then for any $\alpha \leq 1/2$ and any edge E of T containing P, we have*

$$\|v\|_{0,E}^2 \quad\leq\quad ch_T^{1-2\alpha}||r^{\alpha-1}v||_{0,T}\left(h_T^{-1}||r^{\alpha}v||_{0,T} + ||r^{\alpha}\nabla v||_{0,T}\right) \qquad \forall v \in H^{1,\alpha}(T). \tag{5.8}$$

Proof. We use the same arguments than before. If (5.8) holds for $\hat{T}$ then it holds for any T by change of variables. For the reference element $\hat{T}$, we follow the same lines except for v_i, where we do not multiply by $x^{2\alpha-1}$, this yields

$$\|v_i\|_{0,\hat{E}}^2 \quad\leq\quad c\int_{\hat{T}} v_i|\nabla v_i|\,dxdy.$$

Since $\hat{r}$ is bounded and $2\alpha-1 \leq 0$, we have $\hat{r}^{2\alpha-1} \geq c_1$, for some $c_1 > 0$ and by Cauchy-Schwarz's inequality we conclude that

$$\|v_i\|_{0,\hat{E}}^2 \quad\leq\quad c||\hat{r}^{\alpha-1}v_i||_{0,\hat{T}}||\hat{r}^{\alpha}\nabla v_i||_{0,\hat{T}}.$$

As before this leads to (5.8) for $\hat{T}$. □

For functions v satisfying $v^i \in H^1(\Omega_i)$ and $r^{\alpha-1/2}\frac{\partial v^i}{\partial n_i} \in L_2(\Gamma)$ $(i = 1,2)$, we introduce the mesh-dependent norm $\|\,.\,\|_{h,\Omega}$ by

$$\begin{aligned} \|v\|_{h,\Omega}^2 \quad := &\quad \sum_{i=1}^{2}\left(\left\|\nabla v^i\right\|_{0,\Omega_i}^2 + \sum_{E\in\mathcal{E}_h} h_E^{2(1-\alpha_E)}\left\|r^{\alpha_E-1/2}\alpha_i\frac{\partial v^i}{\partial n_i}\right\|_{0,E}^2\right) \qquad (5.9)\\ &+ \sum_{E\in\mathcal{E}_h} h_E^{-1}\left\|v^1 - v^2\right\|_{0,E}^2, \end{aligned}$$

where $\alpha_E = 1/2$ except if $\alpha > 1/2$ and E contains the singular corner P where $\alpha_E = \alpha$ from Theorem 2.1. In that last case, the introduction of the weight $r^{\alpha-1/2}$ is motivated by the fact that $\frac{\partial u^i}{\partial n_i}$ (u solution of (2.2)) no more belongs to $L_2(E)$, the factor $h_E^{2(1-\alpha)}$ will be justified in the next Lemmas 5.4 and 5.5. Note that if $\alpha \leq 1/2$ the above norm is the one introduced in [16]. Let us further remark that the above norm is equivalent to the norm $||\cdot||_{1,h}$ on V_h as the next result shows.

Lemma 5.4. *For all $v_h \in V_h$, one has*

$$||v_h||_{1,h} \leq \|v_h\|_{h,\Omega} \leq C||v_h||_{1,h}.$$

Proof. The first estimate directly follows from the definition of the norms. For the second one we remark that if $\alpha > 1/2$ and E contains a singular corner, then

$$h_E^{2(1-\alpha_E)} \left\| r^{\alpha_E - 1/2} \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 = h_E^{2(1-\alpha)} \left\| r^{\alpha-1/2} \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \leq h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2,$$

since $r \leq h_E$ on E. Therefore we have

$$\sum_{E \in \mathcal{E}_h} h_E^{2(1-\alpha_E)} \left\| r^{\alpha_E - 1/2} \alpha_i \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \leq C \sum_{E \in \mathcal{E}_h} h_E \left\| \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2.$$

By Lemma 4.2, we get

$$\sum_{E \in \mathcal{E}_h} h_E^{2(1-\alpha_E)} \left\| r^{\alpha_E - 1/2} \alpha_i \frac{\partial v_h^i}{\partial n_i} \right\|_{0,E}^2 \leq C||\nabla v_h^i||_{0,\Omega_i}.$$

This estimate then leads to the conclusion. □

We now bound $\|u - u_h\|_{1,h}$ and $\|u - u_h\|_{h,\Omega}$ by the norm $||\,.\,||_{h,\Omega}$ of the interpolation error $u - I_h u$, where $I_h u := (I_h u^1, I_h u^2)$ is the Lagrange interpolant of u, in the sense that $I_h u^i \in V_h^i$ is the standard Lagrange interpolant of u^i in the space V_h^i, $i = 1, 2$.

Lemma 5.5. *Under the assumptions of Theorem 4.3, let u be the solution of (2.1) and $u_h \in V_h$ the solution of (3.8). Then it holds*

$$\|u - u_h\|_{1,h} \leq \|u - u_h\|_{h,\Omega} \leq c\, \|u - I_h u\|_{h,\Omega}\,. \tag{5.10}$$

Proof. Since $I_h u \in V_h$, the triangular inequality yields

$$\|u - u_h\|_{h,\Omega} \;\leq\; \|u - I_h u\|_{h,\Omega} + \|I_h u - u_h\|_{h,\Omega}\,.$$

Since $I_h u - u_h \in V_h$ by Lemma 5.4, this estimate becomes

$$\|u - u_h\|_{h,\Omega} \;\leq\; \|u - I_h u\|_{h,\Omega} + \|I_h u - u_h\|_{1,h}\,. \tag{5.11}$$

Owing to $I_h u - u_h \in V_h$ and to the V_h-ellipticity of $\mathcal{B}_h(.\,,.)$, we have

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1} \left(\mathcal{B}_h(I_h u, I_h u - u_h) \;-\; \mathcal{B}_h(u_h, I_h u - u_h)\right). \tag{5.12}$$

Thanks to (4.2), we get

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1}\,\mathcal{B}_h(I_h u - u, I_h u - u_h). \tag{5.13}$$

Note that the right-hand side is meaningful due to Theorem 2.1 and Hardy's inequality (see above).

If we show that

$$|\mathcal{B}_h(I_h u - u, I_h u - u_h)| \;\leq\; c_1 \,\|I_h u - u\|_{h,\Omega}\, \|I_h u - u_h\|_{1,h}\,, \tag{5.14}$$

then together with (5.13) we obtain

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1} c_1 \,\|I_h u - u\|_{h,\Omega}\, \|I_h u - u_h\|_{1,h}\,. \tag{5.15}$$

This inequality combined with (5.11) and with the obvious estimate $\|u - u_h\|_{1,h} \leq \|u - u_h\|_{h,\Omega}$ proves (5.10).

It then remains to prove the estimate (5.14). Writing for shortness $w = I_h u - u$ and $v_h = I_h u - u_h$, by Cauchy-Schwarz's inequality we may estimate:

$$\begin{aligned}
|\mathcal{B}_h(w, v_h)| \;&\leq\; C \left(\sum_{i=1}^{2} ||\nabla w^i||_{0,\Omega_i}^2\right)^{1/2} \left(\sum_{i=1}^{2} ||\nabla v_h^i||_{0,\Omega_i}^2\right)^{1/2} \\
&+\; C \sum_{E\in\mathcal{E}_h} \sum_{i=1}^{2} ||r^{\alpha_E - 1/2}\frac{\partial w^i}{\partial n_i}||_{0,E} ||r^{1/2-\alpha_E}(v_h^1 - v_h^2)||_{0,E} \\
&+\; C \left(\sum_{E\in\mathcal{E}_h} h_E \sum_{i=1}^{2} ||\frac{\partial v_h^i}{\partial n_i}||_{0,E}^2\right)^{1/2} \left(\sum_{E\in\mathcal{E}_h} h_E^{-1} ||w^1 - w^2||_{0,E}^2\right)^{1/2} \\
&+\; \gamma \left(\sum_{E\in\mathcal{E}_h} h_E^{-1} ||w^1 - w^2||_{0,E}^2\right)^{1/2} \left(\sum_{E\in\mathcal{E}_h} h_E^{-1} ||v_h^1 - v_h^2||_{0,E}^2\right)^{1/2} .
\end{aligned} \tag{5.16}$$

Since on an edge E of $\mathcal{E}_h$, $v_h^1 - v_h^2$ belongs to $P_1(E)$, by change of variable one has

$$\begin{aligned}
||r^{1/2-\alpha_E}(v_h^1 - v_h^2)||_{0,E} \;&\cong\; |E|^{1/2} h_E^{1/2-\alpha_E} ||\hat{r}^{1/2-\alpha_E}(\hat{v}_h^1 - \hat{v}_h^2)||_{0,\hat{E}} \\
&\cong\; |E|^{1/2} h_E^{1/2-\alpha_E} ||\hat{v}_h^1 - \hat{v}_h^2||_{0,\hat{E}} \cong h_E^{1/2-\alpha_E} ||v_h^1 - v_h^2||_{0,E}.
\end{aligned}$$

This equivalence yields

$$\begin{aligned}
\sum_{E\in\mathcal{E}_h} \sum_{i=1}^{2} ||r^{\alpha_E-1/2}\frac{\partial w^i}{\partial n_i}||_{0,E} ||r^{1/2-\alpha_E}(v_h^1 - v_h^2)||_{0,E} &\leq C \sum_{E\in\mathcal{E}_h} \sum_{i=1}^{2} h_E^{1-\alpha_E} \cdot \\
&\qquad ||r^{\alpha_E-1/2}\frac{\partial w^i}{\partial n_i}||_{0,E} h_E^{-1/2} ||v_h^1 - v_h^2||_{0,E} \\
\leq C \left(\sum_{E\in\mathcal{E}_h} \sum_{i=1}^{2} h_E^{2(1-\alpha_E)} ||r^{\alpha_E-1/2}\frac{\partial w^i}{\partial n_i}||_{0,E}^2\right)^{1/2} &\left(\sum_{E\in\mathcal{E}_h} h_E^{-1} ||v_h^1 - v_h^2||_{0,E}^2\right)^{1/2} .
\end{aligned}$$

This estimate in (5.16) leads to (5.14) thanks to Lemma 4.2. □

The above results allow to give the following error estimate.

Theorem 5.6. *Let the assumption of Theorem 2.1 be satisfied. If the triangulation satisfies Assumption 5.1 with the refinement parameter* μ *such that*

$$\mu < \lambda, \tag{5.17}$$

then it holds

$$||u - u_h||_{1,h} \leq \|u - u_h\|_{h,\Omega} \leq Ch||f||_{0,\Omega}. \tag{5.18}$$

Proof. In view of (5.10) it suffices to show that

$$||u - I_h u||_{h,\Omega} \leq Ch||f||_{0,\Omega}. \tag{5.19}$$

The estimate

$$||\nabla(u - I_h u)||_{h,\Omega} \leq Ch||u||_{2,\alpha} \tag{5.20}$$

is standard (see for instance Theorem 3.7 of [17]). It then remains to estimate the interface terms in (5.9). We now observe that, due to the assumptions on the mesh, these terms may be rewritten in term of the edge F of the triangle $T \subset \overline{\Omega}_i$ $(T = T_F)$ with $T \cap \Gamma = F \in \mathcal{E}_h^i$, for $i = 1$ or $i = 2$, namely

$$\sum_{E \in \mathcal{E}_h} h_E^{-1} \|I_h u^i - u^i\|_{0,E}^2 \quad \leq \quad c_1 \sum_{F \in \mathcal{E}_h^i} h_F^{-1} \|I_h u^i - u^i\|_{0,F}^2, \tag{5.21}$$

$$\sum_{E \in \mathcal{E}_h} h_E^{2(1-\alpha_E)} \|\alpha_i r^{\alpha_E - 1/2} \frac{\partial \left(I_h u^i - u^i\right)}{\partial n_i}\|_{0,E}^2 \quad \leq \quad c_2 \sum_{F \in \mathcal{E}_h^i} h_F^{2(1-\alpha_F)} \|r^{\alpha_F - 1/2} \nabla \left(I_h u^i - u^i\right)\|_{0,F}^2, \tag{5.22}$$

where $\alpha_F = \alpha$ if the edge F contains the singular corner and $\alpha > 1/2$, otherwise $\alpha_F = 1/2$.

For the estimation of the right-hand side of (5.21), we apply the refined trace theorem

$$\|v\|_{0,F}^2 \quad \leq \quad c\|v\|_{0,T} \left(h_T^{-1} \|v\|_{0,T} + \|\nabla v\|_{0,T}\right) \qquad \text{for } v \in H^1(T), \tag{5.23}$$

which is proved in [28], cf. also [27] or Theorem 5.2. Taking $v = I_h u^i - u^i$, we get

$$\left\|I_h u^i - u^i\right\|_{0,F}^2 \leq c \left\|I_h u^i - u^i\right\|_{0,T} \left(h_T^{-1} \left\|I_h u^i - u^i\right\|_{0,T} + ||\nabla(I_h u^i - u^i)||_{0,T}\right). \tag{5.24}$$

Again standard arguments yield

$$||\nabla(I_h u^i - u^i)||_{0,T} \quad \leq \quad C h_T^{1-\beta} ||r^\beta \nabla^2 u^i||_{0,T}, \tag{5.25}$$

$$||I_h u^i - u^i||_{0,T} \quad \leq \quad C h_T^{2-\beta} ||r^\beta \nabla^2 u^i||_{0,T}, \tag{5.26}$$

with $\beta = \alpha$ is T contains the singular corner and $\beta = 0$ else. These estimates in (5.24) lead to

$$h_F^{-1} \left\|I_h u^i - u^i\right\|_{0,F}^2 \leq C h_T^{2(1-\beta)} ||r^\beta \nabla^2 u^i||_{0,T}^2. \tag{5.27}$$

If T contains the singular corner, by the refinement rule $h_T \cong h^{\frac{1}{\mu}}$, the above estimate becomes

$$h_F^{-1} \left\|I_h u^i - u^i\right\|_{0,F}^2 \leq C h^{\frac{2(1-\alpha)}{\mu}} ||r^\alpha \nabla^2 u^i||_{0,T}^2.$$

Since α is such that $\alpha > 1 - \lambda$, we may choose α such that

$$\mu < 1 - \alpha < \lambda. \tag{5.28}$$

For such a choice, we get

$$h_F^{-1} \left\| I_h u^i - u^i \right\|_{0,F}^2 \leq C h^2 ||r^\alpha \nabla^2 u^i||_{0,T}^2. \tag{5.29}$$

If T does not contain the singular corner, then (5.27) may be written

$$h_F^{-1} \left\| I_h u^i - u^i \right\|_{0,F}^2 \leq C h_T^2 ||\nabla^2 u^i||_{0,T}^2 \leq C h_T^2 r_T^{-2\alpha} ||r^\alpha \nabla^2 u^i||_{0,T}^2.$$

Again the condition (5.28) and the refinement rule (5.1) lead to

$$h_T r_T^{-\alpha} \leq Ch,$$

and therefore (5.29) still holds in that case.

Summing the estimate (5.29) on $F \in \mathcal{E}_h^i$ yield

$$\sum_{F \in \mathcal{E}_h^i} h_F^{-1} \left\| I_h u^i - u^i \right\|_{0,F}^2 \leq C h^2 ||r^\alpha \nabla^2 u^i||_{0,\Omega_i}^2. \tag{5.30}$$

We now estimate the right-hand side of (5.22). We first consider the case when F contain the singular corner. If $\alpha > 1/2$, we may write

$$h_F^{2(1-\alpha_F)} \left\| r^{\alpha_F - 1/2} \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2 = h_F^{2(1-\alpha)} \left\| r^{\alpha - 1/2} \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2.$$

By Theorem 5.2 (and the equivalence $h_F \cong h_T$), we then have

$$\begin{aligned} h_F^{2(1-\alpha_F)} \left\| r^{\alpha_F - 1/2} \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2 \quad & \leq C h_T^{2(1-\alpha)} ||r^{\alpha-1} \nabla \left(I_h u^i - u^i \right) ||_{0,T} \cdot \\ & \left(h_T^{-1} ||r^\alpha \nabla \left(I_h u^i - u^i \right) ||_{0,T} + ||r^\alpha \nabla^2 u^i||_{0,T} \right). \end{aligned} \tag{5.31}$$

On the other hand if $\alpha \leq 1/2$, then

$$h_F^{2(1-\alpha_F)} \left\| r^{\alpha_F - 1/2} \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2 = h_F \left\| \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2,$$

and Corollary 5.3 yields

$$\begin{aligned} h_F^{2(1-\alpha_F)} \left\| r^{\alpha_F - 1/2} \nabla \left(I_h u^i - u^i \right) \right\|_{0,F}^2 \quad & \leq C h_F h_T^{1-2\alpha} ||r^{\alpha-1} \nabla \left(I_h u^i - u^i \right) ||_{0,T} \cdot \\ & \left(h_T^{-1} ||r^\alpha \nabla \left(I_h u^i - u^i \right) ||_{0,T} + ||r^\alpha \nabla^2 u^i||_{0,T} \right). \end{aligned}$$

This shows that (5.31) holds in this case too since $h_F \cong h_T$.

We now estimate the right-hand side of (5.31). First by a standard change of variables, we have

$$||r^{\alpha-1} \nabla \left(I_h u^i - u^i \right) ||_{0,T} \leq C h_T^{\alpha-1} ||\hat{r}^{\alpha-1} \nabla (\hat{I} \hat{u}^i - \hat{u}^i)||_{0,\hat{T}}.$$

As $\alpha > 0$, by Hardy's inequality (see [14, p.28]) we have

$$||\hat{r}^{\alpha-1}\nabla(\hat{I}\hat{u}^i - \hat{u}^i)||_{0,\hat{T}} \leq C\left(||\nabla(\hat{I}\hat{u}^i - \hat{u}^i)||_{0,\hat{T}} + ||\hat{r}^{\alpha}\nabla^2\hat{u}^i||_{0,\hat{T}}\right).$$

Moreover as $\alpha < 1$ the space $H^{2,\alpha}(\hat{T})$ is compactly embedded into $H^1(\hat{T})$ (see Corollary 2.6 of [17]), therefore

$$||\nabla(\hat{I}\hat{u}^i - \hat{u}^i)||_{0,\hat{T}} \leq C||\hat{r}^{\alpha}\nabla^2\hat{u}^i||_{0,\hat{T}}.$$

The three above inequalities show that

$$||r^{\alpha-1}\nabla\left(I_h u^i - u^i\right)||_{0,T} \leq Ch_T^{\alpha-1}||\hat{r}^{\alpha}\nabla^2\hat{u}^i||_{0,\hat{T}}.$$

Going back to T in the right-hand side finally yields

$$||r^{\alpha-1}\nabla\left(I_h u^i - u^i\right)||_{0,T} \leq C||r^{\alpha}\nabla^2 u^i||_{0,T}. \tag{5.32}$$

We prove similarly that

$$||r^{\alpha}\nabla\left(I_h u^i - u^i\right)||_{0,T} \leq Ch_T||r^{\alpha}\nabla^2 u^i||_{0,T}. \tag{5.33}$$

The estimates (5.32) and (5.33) into (5.31) give

$$h_F^{2(1-\alpha_F)}\left\|r^{\alpha_F-1/2}\nabla\left(I_h u^i - u^i\right)\right\|_{0,F}^2 \leq Ch_T^{2(1-\alpha)}||r^{\alpha}\nabla^2 u^i||_{0,T}.$$

With the refinement rule we arrive at

$$h_F^{2(1-\alpha_F)}\left\|r^{\alpha_F-1/2}\nabla\left(I_h u^i - u^i\right)\right\|_{0,F}^2 \leq Ch^2||r^{\alpha}\nabla^2 u^i||_{0,T}. \tag{5.34}$$

If the edge F does not contain the singular corner, then the estimate (5.23) and the above arguments with $\alpha = 0$ (i.e. in $H^2(T)$) yield

$$h_F^{2(1-\alpha_F)}\left\|r^{\alpha_F-1/2}\nabla\left(I_h u^i - u^i\right)\right\|_{0,F}^2 = h_F\left\|\nabla\left(I_h u^i - u^i\right)\right\|_{0,F}^2 \leq Ch_T^2||\nabla^2 u^i||_{0,T}.$$

As before the refinement rule (5.1) leads to (5.34).

Summing up the estimate (5.34) on all edges $F \in \mathcal{E}_h^i$, we have proved that

$$\sum_{F\in\mathcal{E}_h^i} h_F^{2(1-\alpha_F)}\left\|r^{\alpha_F-1/2}\nabla\left(I_h u^i - u^i\right)\right\|_{0,F}^2 \leq Ch^2||r^{\alpha}\nabla^2 u^i||_{0,\Omega_i}^2. \tag{5.35}$$

The estimates (5.20), (5.21), (5.22), (5.30), (5.35) and Theorem 2.1 imply (5.19). □

Remark 5.7. *If the refinement parameter μ satisfies*

$$\mu \geq \lambda,$$

then the above proof shows that

$$||u - u_h||_{1,h} \leq \|u - u_h\|_{h,\Omega} \leq Ch^{\frac{\lambda}{\mu}-\epsilon}||f||_{0,\Omega},$$

for arbitrary small ϵ. In particular, for quasi-uniform meshes ($\mu = 1$) we have at least an approximation order $\mathcal{O}(h^{\lambda-\epsilon})$.

6 The $\mathbf{L_2}$-error estimate

We start with the following boundedness property of $\mathcal{B}_h$ that we need later on (compare with Theorem 4.4).

Lemma 6.1. *Let the assumptions 3.1 and 3.2 be satisfied. Then there is a constant $\mu_3 > 0$ such that*

$$|\mathcal{B}_h\,(w,v)| \;\leq\; \mu_3\,\|w\|_{h,\Omega}\,\|v\|_{h,\Omega}\,, \tag{6.1}$$

for all $v, w \in V$ such that $r^{\alpha-1/2}\frac{\partial v^i}{\partial n_i}, r^{\alpha-1/2}\frac{\partial w^i}{\partial n_i} \in L_2(\Gamma), i = 1, 2.$

Proof. By Cauchy-Schwarz's inequality we may write

$$\begin{aligned}
|\mathcal{B}_h(w,v)| &\leq C\left(\sum_{i=1}^{2}||\nabla w^i||^2_{0,\Omega_i}\right)^{1/2}\left(\sum_{i=1}^{2}||\nabla v^i||^2_{0,\Omega_i}\right)^{1/2}\\
&+\; C\left(\sum_{E\in\mathcal{E}_h}\sum_{i=1}^{2}h_E^{2(1-\alpha_E)}||r^{\alpha_E-1/2}\alpha_i\frac{\partial w^i}{\partial n_i}||^2_{0,E}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h}h_E^{2(\alpha_E-1)}||r^{1/2-\alpha_E}(v^1-v^2)||^2_{0,E}\right)^{1/2}\\
&+\; C\left(\sum_{E\in\mathcal{E}_h}\sum_{i=1}^{2}h_E^{2(1-\alpha_E)}||r^{\alpha_E-1/2}\alpha_i\frac{\partial v^i}{\partial n_i}||^2_{0,E}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h}h_E^{2(\alpha_E-1)}||r^{1/2-\alpha_E}(w^1-w^2)||^2_{0,E}\right)^{1/2}\\
&+\; \gamma\left(\sum_{E\in\mathcal{E}_h}h_E^{-1}||w^1-w^2||^2_{0,E}\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h}h_E^{-1}||v^1-v^2||^2_{0,E}\right)^{1/2}.
\end{aligned}$$

The conclusion follows from the estimate

$$\sum_{E\in\mathcal{E}_h}h_E^{2(\alpha_E-1)}||r^{1/2-\alpha_E}(v^1-v^2)||^2_{0,E} \leq C\sum_{E\in\mathcal{E}_h}h_E^{-1}||v^1-v^2||^2_{0,E} + C\sum_{i=1}^{2}||\nabla v^i||^2_{0,\Omega_i}, \tag{6.2}$$

that we now show. Indeed if $\alpha \leq 1/2$ or if $E \in \mathcal{E}_h$ does not contain the singular corner P, then $\alpha_E = 1/2$ and we directly get

$$h_E^{2(\alpha_E-1)}||r^{1/2-\alpha_E}(v^1-v^2)||^2_{0,E} = h_E^{-1}||v^1-v^2||^2_{0,E}. \tag{6.3}$$

On the contrary if $\alpha > 1/2$ and $E \in \mathcal{E}_h$ contains the singular corner P, then $\alpha_E = \alpha$ and the weight $h_E^{\alpha_E-1}r^{1/2-\alpha_E}$ is no more bounded by $h_E^{-1/2}$. In this case by the change of variable $r = h_E\hat{r}$ sending E to $\hat{E} = [0,1]$, we have

$$h_E^{\alpha_E-1}||r^{1/2-\alpha_E}(v^1-v^2)||_{0,E} = ||\hat{r}^{1/2-\alpha_E}(\hat{v}^1-\hat{v}^2)||_{0,\hat{E}}.$$

The embedding $H^{1/2}(\hat{E}) \hookrightarrow H^{1/2-\epsilon}(\hat{E})$, for all $\epsilon > 0$ and Corollaries 1.4.4.5 and 1.4.4.10 of [14] imply that

$$||\hat{r}^{-1/2+\epsilon}(\hat{v}^1-\hat{v}^2)||_{0,\hat{E}} \leq C||\hat{v}^1-\hat{v}^2||_{1/2,\hat{E}},$$

where $C > 0$ depends on ϵ. Since $\alpha < 1$, choosing ϵ small enough, we obtain

$$||\hat{r}^{1/2-\alpha_E}(\hat{v}^1-\hat{v}^2)||_{0,\hat{E}} \leq C||\hat{v}^1-\hat{v}^2||_{1/2,\hat{E}}.$$

By the definition of the $H^{1/2}$-norm and the triangle inequality we get

$$||\hat{r}^{1/2-\alpha_E}(\hat{v}^1 - \hat{v}^2)||_{0,\hat{E}} \leq C||\hat{v}^1 - \hat{v}^2||_{0,\hat{E}} + C\sum_{i=1}^{2} |\hat{v}^i|_{1/2,\hat{E}}.$$

Going back to E, we have shown that

$$h_E^{\alpha_E-1}||r^{1/2-\alpha_E}(v^1 - v^2)||_{0,E} \leq Ch_E^{-1/2}||v^1 - v^2||_{0,E} + C\sum_{i=1}^{2} |v^i|_{1/2,E}. \tag{6.4}$$

Summing the square of the estimates (6.3) and (6.4), we deduce that

$$\sum_{E\in\mathcal{E}_h} h_E^{2(\alpha_E-1)}||r^{1/2-\alpha_E}(v^1 - v^2)||_{0,E}^2 \leq C\sum_{E\in\mathcal{E}_h} h_E^{-1}||v^1 - v^2||_{0,E}^2 + C\sum_{i=1}^{2} |v^i|_{1/2,\Gamma}^2.$$

As v^i belongs to $H^1(\Omega_i)$, we clearly have

$$|v^i|_{1/2,\Gamma} \leq C||\nabla v^i||_{0,\Omega_i}.$$

Since the estimate (6.2) follows from the two above estimates, the proof is complete. □

Adapting the standard Aubin-Nitsche trick, we are now able to show that the L_2-error is twice of that in the $||\cdot||_{h,\Omega}$-norm.

Lemma 6.2. *Under the assumption of Theorem 5.6, the estimate*

$$\|u - u_h\|_{0,\Omega} \leq Ch^2\,\|f\|_{0,\Omega} \tag{6.5}$$

holds.

Proof. Take the error $u - u_h$, where u is the solution of (2.2) and u_h its finite-element approximation from (3.8), and consider the solution u_e of the variational equation

$$\text{find} \quad u_e \in H_0^1(\Omega): \qquad a(u_e, v) = \int_\Omega (u - u_h)\, v\, dx \qquad \forall\, v \in H_0^1(\Omega)\,. \tag{6.6}$$

Then, the finite-element approximation of u_e in V_h is defined by

$$\text{find} \quad u_{eh} \in V_h: \qquad \mathcal{B}_h(u_{eh}, v_h) = \int_\Omega (u - u_h)\, v_h\, dx \qquad \forall\, v_h \in V_h\,. \tag{6.7}$$

As $u - u_h$ belongs to $L_2(\Omega)$, the regularity results from section 2 may be applied to u_e. Consequently Theorem 5.6 implies the error estimate

$$\|u_e - u_{eh}\|_{h,\Omega} \leq Ch\,\|u - u_h\|_{0,\Omega}\,. \tag{6.8}$$

Clearly, the consistency proved by Theorem 4.1 yields here

$$\mathcal{B}_h(u_e, v_h) = \int_\Omega (u - u_h)\, v_h\, dx \qquad \forall\, v_h \in V_h\,. \tag{6.9}$$

Moreover, observe that for the solutions u and u_e from (2.2) and (6.6), respectively, the relation

$$\mathcal{B}_h(u_e, u) = a(u_e, u) = \int_\Omega (u - u_h)\, u \, dx \qquad (6.10)$$

holds. Using (6.9) (put $v_h = u_h$), (6.10) and the orthogonality relation $\mathcal{B}_h(u - u_h, v_h) = 0$ for $v_h = u_{eh}$ from (4.2) as well as the symmetry of $\mathcal{B}_h(.,.)$, we get the identities

$$\|u - u_h\|^2_{0,\Omega} = \mathcal{B}_h(u_e, u - u_h) = \mathcal{B}_h(u - u_h, u_e - u_{eh})\,. \qquad (6.11)$$

By Lemma 6.1, we obtain

$$\|u - u_h\|^2_{0,\Omega} \leq \mu_3 \, \|u - u_h\|_{h,\Omega} \, \|u_e - u_{eh}\|_{h,\Omega}\,.$$

Theorem 5.6 then yields (see (6.8))

$$\|u - u_h\|^2_{0,\Omega} \leq C h^2 \, \|f\|_{0,\Omega} \, \|u - u_h\|_{0,\Omega}\,.$$

This proves (6.5). □

Remark 6.3. *If the refinement parameter μ satisfies $\mu \geq \lambda$, then the error estimate*

$$||u - u_h||_{0,\Omega} \leq C h^{\frac{2\lambda}{\mu} - \epsilon} ||f||_{0,\Omega}$$

holds for arbitrary small ϵ. In particular, for quasi-uniform meshes ($\mu = 1$), we have at least an approximation order $\mathcal{O}(h^{2\lambda - \epsilon})$.

The proof is quite analogous to that of Lemma 6.2.

7 Numerical experiments

In the following we shall illustrate the behaviour of the Nitsche type finite element approximation for some transmission problem which yields a solution with one singularity function. Moreover, we shall study the rate of convergence of this approximation in the $\{1, h\}$-norm and

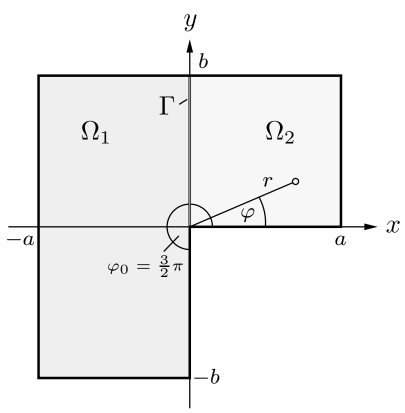

Figure 4: The L-shaped domain Ω.

p_1	30.83623	5.39245	2.23607	1.00000	0.70130	0.23606
p_2	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
λ	0.51	0.55	0.6	0.66667	0.7	0.8

Table 1: Solutions λ of equation (7.3) for some pairs of p_1, p_2

in the L_2-norm when local mesh refinement via mesh grading is applied. For that consider the BVP

$$-\mathrm{div}(p\,\nabla\, u) \;=\; f \quad \text{in}\ \Omega\,, \quad u \;=\; 0 \quad \text{on}\ \partial\Omega\,, \tag{7.1}$$

where Ω is the L-shaped domain of Figure 4 with $\Omega_1 = (-a, 0) \times (-b, b)$ and $\Omega_2 = (0, a) \times (0, b)$ for some $a, b > 0$. For Ω_1, Ω_2, see also Figure 4. The line of discontinuity of the coefficient p is given by the straight segment between $(0, 0)$ and $(0, b)$. The right-hand side f is chosen such that the exact solution u of (7.1) is given by

$$u(x, y) \;=\; (a^2 - x^2)(b^2 - y^2)r^\lambda \begin{cases} \sin(\lambda\varphi) & \text{for } x > 0 \\ \dfrac{\sin(\lambda(\frac{3\pi}{2} - \varphi))}{2\cos(\lambda\frac{\pi}{2})} & \text{for } x < 0 \end{cases} \tag{7.2}$$

where $r^2 = x^2 + y^2$, $0 \le \varphi \le \varphi_0$, $\varphi_0 = \frac{3}{2}\pi$. The parameter λ denotes the exponent of the singularity function with centre at $P_0 = (0, 0)$. Obviously, $u \in H^{1+\lambda-\varepsilon}(\Omega)$ ($\varepsilon > 0$ arbitrary), $u|_{\partial\Omega} = 0$ and $f \in L_2(\Omega)$ are satisfied. The value of λ in (7.2) is determined by the ratio of the values $p_1 = p|_{\Omega_1}$ and $p_2 = p|_{\Omega_2}$ of the coefficient p over the subdomains Ω_1 and Ω_2, respectively, and will be obtained as the smallest positive solution of the equation

$$-p_1 \sin(\tfrac{\lambda\pi}{2}) \cos(\lambda\pi) = p_2 \cos(\tfrac{\lambda\pi}{2}) \sin(\lambda\pi). \tag{7.3}$$

This follows e.g. from [19]. In Table 1, for some pairs of p_1, p_2, the corresponding values of λ are presented.

For the application of the Nitsche type mortaring method to this BVP we use an initial mesh as shown in Figure 5. The interface Γ of the non-matching meshes coincides with the physical interface of the jumping coefficients. Consequently, the singularity acts also in the region of non-matching meshes. For the value $\lambda = 0.6$ the Nitsche type finite element approximation $u_h \in V_h$ according to (3.8) is visualized in Figure 6. The initial mesh is refined globally by dividing each triangle into four equal triangles such that the mesh parameters form a sequence $\{h_1, h_2, \ldots\}$ defined by $\{h_1, \frac{h_1}{2}, \ldots\}$. The ratio of the number of mesh segments (without grading) on the mortar interface is given by $2 : 3$ (see Figure 5).

In the numerical experiments, different values of α_1 and α_2 ($\alpha_1 + \alpha_2 = 1$) as well as different partitions $\mathcal{E}_h$ on Γ are chosen. For $\alpha_i = 1$ ($i = 1$ or $i = 2$), the partition of Γ is obtained by taking the trace of the triangulation $\mathcal{T}_h^i$ of $\bar{\Omega}_i$ on the interface: $\mathcal{E}_h$=$\mathcal{E}_h^i$. In this case, the choice $\gamma = 3p_i$ is sufficient to ensure stability. Otherwise, for the case $0 < \alpha_i < 1$ ($i = 1, 2$) the intersection of the mesh traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on the interface Γ is (for this example) a convenient partition $\mathcal{E}_h$. Moreover, local refinement by mesh grading around the corner vertex P_0 according to Assumption 5.1 is applied. If u_h denotes the finite element approximation of

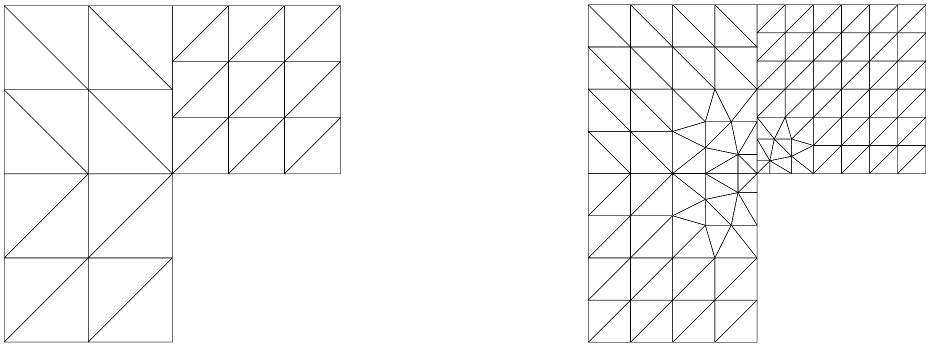

Figure 5: Non-matching triangulations with mesh ratio 2 : 3; initial mesh (h_1–level, left) and mesh on the h_2–level with local grading at the interface corner (right).

the exact solution u according to (3.8), then the error estimate in the discrete norm $||\,.\,||_{1,h}$ from (4.5) is given by (5.18) ($\mu < \lambda$). For measuring the convergence rate, it is supposed that

$$\|u - u_h\|_{1,h} \;\approx\; Ch^{\alpha} \tag{7.4}$$

holds, with some constant C which is approximately the same for two consecutive levels of h, like h and $\frac{h}{2}$. The smaller h is, the better this assumption will be justified. Then $\alpha = \alpha_{obs}$ (observed value) is derived from (7.4) by $\alpha_{obs} := \log_2 q_h$ with $q_h := \|u - u_h\|(\|u - u_{h/2}\|)^{-1}$. For calculating the norms, cubature formulas with high accuracy are applied. By analogy to (7.4), for the L_2–norm we suppose that the relation $\|u - u_h\|_{0,\Omega} \;\approx\; Ch^{\beta}$ is satisfied. For the choice $\alpha_1 = 1$ and different values of λ and for quasi-uniform meshes ($\mu = 1$) and for meshes with grading ($\mu = 0.7\lambda$), the observed and expected values of α and β are given in Table 2 and 3. Further experiments with parameters $\alpha_2 = 1$ and $0 < \alpha_i < 1$ ($i = 1, 2$) led to values α and β which differed only slightly from the values associated with the parameter $\alpha_1 = 1$.

norm $\|\,.\,\|_{1,h}$	$\lambda = 0.51$	$\lambda = 0.55$	$\lambda = 0.6$	$\lambda = \frac{2}{3}$	$\lambda = 0.7$	$\lambda = 0.8$	expected values
α_{obs} ($\mu = 1$)	0.53	0.58	0.65	0.76	0.82	0.94	$\alpha_{exp} \approx \lambda$
α_{obs} ($\mu < \lambda$)	0.98	0.99	0.99	0.99	0.98	0.99	$\alpha_{exp} = 1$

Table 2: Observed convergence order α_{obs} for the error in the norm $\|\,.\,\|_{1,h}$ for different λ, fixed $\alpha_1 = 1$, $\mathcal{E}_h = \mathcal{E}_h^1$, and for the grading parameters $\mu = 1$ and $\mu = 0.7 \cdot \frac{2}{3} < \lambda$.

norm $\|\,.\,\|_{0,\Omega}$	$\lambda = 0.51$	$\lambda = 0.55$	$\lambda = 0.6$	$\lambda = \frac{2}{3}$	$\lambda = 0.7$	$\lambda = 0.8$	expected values
β_{obs} ($\mu = 1$)	1.05	1.14	1.25	1.42	1.51	1.80	$\beta_{exp} \approx 2\lambda$
β_{obs} ($\mu < \lambda$)	1.93	1.96	1.97	1.97	1.97	1.98	$\beta_{exp} = 2$

Table 3: Observed convergence order β_{obs} for the error in the norm $||\,.\,||_{0,\Omega}$ for different λ, fixed $\alpha_1 = 1$, $\mathcal{E}_h = \mathcal{E}_h^1$, and for the grading parameters $\mu = 1$ and $\mu = 0.7 \cdot \frac{2}{3} < \lambda$.

The numerical experiments show that the observed rates of convergence α_{obs} and β_{obs} are approximately equal to the values expected from the theory. In particular, $\alpha_{obs} \approx 1$ and $\beta_{obs} \approx 2$ was observed on meshes with appropriate grading. Furthermore, on those meshes the local error as well as the error norms are diminished; cf. the error in Figure 6.
Thus, local mesh grading is suited to overcome the decrease of convergence order and the loss of accuracy on non-matching meshes caused by interface singularities.

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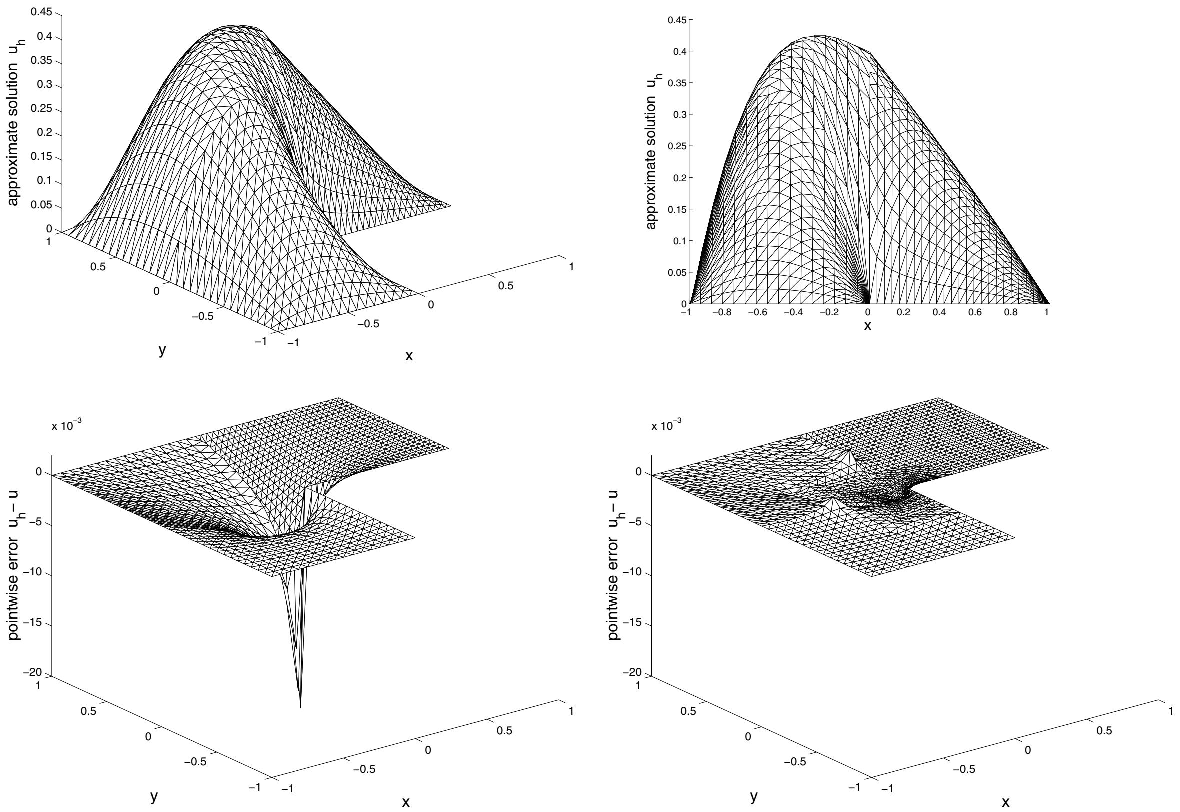

Figure 6: The approximate solution u_h for the coefficient pair $p_1 = 2.23$ and $p_2 = 1.00$ in two different perspectives (top), the corresponding local (pointwise) error on the h-level h_4 for a quasi-uniform mesh (bottom left) and for a mesh with grading (bottom right).

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