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Nitsche type mortaring for some elliptic problem with corner singularities

B. Heinrich K. Pietsch

Abstract

The paper deals with Nitsche type mortaring as a finite element method (FEM) for treating non-matching meshes of triangles at the interface of some domain decomposition. The approach is applied to the Poisson equation with Dirichlet conditions (as a model problem) under the aspects that the interface passes re-entrant corners of the domain. For such problems and non-matching meshes with and without local refinement near the re-entrant corner, some properties of the finite element scheme and error estimates are proved. They show that appropriate mesh grading yields convergence rates as known for the classical FEM in presence of regular solutions. Finally, a numerical example illustrates the approach and confirms the theoretical results.

Key words. finite element method, non-matching meshes, mortar finite elements, corner singularities, Nitsche type mortaring

AMS subject classification. 65N30, 65N55

1 Introduction

For the efficient numerical treatment of boundary value problems (BVPs), domain decomposition methods are widely used. They allow to work in parallel: generating the mesh in subdomains, calculating the corresponding parts of the stiffness matrix and of the right-hand side, and solving the system of finite element equations.

There is a particular interest in triangulations which do not match at the interface of the subdomains. Such non-matching meshes arise, for example, if the meshes in different subdomains are generated independently from each other, or if a local mesh with some structure is to be coupled with a global unstructured mesh, or if an adaptive remeshing in some subdomain is of primary interest. This is often caused by extremely different data (material properties or right-hand sides) of the BVP in different subdomains or by a complicated geometry of the domain, which have their response in a solution with singular or anisotropic behaviour. Moreover, non-matching meshes are also applied if different discretization approaches are used in different subdomains.

There are several approaches to work with non-matching meshes. The task to satisfy some continuity requirements on the interface (e.g. of the solution and its conormale derivative) can be done by iterative procedures (e.g. Schwarz's method) or by direct methods like the Lagrange multiplier technique.

There are many papers on the Lagrange multiplier mortar technique, see e.g. [5, 6, 9, 25] and the literature quoted in these papers. There, one has new unknowns (the Lagrange multipliers) and the stability of the problem has to be ensured by satisfying some inf-sup condition (for the actual mortar method) or by stabilization techniques.

Another approach which is of particular interest here is related to the classical Nitsche method [16] of treating essential boundary conditions. This approach has been worked out more generally in [23, 20] and transferred to interior continuity conditions by Stenberg [21] (Nitsche type mortaring), cf. also [1]. As shown in [4] and [10], the Nitsche type mortaring can be interpreted as a stabilized variant of the mortar method based on a saddle point problem.

Compared with the classical mortar method, the Nitsche type mortaring has several advantages. Thus, the saddle point problem, the inf–sup–condition as well as the calculation of additional variables (the Lagrange multipliers) are circumvented. The method employs only a single variational equation which is, compared with the usual equations (without any mortaring), slightly modified by an interface term. This allows to apply existing software tools by slight modifications. Moreover, the Nitsche type method yields symmetric and positive definite discretization matrices in correspondence to symmetry and ellipticity of the operator of the BVP. Although the approach involves a stabilizing parameter γ, it is not a penalty method since it is consistent with the solution of the BVP. The parameter γ can be estimated easily (see below). The mortar subdivision of the chosen interface Γ can be done in a more general way than known for the classical mortar method. This can be advantageous for solving the system of finite element equations by iterative domain decomposition methods.

Basic aspects of the Nitsche type mortaring and error estimates for regular solutions $u \in H^k(\Omega)$ ($k \geq 2$) on quasi-uniform meshes are published in [21, 4]. Compared with these papers, we extend the application of the Nitsche type mortaring to problems with non-regular solutions and to meshes being locally refined and not quasi-uniform.

We consider the model problem of the Poisson equation with Dirichlet data in the presence of re-entrant corners and admit that the interface with non-matching meshes passes the vertex of such corners. For the appropriate treatment of corner singularities we employ local mesh refinement around the corner by mesh grading in correspondence with the degree of the singularity. Therefore, the Nitsche type mortaring is to be analyzed on more general triangulations. For meshes with and without grading, basic inequalities, stability and boundedness of the bilinear form as well as error estimates in a discrete H^1-norm are proved. The rate of convergence in L_2 is twice of that in the H^1-norm. For an appropriate choice of some mesh grading parameter, the rate of convergence is proved to be the same as for regular solutions on quasi-uniform meshes. Finally, some numerical experiments are given which confirm the rates of convergence derived.

2 Analytical preliminaries

In the following, $H^s(X)$, s real (X some domain, $H^0 = L_2$), denotes the usual Sobolev spaces, with the corresponding norms and the abbreviation $\| \,.\, \|_{s,X} := \| \,.\, \|_{H^s(X)}$. Constants C or c occuring in inequalities are generic constants.

For simplicity we consider the Poisson equation with homogeneous Dirichlet boundary conditions as a model problem:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{aligned} \tag{2.1}$$

Here, Ω is a bounded polygonal domain in $\mathbb{R}^2$, with Lipschitz-boundary $\partial\Omega$ consisting of straight line segments. Suppose further that $f \in L_2(\Omega)$ holds. The variational equation of (2.1) is given as follows. Find $u \in H_0^1(\Omega) := \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ such that

$$a(u,v) \;=\; f(v) \qquad \forall\, v \in H_0^1(\Omega), \tag{2.2}$$

$$\text{with } a(u,v) \;:=\; \int_\Omega (\nabla u, \nabla v)\, dx\,, \qquad f(v) \;:=\; \int_\Omega f v\, dx.$$

We now decompose the domain Ω into non-overlapping subdomains. For simplicity of notation we consider two subdomains Ω_1 and Ω_2 with interface Γ, where

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma,$$

holds ($\overline{X}$: closure of the set X). We assume that the boundaries $\partial\Omega_i$ of Ω_i $(i = 1, 2)$ are also Lipschitz-continuous and formed by open straight line segments Γ_j such that

$$\Gamma \;=\; \bigcup_{j=1}^{J} \overline{\Gamma}_j.$$

We distinguish two important types of interfaces Γ:

case I1: the intersection $\Gamma \cap \partial\Omega$ consists of two points P_1, P_2 $(P_1 \neq P_2)$ being the endpoints of Γ, and at least one point is the vertex of a re-entrant corner, like in Figure 1,

case I2: $\Gamma \cap \partial\Omega = \emptyset$, i.e., Γ does not touch the boundary $\partial\Omega$, like in Figure 2.

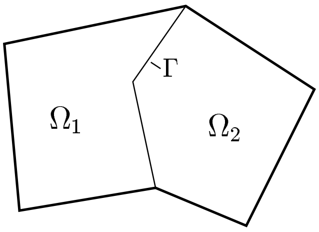

Figure 1:

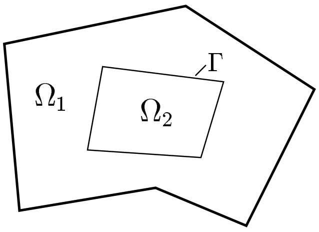

Figure 2:

For the presentation of the method and error estimates we need the degree of regularity of the solution u. Clearly, the functionals $a(.,.)$ and $f(.)$ satisfy the standard assumptions of the Lax-Milgram theorem and we have the existence of a solution $u \in H_0^1(\Omega)$ of problem (2.2) as well as the a priori estimate $\|u\|_{1,\Omega} \le C\, \|f\|_{0,\Omega}$.

Furthermore, the regularity theory of (2.2) yields $u \in H^2(\Omega)$ and $\|u\|_{2,\Omega} \le C\, \|f\|_{0,\Omega}$ if Ω is convex. If $\partial\Omega$ has re-entrant corners with angles $\varphi_{0j} : \pi < \varphi_{0j} < 2\pi$ $(j = 1, \ldots, I)$, then u can be represented by

$$u \;=\; \sum_{j=1}^{I} \eta_j\, a_j\, r_j^{\lambda_j} \sin(\lambda_j \varphi_j) + w, \tag{2.3}$$

with a regular remainder $w \in H^2(\Omega)$. Here, (r_j, φ_j) denote the local polar coordinates of a point $P \in \Omega$ with respect to the vertex $P_j \in \partial\Omega$, where $0 < r_j \le r_{0j}$ and $0 < \varphi_j < \varphi_{0j}$ hold;

r_{0j} is the radius of some circle neighborhood with center at P_j. Moreover, we have $\lambda_j = \frac{\pi}{\varphi_{0j}}$ ($\frac{1}{2} < \lambda_j < 1$), a_j is some constant, and η_j is a locally acting (smooth) cut-off function around the vertex P_j, with

$$0 \le \eta_j \le 1, \qquad \eta_j = \begin{cases} 1 & \text{for } 0 \le r_j \le \dfrac{r_{0j}}{3} \\ 0 & \text{for } \dfrac{2r_{0j}}{3} \le r_j \le r_{0j} \,. \end{cases}$$

The solution $u \in H_0^1(\Omega)$ satisfies the relations

$$\sum_{j=1}^{I} |a_j| + \|w\|_{2,\Omega} \le C\,\|f\|_{0,\Omega}\,, \qquad u \in H^1(\Delta,\Omega) := \{v \in H^1(\Omega) : \Delta v \in L_2(\Omega)\} \tag{2.4}$$

and, owing to (2.3), also $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ for any ε: $0 < \varepsilon < \varepsilon_0$, ε_0 sufficiently small. For these results, see e.g. [13, 7].

In the context of dividing Ω into subdomains Ω_1, Ω_2, we introduce the restrictions $v^i := v|_{\Omega_i}$ of some function v on Ω_i as well as the vectorized form of v by $v = (v^1, v^2)$, i.e. we have $v^i(x) = v(x)$ for $x \in \Omega_i$ $(i = 1, 2)$. It should be noted that we shall use here the same symbol v for denoting the function on Ω as well as the vector (v^1, v^2). This will not lead to confusion, since the meaning will be clear from the context. The one-to-one correspondence between the "field function" v and the "vector function" v is given on $\Omega_1 \cup \Omega_2$. Moreover, $v|_\Gamma$ is defined by the trace. We shall keep the notation also in cases, where the traces $v^1|_\Gamma, v^2|_\Gamma$ on the interface Γ are different (e.g. for interpolants on Ω_i).

Using this notation, it is obvious that the solution of the BVP (2.1) is equivalent to the solution of the following interface problem: Find (u^1, u^2) such that

$$\begin{aligned} -\Delta u^i &= f \quad \text{in } \Omega_i\,, && i = 1,2\,, \\ u^i &= 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega\,, && i = 1,2\,, \\ u^1 &= u^2 \quad \text{on } \Gamma\,, \\ \frac{\partial u^1}{\partial n_1} + \frac{\partial u^2}{\partial n_2} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.5}$$

are satisfied, where n_i $(i = 1, 2)$ denotes the outward normal to $\partial\Omega_i \cap \Gamma$. Introducing the spaces V^i $(i = 1, 2)$ given by

$$\begin{aligned} &\text{case I1:} && V^i := \left\{ v^i : v^i \in H^1(\Omega_i),\, v^i|_{\partial\Omega\cap\partial\Omega_i} = 0 \right\} && \text{for } \partial\Omega \cap \partial\Omega_i \ne \emptyset, \\ &\text{case I2:} && V^i := H^1(\Omega_i) && \text{for } \partial\Omega \cap \partial\Omega_i = \emptyset, \end{aligned} \tag{2.6}$$

and the space $V := V^1 \times V^2$, the BVP (2.5) can be formulated in a weak form (see e.g. [2]). Clearly, we have $u^i \in V^i$ and $u^i \in H^1(\Delta, \Omega_i)$ $(i = 1, 2)$ as well as $u = (u^1, u^2) \in V$. The continuity of the solution u and of its normal derivative $\frac{\partial u^i}{\partial n}$ on Γ ($n = n_1$ or $n = n_2$) is to be required in the sense of $H_*^{\frac{1}{2}}(\Gamma)$ and $H_*^{-\frac{1}{2}}(\Gamma)$ (the dual space of $H_*^{\frac{1}{2}}(\Gamma)$), respectively.

Define $H_*^{\frac{1}{2}}(\partial\Omega_i)$ $(H_{00}^{\frac{1}{2}})$ by the range of V^i by the trace operator and to be provided with the quotient norm, see e.g. [9, 13]. So we use in case I1: $H_*^{\frac{1}{2}}(\partial\Omega_i) \simeq H_{00}^{\frac{1}{2}}(\partial\Omega_i \setminus \partial\Omega)$ for $\partial\Omega \cap \partial\Omega_i \neq \emptyset$, in case I2: $H_*^{\frac{1}{2}}(\partial\Omega_i) = H^{\frac{1}{2}}(\partial\Omega_i)$ for $\partial\Omega \cap \partial\Omega_i = \emptyset$. Here $\simeq$ means that we identify the corresponding spaces. By $\langle\,.\,,.\,\rangle_{\partial\Omega_i}$ we shall denote the duality pairing of $H_*^{-\frac{1}{2}}(\partial\Omega_i)$ and $H_*^{\frac{1}{2}}(\partial\Omega_i)$.

3 Non-matching mesh finite element discretization

We cover $\overline{\Omega}_i$ $(i = 1,2)$ by a triangulation $\mathcal{T}_h^i$ $(i = 1,2)$ consisting of triangles. The triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are independent of each other. Moreover, compatibility of the nodes of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ along $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ is not required, i.e., non-matching meshes on Γ are admitted. Let h denote the mesh parameter of these triangulations, with $0 < h \leq h_0$ and sufficiently small h_0. Take e.g. $h = \max\{h_T : T \in \mathcal{T}_h^1 \cup \mathcal{T}_h^2\}$, where T $(T = \overline{T})$ denotes a triangle and $h_T := \operatorname{diam} T$ its diameter. Let $\mathcal{E}_h^1, \mathcal{E}_h^2$ denote the triangulations of Γ defined by the traces of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ on Γ, respectively.

Assumption 3.1

(i) For $i = 1,2$, it holds
$$\overline{\Omega}_i = \bigcup_{T \in \mathcal{T}_h^i} T. \tag{3.1}$$

(ii) Two arbitrary triangles $T, T' \in \mathcal{T}_h^i$ $(T \neq T', i = 1,2)$ are either disjoint or have a common vertex, or a common edge.

(iii) The mesh in $\overline{\Omega}_i$ $(i = 1,2)$ is shape regular, *i.e., for the diameter h_T of T and the diameter ϱ_T of the largest inscribed sphere of T, we have*
$$\frac{h_T}{\varrho_T} \leq C \text{ for any } T \in \mathcal{T}_h^i, \tag{3.2}$$
where C is independent of T and h.

Clearly, relation (3.2) implies that the angle θ at any vertex and the length h_F of any side F of the triangle T satisfy the inequalities
$$0 < \theta_0 \leq \theta \leq \pi - \theta_0, \qquad \varepsilon_1 h_T \leq h_F \leq h_T, \quad (0 < \varepsilon_1 < 1),$$
with constants θ_0 and ε_1 being independent of h and T. Owing to (3.2), the triangulations $\mathcal{T}_h^i$ $(i = 1,2)$ do not have to be quasi-uniform in general.

For $i = 1,2$ and according to V^i from (2.6) introduce finite element spaces V_h^i of functions v^i on Ω_i by
$$V_h^i \; := \; \left\{ v^i \in H^1(\Omega_i) : v^i\big|_T \in \mathbb{P}_k(T) \;\; \forall\, T \in \mathcal{T}_h^i,\; v^i\big|_{\partial\Omega \cap \partial\Omega_i} = 0 \right\}, \quad i = 1,2, \tag{3.3}$$
where $\mathbb{P}_k(T)$ denotes the set of all polynomials on T with degree $\leq k$. We do not employ different polynomial degrees on $\overline{\Omega}_1, \overline{\Omega}_2$, which could also be done. The finite element space V_h of vectorized functions v_h with components v_h^i on Ω_i is given by
$$V_h \; := \; V_h^1 \times V_h^2 \; = \; \left\{ v_h = \left(v_h^1, v_h^2\right) : v_h^1 \in V_h^1,\, v_h^2 \in V_h^2 \right\}. \tag{3.4}$$

In general, $v_h \in V_h$ is not continuous across Γ.

Consider further some triangulation $\mathcal{E}_h$ of Γ by intervals E ($E = \overline{E}$), i.e. $\Gamma = \bigcup_{E \in \mathcal{E}_h} E$, where h_E denotes the diameter of E. Furthermore, let γ be some positive constant (to be specified subsequently) and α_1, α_2 real parameters with

$$0 \leq \alpha_i \leq 1 \quad (i = 1, 2), \qquad \alpha_1 + \alpha_2 = 1. \tag{3.5}$$

Following [21] we now introduce the bilinear form $\mathcal{B}_h(.,.)$ on $V_h \times V_h$ and the linear form $\mathcal{F}_h(.)$ on V_h as follows:

$$\begin{aligned}
\mathcal{B}_h(u_h, v_h) \;&:=\; \sum_{i=1}^{2} \left(\nabla u_h^i, \nabla v_h^i\right)_{\Omega_i} - \left\langle \alpha_1 \frac{\partial u_h^1}{\partial n_1} - \alpha_2 \frac{\partial u_h^2}{\partial n_2}, v_h^1 - v_h^2 \right\rangle_{\Gamma} \\
&\quad - \left\langle \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}, u_h^1 - u_h^2 \right\rangle_{\Gamma} + \gamma \sum_{E \in \mathcal{E}_h} h_E^{-1} \left\langle u_h^1 - u_h^2, v_h^1 - v_h^2 \right\rangle_E, \qquad (3.6) \\
\mathcal{F}_h(v_h) \;&:=\; \sum_{i=1}^{2} \left(f, v_h^i\right)_{\Omega_i} \qquad \text{for } u_h, v_h \in V_h.
\end{aligned}$$

(Note that in [4] a similar bilinear form with $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $h_E = h$ is employed.) The finite element approximation u_h of u on the non-matching triangulation $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ is now defined by $u_h = (u_h^1, u_h^2) \in V_h = V_h^1 \times V_h^2$ satisfying the equation

$$\mathcal{B}_h(u_h, v_h) \;=\; \mathcal{F}_h(v_h) \qquad \forall\, v_h \in V_h. \tag{3.7}$$

Here, $(.,.)_{\Omega_i}$ denotes the $L_2(\Omega_i)$-scalar product, $\langle .,. \rangle_\Gamma$ the $H_*^{-\frac{1}{2}}(\Gamma) \times H_*^{\frac{1}{2}}(\Gamma)$-duality pairing and $\langle .,. \rangle_E$ the $L_2(E)$-scalar product. Owing to $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$, the trace theorem yields $\left.\frac{\partial u^i}{\partial n_i}\right|_\Gamma \in L_2(\Gamma)$. Furthermore, $\left.\frac{\partial v_h^i}{\partial n_i}\right|_\Gamma \in L_2(\Gamma)$ holds also for $v_h = (v_h^1, v_h^2) \in V_h$. This will be used subsequently for evaluating $\langle .,. \rangle_\Gamma$ by the $L_2(\Gamma)$-scalar product. A natural choice for the triangulation $\mathcal{E}_h$ of Γ is $\mathcal{E}_h := \mathcal{E}_h^1$ ($\alpha_1 = 1$) or $\mathcal{E}_h := \mathcal{E}_h^2$ ($\alpha_2 = 1$), where

$$\mathcal{E}_h^i \;=\; \{E : E = \partial T \cap \Gamma, \text{ if } E \text{ is a segment}, \ T \in \mathcal{T}_h^i\}, \quad \text{for } i = 1, 2, \tag{3.8}$$

cf. Figure 3.

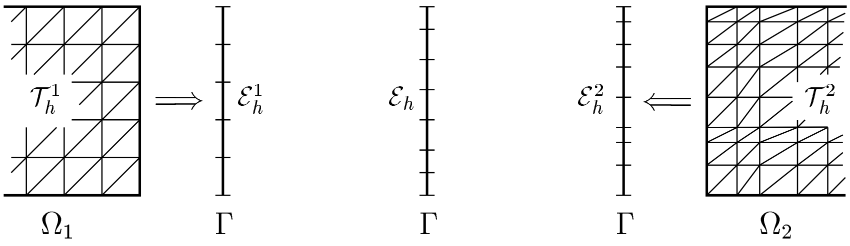

Figure 3:

We require the asymptotic behaviour of the triangulations $\mathcal{T}_h^1, \mathcal{T}_h^2$ and of $\mathcal{E}_h$ to be consistent on Γ in the sense of the following assumption.

Assumption 3.2 *For $T \in \mathcal{T}_h^i$ $(i = 1,2)$ and $E \in \mathcal{E}_h$ with $\partial T \cap E \neq \emptyset$, there are positive constants C_1 and C_2 independent of h_T, h_E and h $(0 < h \leq h_0)$ such that the condition*

$$C_1 h_T \;\leq\; h_E \;\leq\; C_2 h_T \tag{3.9}$$

is satisfied.

Relation (3.9) guarantees that the diameter h_T of the triangle T touching the interface Γ at E is asymptotically equivalent to the diameter h_E of the segment E, i.e., the equivalence of h_T, h_E is required only locally.

4 Properties of the discretization

First we show that the solution u of the BVP (2.1) satisfies the variational equation (3.7), i.e., u is consistent with the approach (3.7).

Theorem 4.1 *Let u be the solution of the BVP (2.1). Then $u = (u^1, u^2)$ solves (3.7), i.e., we have*

$$\mathcal{B}_h(u, v_h) \;=\; \mathcal{F}_h(v_h) \qquad \forall\, v_h \in V_h. \tag{4.1}$$

Proof. Insert the solution u into $\mathcal{B}_h(.,v_h)$. Owing to the properties of u, $\mathcal{B}_h(u,v_h)$ is well defined and, since $u^1|_\Gamma = u^2|_\Gamma$ and $\frac{\partial u^1}{\partial n_1}\Big|_\Gamma = -\frac{\partial u^2}{\partial n_2}\Big|_\Gamma$ hold, cf. (2.5), we get

$$\mathcal{B}_h(u, v_h) \;=\; \sum_{i=1}^{2} \left(\nabla u^i, \nabla v_h^i\right)_{\Omega_i} - \left\langle \frac{\partial u^1}{\partial n_1}, v_h^1 \right\rangle_\Gamma - \left\langle \frac{\partial u^2}{\partial n_2}, v_h^2 \right\rangle_\Gamma .$$

Taking into account (2.4) and using Green's formula on the domains Ω_i, the relations

$$\mathcal{B}_h(u, v_h) \;=\; -\sum_{i=1}^{2} \left(\Delta u^i, v_h^i\right)_{\Omega_i} \;=\; \sum_{i=1}^{2} \left(f, v_h^i\right)_{\Omega_i} \;=\; \mathcal{F}_h(v_h)$$

are derived for any $v_h \in V_h$. This proves the assertion. □

Note that due to (4.1) and (3.7) we also have the $\mathcal{B}_h$-orthogonality of the error $u - u_h$ on V_h, i.e.,

$$\mathcal{B}_h(u - u_h, v_h) \;=\; 0 \qquad \forall\, v_h \in V_h. \tag{4.2}$$

For further results on stability and convergence of the method, the following "weighted discrete trace theorem" will be useful, which describes also an inverse inequality.

Lemma 4.2 *Let Assumption 3.1 and 3.2 be satisfied. Then, for any $v_h \in V_h$ the inequality*

$$\sum_{E \in \mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \;\leq\; C_I \sum_{i=1}^{2} \alpha_i^2 \sum_{F \in \mathcal{E}_h^i} \left\| \nabla v_h^i \right\|_{0,T_F}^2 \tag{4.3}$$

holds, where $F \in \mathcal{E}_h^i$ is the face of a triangle $T_F \in \mathcal{T}_h^i$ touching Γ by F $(T_F \cap \Gamma = F)$. The constant C_I does not depend on h, h_T, h_E.

Note that extending the norms on the right-hand side of (4.3) to the whole of Ω_i implies

$$\sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \;\le\; C_I \sum_{i=1}^{2} \alpha_i^2 \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2 . \tag{4.4}$$

For inequalities on quasi-uniform meshes related with (4.4) see [23, 21, 4].

Proof. For $i = 1, 2$, $v_h^i \in V_h^i$ yields $\left.\frac{\partial v_h^i}{\partial x_s}\right|_\Gamma \in L_2(\Gamma)$ $(s = 1, 2)$ and $\left.\frac{\partial v_h^i}{\partial n_i}\right|_\Gamma \in L_2(\Gamma)$. Moreover,

$$\left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \le 2 \sum_{i=1}^{2} \alpha_i^2 \left\| \nabla v_h^i \right\|_{0,E}^2$$

holds. Let h_F denote the length of side F belonging to triangle $T = T_F$. Since the shape regularity of T is given, the quantities h_F and h_T are asymptotically equivalent. Owing to $\sum_{E\in\mathcal{E}_h} h_E \left\|\nabla v_h^i\right\|_{0,E}^2 \le c_1 \sum_{F\in\mathcal{E}_h^i} h_F \left\|\nabla v_h^i\right\|_{0,F}^2$ and to inequality

$$\left\|\nabla v_h^i\right\|_{0,F}^2 \le c_2 \frac{1}{h_F} \left\|\nabla v_h^i\right\|_{0,T_F}^2 ,$$

which is derived by means of the trace theorem on T_F and of the inverse inequality, we get

$$\sum_{E\in\mathcal{E}_h} h_E \left\|\nabla v_h^i\right\|_{0,E}^2 \;\le\; c_3 \sum_{F\in\mathcal{E}_h^i} \left\|\nabla v_h^i\right\|_{0,T_F}^2 \qquad \text{for } i = 1, 2, \tag{4.5}$$

where $T_F \subset \overline{\Omega}_i$ has the edge $F \in \mathcal{E}_h^i$. The constants c_i $(i = 1, 2, 3)$ do not depend on h; c_2 is also uniform in T. Inequality (4.5) combined with the previous inequalities yields (4.3). ☐

The constant C_I in the inequalities (4.3) and (4.4) can be estimated easily if special assumptions on $\mathcal{E}_h$ and on the polynomial degree k are made. For example, let us choose $\mathcal{E}_h = \mathcal{E}_h^1$ from (3.8), $\alpha_1 = 1$ and $k = 1$, i.e., $v_h^i|_T \in \mathbb{P}_1$. Then, on the triangle T the derivatives $\frac{\partial v_h^1}{\partial x_s}$ $(s = 1, 2)$ and $\frac{\partial v_h^1}{\partial n_1}$ are constants which can be calculated explicitely, together with their L_2-norms on E and on T_E. Thus, we get

$$h_E \left\| \frac{\partial v_h^1}{\partial n_1} \right\|_{0,E}^2 \;\le\; 2 \frac{h_E}{h_{H_E}} \left\|\nabla v_h^1\right\|_{0,T_E}^2 , \tag{4.6}$$

where h_{H_E} denotes the height of T_E over the side E, h_E the length of E. Taking the sum over $E \in \mathcal{E}_h^1$ for all inequalities (4.6), we obtain the value of C_I to be

$$C_I \;=\; \max_{E\in\mathcal{E}_h^1} \left(2 \frac{h_E}{h_{H_E}} \right) .$$

Thus, for equilateral triangles and isosceles rectangular triangles (see the mesh on the left-hand sides of Figures 6, 7) near Γ, we get $C_I = 4/\sqrt{3}$ and $C_I = 2$, respectively.
For deriving the V_h-ellipticity and V_h-boundedness of the discrete bilinear form $\mathcal{B}_h(.,.)$ from (3.6), we introduce the following discrete norm $\|\cdot\|_{1,h}$:

$$\|v_h\|_{1,h}^2 \;:=\; \sum_{i=1}^{2} \left\|\nabla v_h^i\right\|_{0,\Omega_i}^2 + \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\|v_h^1 - v_h^2\right\|_{0,E}^2 \tag{4.7}$$

cf. [21] and [9, 4] (uniform weights). Then we can prove the following theorem.

Theorem 4.3 *Let Assumptions 3.1 and 3.2 for $\mathcal{T}_h^i$ $(i = 1, 2)$ and for $\mathcal{E}_h$ be satisfied. Choose the constant γ in (3.6) independently of h and such that $\gamma > C_I$ holds, C_I from (4.3). Then,*

$$\mathcal{B}_h(v_h,v_h) \;\; \geq \;\; \mu_1 \left\| v_h \right\|_{1,h}^2 \qquad \forall\, v_h \in V_h \tag{4.8}$$

holds, with a constant $\mu_1 > 0$ independent of h.

Proof. For $\mathcal{B}_h(.,.)$ from (3.6) we have the identity

$$\mathcal{B}_h(v_h,v_h) \;=\; \sum_{i=1}^{2} \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2 - 2 \sum_{E\in\mathcal{E}_h} \left\langle \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2}, v_h^1 - v_h^2 \right\rangle_E + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2 .$$

Using Cauchy's inequality and Young's inequality ($2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$) we get

$$\begin{aligned}\mathcal{B}_h(v_h,v_h) \;\geq\; & \sum_{i=1}^{2} \left\| \nabla v_h^i \right\|_{0,\Omega_i}^2 - \frac{1}{\varepsilon} \sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_1 \frac{\partial v_h^1}{\partial n_1} - \alpha_2 \frac{\partial v_h^2}{\partial n_2} \right\|_{0,E}^2 \\ & - \varepsilon \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2 + \gamma \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v_h^1 - v_h^2 \right\|_{0,E}^2 .\end{aligned}$$

Utilizing inequality (4.3) yields (4.8), with $\mu_1 = \min\{1 - \frac{C_I}{\varepsilon}, \gamma - \varepsilon\} > 0$, if ε is chosen according to $C_I < \varepsilon < \gamma$. ☐

Beside of the V_h-ellipticity of $\mathcal{B}_h(.,.)$ we also prove the V_h-boundedness.

Theorem 4.4 *Let Assumption 3.1 and 3.2 be satisfied. Then there is a constant $\mu_2 > 0$ such that the following relations holds,*

$$\left| \mathcal{B}_h\left(w_h, v_h\right) \right| \;\; \leq \;\; \mu_2 \left\| w_h \right\|_{1,h} \left\| v_h \right\|_{1,h} \qquad \textit{for } w_h, v_h \in V_h . \tag{4.9}$$

Proof. We apply Cauchy's inequality several times (also with distributed weights h_E, h_E^{-1}, $h_E h_E^{-1} = 1$), insert inequality (4.3) and get relation (4.9) with a constant $\mu_2 = \max\{1 + C_I, 1 + \gamma\}$. ☐

5 Error estimates and convergence

Let u be the solution of (2.1) and u_h from (3.7) its finite element approximation. We shall study the error $u - u_h$ in the norm $\|\,.\,\|_{1,h}$ given in (4.7). For functions v satisfying $v^i \in H^1(\Omega_i)$ and $\frac{\partial v^i}{\partial n_i} \in L_2(\Gamma)$ $(i = 1, 2)$, introduce the mesh-dependent norm $\|\,.\,\|_{h,\Omega}$ by

$$\|v\|_{h,\Omega}^2 \;:=\; \sum_{i=1}^{2} \left(\left\| \nabla v^i \right\|_{0,\Omega_i}^2 + \sum_{E\in\mathcal{E}_h} h_E \left\| \alpha_i \frac{\partial v^i}{\partial n_i} \right\|_{0,E}^2 \right) + \sum_{E\in\mathcal{E}_h} h_E^{-1} \left\| v^1 - v^2 \right\|_{0,E}^2 . \tag{5.1}$$

First we bound $\|u - u_h\|_{1,h}$ by the norm $\|\,.\,\|_{h,\Omega}$ of the interpolation error $u - I_h u$, where $I_h u := (I_h u^1, I_h u^2)$, $I_h u^i \in V_h^i$, and $I_h u^i$ denotes the usual Lagrange interpolant of u^i in the space V_h^i, $i = 1, 2$.

Lemma 5.1 *Let Assumption 3.1 and 3.2 be satisfied. For u, u_h from (2.1), (3.7), respectively, and $\gamma > C_I$, the following estimate holds,*

$$\|u - u_h\|_{1,h} \;\leq\; c\,\|u - I_h u\|_{h,\Omega}\,. \tag{5.2}$$

Proof. Obviously, $I_h u \in V_h$ holds, and the triangle inequality yields

$$\|u - u_h\|_{1,h} \;\leq\; \|u - I_h u\|_{1,h} + \|I_h u - u_h\|_{1,h}\,. \tag{5.3}$$

Owing to $I_h u - u_h \in V_h$ and to the V_h-ellipticity of $\mathcal{B}_h(.\,,.)$, we have

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1}\left(\mathcal{B}_h(I_h u, I_h u - u_h) \;-\; \mathcal{B}_h(u_h, I_h u - u_h)\right). \tag{5.4}$$

In relation (5.4) we utilize (4.2) and get

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1}\,\mathcal{B}_h(I_h u - u, I_h u - u_h). \tag{5.5}$$

For abbreviation we use here $w := I_h u - u$ and $v_h := I_h u - u_h$. Clearly $u \in H^{\frac{3}{2}+\varepsilon}(\Omega)$ yields $\left.\frac{\partial u^i}{\partial n_i}\right|_\Gamma \in L_2(\Gamma)$. Because of $I_h u$, $u_h \in V_h$, we also have $\left.\frac{\partial v_h^i}{\partial n_i}\right|_\Gamma \in L_2(\Gamma)$ (although $I_h u^i$ denoting the interpolant of u^i in V_h^i and u_h^i belong only to $H^{\frac{3}{2}-\varepsilon}(\Omega_i)$). Unfortunately, $w \notin V_h$ holds, but $\mathcal{B}_h(w, v_h)$ is well-defined.
We now apply the same inequalities as used for the proof of Theorem 4.4, with the modification that inequality (4.3) is only employed with respect to the function v_h. This leads to the estimate

$$|\mathcal{B}_h(w, v_h)| \quad\leq\quad c_1\,\|w\|_{h,\Omega}\,\|v_h\|_{1,h}\,,$$

which gives together with (5.5) the inequality

$$\|I_h u - u_h\|_{1,h}^2 \;\leq\; \mu_1^{-1} c_1\,\|I_h u - u\|_{h,\Omega}\,\|I_h u - u_h\|_{1,h}\,.$$

This inequality combined with (5.3) and with the obvious estimate $\|I_h u - u\|_{1,h} \leq \|I_h u - u\|_{h,\Omega}$ confirms assertion (5.2). The positive constant c_1 depends on γ and C_I. ☐

An estimate of the error $\|u - u_h\|_{1,h}$ for regular solutions u is given in [20] and in [4] by citation of results contained in [23]. Nevertheless, since we consider a more general case, and since we need a great part of the proof for regular solutions also for singular solutions, the following theorem is proved.

Theorem 5.2 *Let $u \in H^l(\Omega)$ ($l \geq 2$) be the solution of (2.1) and $u_h \in V_h$ its finite element approximation according to (3.7), with $\gamma > C_I$. Furthermore, let the mesh from Assumptions 3.1, 3.2 be quasi-uniform, i.e. $\frac{\max_{T \in \mathcal{T}_h} h_T}{\min_{T \in \mathcal{T}_h} \varrho_T} \leq C$. Then the following error estimate holds,*

$$\|u - u_h\|_{1,h} \;\leq\; c\,h^{l-1}\,\|u\|_{l,\Omega} \qquad \textit{for } 2 \leq l \leq k+1, \tag{5.6}$$

with $k \geq 1$ being the polynomial degree in V_h^i, $i = 1, 2$.

Proof. We start from inequality (5.2) which bounds $\|u-u_h\|_{1,h}$ by the interpolation error $\|I_hu-u\|_{h,\Omega}$ and, in the following, take into account tacitly the assumptions on the mesh. Note that the traces on Γ of the interpolants I_hu^i of u^i in V_h^i $(i=1,2)$ do not coincide, in general. First we observe that the weighted squared norms $\|\,.\,\|_{0,E}^2$ can be rewritten such that interpolation estimates involve the edge F of the triangle $T\subset\overline{\Omega}_i$ $(T=T_F)$ with $T\cap\Gamma=F\in\mathcal{E}_h^i$, for $i=1$ or $i=2$:

$$\sum_{E\in\mathcal{E}_h} h_E^{-1}\left\|I_hu^i-u^i\right\|_{0,E}^2 \;\le\; c_1\sum_{F\in\mathcal{E}_h^i} h_F^{-1}\left\|I_hu^i-u^i\right\|_{0,F}^2, \tag{5.7}$$

$$\sum_{E\in\mathcal{E}_h} h_E\left\|\frac{\partial\left(I_hu^i-u^i\right)}{\partial n_i}\right\|_{0,E}^2 \;\le\; c_2\sum_{F\in\mathcal{E}_h^i} h_F\left\|\nabla\left(I_hu^i-u^i\right)\right\|_{0,F}^2. \tag{5.8}$$

Moreover, we apply the refined trace theorem

$$\|v\|_{0,F}^2 \;\le\; c\left(h_T^{-1}\|v\|_{0,T}^2+\|v\|_{0,T}\|\nabla v\|_{0,T}\right) \qquad \text{for } v\in H^1(T), \tag{5.9}$$

which is proved in [24], cf. also [23]. Replace v by $I_hu^i-u^i$ and $\frac{\partial(I_hu^i-u^i)}{\partial x_s}$ $(s=1,2)$. Then, using (5.9) and some simple estimates, we get

$$\left\|I_hu^i-u^i\right\|_{0,F}^2 \;\le\; c\left(h_T^{-1}\left\|I_hu^i-u^i\right\|_{0,T}^2+\left\|I_hu^i-u^i\right\|_{0,T}\left|I_hu^i-u^i\right|_{1,T}\right), \tag{5.10}$$

$$\left\|\nabla\left(I_hu^i-u^i\right)\right\|_{0,F}^2 \;\le\; c\left(h_T^{-1}\left|I_hu^i-u^i\right|_{1,T}^2+\left|I_hu^i-u^i\right|_{1,T}\left|I_hu^i-u^i\right|_{2,T}\right). \tag{5.11}$$

Taking the well-known interpolation error estimate on triangles T,

$$\left\|I_hu^i-u^i\right\|_{j,T} \;\le\; ch_T^{l-j}\left\|u^i\right\|_{l,T} \qquad \text{for } 2\le l\le k+1 \text{ and } j=0,1,2, \tag{5.12}$$

see e.g. [8, 11], we derive from the inequalities (5.10) and (5.11) the estimates

$$\left\|I_hu^i-u^i\right\|_{0,F}^2 \;\le\; ch_T^{2l-1}\left\|u^i\right\|_{l,T}^2, \qquad \left\|\nabla\left(I_hu^i-u^i\right)\right\|_{0,F}^2 \;\le\; ch_T^{2l-3}\left\|u^i\right\|_{l,T}^2.$$

Using these estimates and (5.7), (5.8), we realize that

$$\sum_{E\in\mathcal{E}_h}\left(h_E^{-1}\left\|I_hu^i-u^i\right\|_{0,E}^2+h_E\left\|\frac{\partial\left(I_hu^i-u^i\right)}{\partial n_i}\right\|_{0,E}^2\right) \;\le\; ch^{2l-2}\sum_{\substack{T\in\mathcal{T}_h^i:\\ T\cap\Gamma\neq\emptyset}}\left\|u^i\right\|_{l,T}^2 \tag{5.13}$$

holds. For the interpolation error $I_hu^i-u^i$ on Ω_i, the estimate

$$\left\|\nabla\left(I_hu^i-u^i\right)\right\|_{0,\Omega_i}^2 \;=\; \left|I_hu^i-u^i\right|_{1,\Omega_i}^2 \;\le\; ch^{2l-2}\left|u^i\right|_{l,\Omega_i}^2 \tag{5.14}$$

obviously follows from (5.12). Clearly, (5.13) and (5.14) lead via (5.2) to (5.6). ☐

6 Treatment of corner singularities

We now study the finite element approximation with non-matching meshes for the case that Γ has endpoints at vertices of re-entrant corners (case I1). Since the influence region of corner singularities is a local one (around the vertex P_0), it suffices to consider one corner. For basic approaches of treating corner singularities by finite element methods see e.g. [3, 7, 13, 17, 19, 22]. For simplicity, we study solutions $u \notin H^2(\Omega)$ in correspondence with continuous piecewise linear elements, i.e. $k = 1$ in V_h^i from (3.3). We shall consider the error $u - u_h$ on quasi-uniform meshes as well as on meshes with appropriate local refinement at the corner.

Let (x_0, y_0) be the coordinates of the vertex P_0 of the corner, (r, φ) the local polar coordinates with center at P_0, i.e. $x - x_0 = r\cos(\varphi + \varphi_r)$, $y - y_0 = r\sin(\varphi + \varphi_r)$, cf. Figure 4.

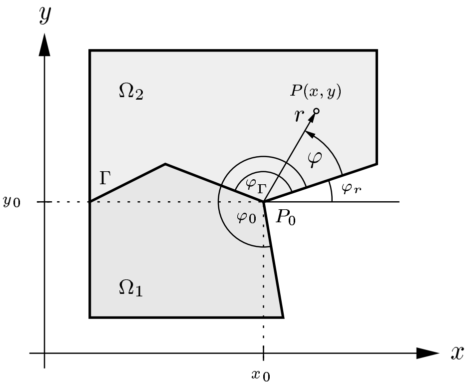

Figure 4:

Define some circular sector $\overline{G}$ around P_0, with the radius $r_0 > 0$ and the angle φ_0 (here: $\pi < \varphi_0 < 2\pi$):

$$\overline{G} \;:=\; \left\{(x,y) \in \overline{\Omega} : 0 \le r \le r_0,\, 0 \le \varphi \le \varphi_0\right\}, \quad G := \overline{G} \setminus \partial G, \tag{6.1}$$

∂G boundary of G. For defining a mesh with grading, we employ the real grading parameter μ, $0 < \mu \le 1$, the grading function R_i $(i = 0, 1, \ldots, n)$ with some real constant $b > 0$, and the step size h_i for the mesh associated with layers $[R_{i-1}, R_i] \times [0, \varphi_0]$ around P_0:

$$R_i \;:=\; b\,(ih)^{\frac{1}{\mu}} \quad (i = 0, 1, \ldots, n), \qquad h_i \;:=\; R_i - R_{i-1} \quad (i = 1, 2, \ldots, n). \tag{6.2}$$

Here $n := n(h)$ denotes an integer of the order h^{-1}, $n := \left[\beta h^{-1}\right]$ for some real $\beta > 0$ ($[.]$: integer part). We shall choose the numbers β, $b > 0$ such that $\frac{2}{3}r_0 < R_n < r_0$ holds, i.e., the mesh grading is located within $\overline{G}$ from (6.1).

Lemma 6.1 *For h, h_i, R_i, and μ $(0 < h \le h_0,\, 0 < \mu < 1)$ the following relations hold*

$$\begin{array}{llll} b^\mu h R_i^{1-\mu} \;\le\; h_i \;\le\; \frac{b^\mu}{\mu} h R_i^{1-\mu}, & b R_i \frac{1}{i} \;\le\; h_i \;\le\; \frac{b}{\mu} R_i \frac{1}{i}, & (i = 1, 2, \ldots, n), \\ h_{i-1} \;<\; h_i \;\le\; (2^{\frac{1}{\mu}} - 1) h_{i-1}, & R_{i-1} \;<\; R_i \;\le\; 2^{\frac{1}{\mu}} R_{i-1}, & (i = 2, 3, \ldots, n). \end{array} \tag{6.3}$$

We skip the proof of Lemma 6.1 since it is comparatively simple.

Using the step size h_i $(i = 1, 2, \ldots, n)$, define in the neighbourhood of the vertex P_0 of the corner a mesh with grading, and for the remaining domain we employ a mesh which is quasi-uniform. The triangulation $\mathcal{T}_h^\mu$ is now characterized by the mesh size h and the grading parameter μ, with $0 < h \le h_0$ and $0 < \mu \le 1$. We summarize the properties of $\mathcal{T}_h^\mu$ in the following assumption.

Assumption 6.2 *The triangulation $\mathcal{T}_h^\mu$ satisfies Assumption 3.1, Assumption 3.2 and is provided with a grading around the vertex P_0 of the corner such that $h_T := \operatorname{diam} T$ depends on the distance R_T of T from P_0, $R_T := \operatorname{dist}(T, P_0) := \inf_{P \in T} |P_0 - P|$, in the following way:*

$$\begin{array}{rcccll} \varrho_1 h^{\frac{1}{\mu}} & \leq & h_T & \leq & \varrho_1^{-1} h^{\frac{1}{\mu}} & \text{for } T \in \mathcal{T}_h^\mu : R_T = 0, \\ \varrho_2 h R_T^{1-\mu} & \leq & h_T & \leq & \varrho_2^{-1} h R_T^{1-\mu} & \text{for } T \in \mathcal{T}_h^\mu : 0 < R_T < R_g, \\ \varrho_3 h & \leq & h_T & \leq & \varrho_3^{-1} h & \text{for } T \in \mathcal{T}_h^\mu : R_g \leq R_T, \end{array} \tag{6.4}$$

with some constants ϱ_i, $0 < \varrho_i \leq 1$ $(i = 1, 2, 3)$ and some real R_g, $0 < \underline{R}_g < R_g < \overline{R}_g$, where $\underline{R}_g, \overline{R}_g$ are fixed and independent of h.

Here, R_g is the radius of the sector with mesh grading and we can assume $R_g = R_n$ (w.l.o.g.). Outside this sector the mesh is quasi-uniform. The value $\mu = 1$ yields a quasi-uniform mesh in the whole region Ω, i.e., $\frac{\max_{T \in \mathcal{T}_h^\mu} h_T}{\min_{T \in \mathcal{T}_h^\mu} \varrho_T} \leq C$ holds. In [3, 17, 19] related types of mesh grading are described. In [15] a mesh generator is given which automatically generates a mesh of type (6.4).

For the error analysis we introduce several subsets of the triangulation $\mathcal{T}_h^\mu$ near the vertex P_0 of the re-entrant corner, viz.

$$\mathcal{C}_{0h} := \{T \in \mathcal{T}_h^\mu : R_T < R_n\}, \quad \mathcal{C}_h := \{T \in \mathcal{T}_h^\mu : R_T \geq R_n\},$$

with R_n from (6.2). The set $\mathcal{C}_{0h}$ is now decomposed into layers (of triangles) $\mathcal{D}_{jh}$, $j = 0, 1, \ldots, n$, such that $\mathcal{C}_{0h} := \bigcup_{j=0}^n \mathcal{D}_{jh}$ holds:

$$\begin{array}{rcl} \mathcal{D}_{0h} & := & \{T \in \mathcal{T}_{h\mu} : \ R_T = 0\}, \quad \mathcal{D}_{1h} := \{T \in \mathcal{T}_{h\mu} : \ 0 < R_T < R_1\}, \\ \mathcal{D}_{jh} & := & \{T \in \mathcal{T}_{h\mu} : \ R_{j-1} \leq R_T < R_j\} \quad \text{for } j = 2, \ldots, n. \end{array}$$

According to $\frac{2}{3} r_0 < R_n < r_0$, the triangles $T \in \mathcal{C}_{0h}$ are located in $\overline{G}$, $\overline{G}$ from (6.1). Owing to Assumption 6.2 (cf. also Lemma 6.1), the asymptotic behaviour of h_T is determined by the relations (given for the case of one corner)

$$\begin{array}{rcccll} \varepsilon_2 h_j & \leq & h_T & \leq & \varepsilon_2^{-1} h_j & \text{for } T \in \mathcal{T}_h^\mu : R_{j-1} \leq R_T < R_j \quad (j = 1, 2, \ldots, n), \\ \varepsilon_3 h & \leq & h_T & \leq & \varepsilon_3^{-1} h & \text{for } T \in \mathcal{T}_h^\mu : R_n \leq R_T, \end{array} \tag{6.5}$$

with $0 < \varepsilon_l \leq 1$ $(l = 2, 3)$, and h_j, R_j as well as n taken from (6.2). Note that the number of all triangles $T \in \mathcal{T}_h^\mu$ $(0 < \mu \leq 1)$ and nodes of the triangulation is of the order $O(h^{-2})$. The number n_j of all triangles $T \in \mathcal{D}_{jh}$ is bounded by $C \cdot j$ $(j = 1, \ldots, n)$, n_0 by C, where C is independent of h, cf. [14].

First we investigate the interpolation error of a singularity function s from (2.3) in the class of polynomials with degree $k = 1$. Employ the restrictions $s^i := s|_{\overline{\Omega}_i}$ and take always into account that $s = 0$ for $r \geq \frac{2}{3} r_0$.

Lemma 6.3 *Let $s = \eta\, a\, r^\lambda \sin(\lambda\varphi)$ $(\lambda = \frac{\pi}{\varphi_0}$, $\frac{1}{2} < \lambda < 1)$ be the singularity function with respect to the corner at vertex P_0. Further, let $\mathcal{T}_h^\mu$ be the triangulation of $\overline{\Omega}$ with mesh* *grading within $\overline{G}$ according to Assumption 6.2 (cf. (6.2)–(6.5)). Then, the interpolation error $s^i - I_h s^i$ in the seminorm $|\,.\,|_{1,\Omega_i}$ can be bounded as follows:*

$$\left|s^i - I_h s^i\right|_{1,\Omega_i} \;\le\; c\,|a|\,\kappa(h,\mu) \quad \textit{for } i = 1,2, \tag{6.6}$$

where $\kappa(h,\mu)$ is given by

$$\kappa(h,\mu) \;=\; \begin{cases} h^{\frac{\lambda}{\mu}} & \textit{for } \lambda < \mu \le 1 \\ h\,|\ln h|^{\frac{1}{2}} & \textit{for } \mu = \lambda \\ h & \textit{for } 0 < \mu < \lambda < 1. \end{cases} \tag{6.7}$$

Proof. According to the mesh layers $\mathcal{D}_{jh}$ $(j = 0, 1, \ldots, n)$, the norms of the global interpolation error $s^i - I_h s^i$ are represented by the local interpolation error $s^i - I_T s^i$ ($I_T v^i := I_h v|_T$ for $T \in \Omega_i$, I_T: local $\mathbb{P}_1$–Lagrange interpolation operator) as follows

$$\left|s^i - I_h s^i\right|^2_{1,\Omega_i} \;=\; \sum_{T \in \mathcal{D}^i_{0h}} \left|s^i - I_T s^i\right|^2_{1,T} + \sum_{j=1}^{n} \sum_{T \in \mathcal{D}^i_{jh}} \left|s^i - I_T s^i\right|^2_{1,T} \quad \text{for } i = 1,2,$$

with $\mathcal{D}^i_{jh} := \{T \in \mathcal{D}_{jh} : T \subset \overline{\Omega}_i\}$ $(j = 0, 1, \ldots, n;\ i = 1,2)$.
(i) case $T \in \mathcal{D}^i_{0h}$ $(i = 1,2)$:
First, we consider triangles $T \in \mathcal{D}^i_{0h}$ and employ the estimate

$$\left|s^i - I_T s^i\right|_{1,T} \;\le\; \left|s^i\right|_{1,T} + \left|I_T s^i\right|_{1,T}. \tag{6.8}$$

Using the explicit representation of s^i and $I_T s^i$, we calculate the norms on the right-hand side of (6.8) and get the following bound:

$$\left|s^i\right|_{1,T} + \left|I_T s^i\right|_{1,T} \;\le\; c\;|a|\,h^{\frac{\lambda}{\mu}}, \qquad \text{for } T \in \mathcal{D}^i_{0h}. \tag{6.9}$$

(ii) case $T \in \mathcal{D}^i_{jh}$ $(j = 1, 2, \ldots, n;\ i = 1,2)$:
We now consider triangles $T \in \mathcal{D}^i_{jh}$ which do not touch the vertex P_0 (center of singularity), i.e. $T \in \mathcal{C}_{0h} \setminus \mathcal{D}^i_{0h}$. In this case, $s \in H^2(T)$ holds owing to $R_T > 0$. Hence, the well-known interpolation error estimate

$$\left|s^i - I_T s^i\right|_{1,T} \;\le\; c h_T \left|s^i\right|_{2,T} \tag{6.10}$$

can be applied, where c is independent of the triangle T. The norm $\left|s^i\right|_{2,T}$ is estimated easily by

$$\left|\eta\, a\, r^{\lambda} \sin(\lambda\varphi)\right|^2_{2,T} \le c\,|a| h_T^2 \left(\inf_{P \in T} r\right)^{2(\lambda-2)} \qquad \text{for } T \in \mathcal{D}^i_{jh}. \tag{6.11}$$

Taking into account the relations between h, h_T, R_T, j and μ from Assumption 6.2, cf. also (6.2),(6.3) ,(6.4) and (6.5), a we find easily bounds of the right-hand side in (6.11). This leads together with (6.10) to the estimates

$$\begin{aligned} |s^i - I_h s^i|^2_{1,T} \;&\le\; c\,|a|^2 h^{\frac{2\lambda}{\mu}}\, j^{\frac{4}{\mu}-4}\,(j-1)^{\frac{2\lambda-4}{\mu}} \quad \forall\, T \in \mathcal{D}^i_{jh},\ j = 2, \ldots, n, \\ |s^i - I_h s^i|^2_{1,T} \;&\le\; c\,|a|^2 h^{\frac{2\lambda}{\mu}} \;\; \forall\; T \in \mathcal{D}^i_{1h}, \end{aligned}$$

where $i = 1, 2$. Since the number of triangles in the layer $\mathcal{D}^i_{jh}$ grows not faster than with j, we get by summation of the error contributions of the triangles $T \in \mathcal{C}_{0h} \setminus \mathcal{D}^i_{0h}$ the estimate

$$\sum_{j=1}^{n} \sum_{T \in \mathcal{D}^i_{jh}} \left|s^i - I_h s^i\right|^2_{1,T} \;\le\; c\,|a|^2\, h^{\frac{2\lambda}{\mu}} \left(1 + \sum_{j=2}^{n} j^{\frac{4}{\mu}-3}\,(j-1)^{\frac{2\lambda-4}{\mu}}\right), \quad i = 1, 2. \tag{6.12}$$

Using monotonicity arguments and the estimation of sums by related integrals, it is not hard to derive the following set of inequalities,

$$\sum_{j=1}^{n} j^{\frac{2\lambda}{\mu}-3} \le C \begin{cases} 1 & \text{for } \lambda < \mu \le 1 \\ \ln n & \text{for } \mu = \lambda \\ n^{\frac{2\lambda}{\mu}-2} & \text{for } 0 < \mu < \lambda < 1. \end{cases} \tag{6.13}$$

Some simple estimates of the right-hand side of (6.12) allow to apply (6.13) and $n \le ch^{-1}$ for getting the inequality

$$\sum_{j=1}^{n} \sum_{T \in \mathcal{D}^i_{jh}} \left|s^i - I_h s^i\right|^2_{1,T} \;\le\; c\,|a|^2\, \kappa^2(h,\mu), \tag{6.14}$$

with $\kappa(h,\mu)$ given at (6.7) and for $i = 1, 2$.

Finally, combining the estimates (6.8), (6.9) from case (i) and (6.14) from case (ii), we easily confirm (6.6). $\square$

We now study the interpolation error $s^i - I_h s^i$ and its first order derivatives in the trace norms.

Lemma 6.4 *Under the assumption of Lemma 6.3 and with $\kappa(h,\mu)$ from (6.7), the following interpolation error estimates hold for the singularity function $s = \eta\, a\, r^\lambda \sin(\lambda\varphi)$ and $i = 1, 2$:*

$$\sum_{E \in \mathcal{E}_h} h_E^{-1} \left\|s^i - I_h s^i\right\|^2_{0,E} \le c\,|a|^2\,\kappa^2(h,\mu), \qquad \sum_{E \in \mathcal{E}_h} h_E \left\|\frac{\partial\left(s^i - I_h s^i\right)}{\partial n_i}\right\|^2_{0,E} \le c\,|a|^2\,\kappa^2(h,\mu). \tag{6.15}$$

Proof. Clearly, due to the assumption on $\mathcal{E}_h$ we have for $v^i = s^i - I_h s^i$ $(i = 1, 2)$ the inequalities

$$\sum_{E \in \mathcal{E}_h} h_E^{-1} \left\|v^i\right\|^2_{0,E} \le c \sum_{F \in \mathcal{E}^i_h} h_F^{-1} \left\|v^i\right\|^2_{0,F}, \qquad \sum_{E \in \mathcal{E}_h} h_E \left\|\frac{\partial v^i}{\partial n_i}\right\|^2_{0,E} \le c \sum_{F \in \mathcal{E}^i_h} h_F \left\|\nabla v^i\right\|^2_{0,F}. \tag{6.16}$$

Consider now faces F of triangles $T = T_F$ touching Γ and the local interpolate $I_T s^i$.
(i) case $T \in \mathcal{D}^i_{0h}$ $(i = 1, 2)$:
Here we use a similar approach like at (6.8) and get by direct evaluation of the norms the following estimates:

$$h_F^{-1} \left\|s^i - I_T s^i\right\|^2_{0,F} \;\le\; 2\Big(h_F^{-1} \left\|s^i\right\|^2_{0,F} + h_F^{-1} \left\|I_T s^i\right\|^2_{0,F}\Big) \le c\,|a|^2 h_F^{2\lambda} \;\le\; c\,|a|^2 h^{\frac{2\lambda}{\mu}}, \tag{6.17}$$

$$h_F \left\|\nabla\left(s^i - I_T s^i\right)\right\|^2_{0,F} \;\le\; 2\Big(h_F \left\|\nabla s^i\right\|^2_{0,F} + h_F \left\|\nabla(I_T s^i)\right\|^2_{0,F}\Big) \le c\,|a|^2 h_F^{2\lambda} \;\le\; c\,|a|^2 h^{\frac{2\lambda}{\mu}}. \tag{6.18}$$

(ii) case $T \in \mathcal{D}^i_{jh}$ $(j = 1, 2, \ldots, n;\ i = 1, 2)$:
For the remaining faces F and adjacent triangles T which do not touch the vertex P_0 of the corner, $s^i \in H^2(T)$ holds. Therefore, inequalities (5.10), (5.11) can be applied. We insert the well-known estimates $|s^i - I_T s^i|_{l,T} \le c h_T^{2-l} |s^i|_{2,T}$ $(l = 0, 1, 2)$ into (5.10), (5.11) and get for any triangle with face $F \subset \Gamma$:

$$h_F^{-1} \left\|s^i - I_T s^i\right\|^2_{0,F} \;\le\; c h_T^2 \left|s^i\right|^2_{2,T}, \qquad h_F \left\|\nabla(s^i - I_T s^i)\right\|^2_{0,F} \;\le\; c h_T^2 \left|s^i\right|^2_{2,T}. \tag{6.19}$$

Calculating and estimating $|s^i|^2_{2,T}$ and summation over all triangles $T \in \mathcal{C}_{0h} \setminus \mathcal{D}_{0h}$ touching Γ near the singularity yields by analogy to (6.14) the estimate

$$\sum_{j=1}^{n} \sum_{\substack{T \in \mathcal{D}^i_{jh}: \\ T \cap \Gamma \neq \emptyset}} h_T^2 \left|s^i\right|^2_{2,T} \;\le\; c\, |a|^2\, \kappa^2(h,\mu) \;\text{ for } i = 1, 2. \tag{6.20}$$

Finally, we combine the inequalities (6.16)–(6.20) and get (6.15). ☐

Lemma 6.5 *Assume that there is one re-entrant corner and that the triangulation $\mathcal{T}_h^\mu$ is provided with mesh grading according to the Assumption 6.2. Then the following estimate holds for the error $u - I_h u$ of the Lagrange interpolant $I_h u \in V_h$, with u from (2.3) and $\kappa(h,\mu)$ from (6.7):*

$$\|u - I_h u\|_{h,\Omega} \;\le\; c\kappa(h,\mu)\, \|f\|_{0,\Omega}. \tag{6.21}$$

Proof. According to (2.3), the solution u of the BVP (2.1) can be represented by $u = s + w = \eta\, a\, r^\lambda \sin(\lambda\varphi) + w$, where $w \in H^2(\Omega)$ denotes the regular part of the solution, and s is the singular part. Apply the triangle inequality $||u - I_h u||_{h,\Omega} \le\ ||s - I_h s||_{h,\Omega} + ||w - I_h w||_{h,\Omega}$. Since $w \in H^2(\Omega) \cap H^1_0(\Omega)$ holds, the norm $||w - I_h w||_{h,\Omega}$ has been already estimated in the proof of Theorem 5.2. Thus, using the estimates (5.13) and (5.14) for $l = k+1 = 2$, together with (2.4), we get

$$\|w - I_h w\|_{h,\Omega} \;\le\; ch\, \|w\|_{2,\Omega} \;\le\; ch\, \|f\|_{0,\Omega}. \tag{6.22}$$

Bounds of the norm $\|s - I_h s\|_{h,\Omega}$ can be derived from Lemma 6.3 and Lemma 6.4. The combination of (6.6), (6.15) and (2.4) yields the inequalities

$$\|s - I_h s\|_{h,\Omega} \;\le\; c\kappa(h,\mu)\, |a| \;\le\; c\kappa(h,\mu)\, \|f\|_{0,\Omega}, \tag{6.23}$$

with $\kappa(h,\mu)$ from (6.7). Estimate (6.21) is obvious by (6.22) and (6.23). ☐

The final error estimate is given in the next theorem.

Theorem 6.6 *Let u and u_h be the solutions of the BVP (2.1) with one re-entrant corner and of the finite element equation (3.7), respectively. Further, for $\mathcal{T}_h^\mu$ let Assumption 6.2 be satisfied. Then the error $u - u_h$ in the norm $\|\,.\,\|_{1,h}$ (4.7) is bounded by*

$$\|u - u_h\|_{1,h} \;\le\; c\kappa(h,\mu)\, \|f\|_{0,\Omega}, \tag{6.24}$$

$$\textit{with}\quad \kappa(h,\mu) \;=\; \begin{cases} h^{\frac{\lambda}{\mu}} & \textit{for } \lambda < \mu \le 1 \\ h\, |\ln h|^{\frac{1}{2}} & \textit{for } \mu = \lambda \\ h & \textit{for } 0 < \mu < \lambda < 1. \end{cases}$$

Proof. The combination of Lemma 5.1 with Lemma 6.5 immediately yields the assertion. ☐

Remark 6.7 Estimate (6.24) holds also for more than one re-entrant corner, with a slightly modified function $\kappa(h,\mu)$. For example, if the mortar interface Γ touches the vertices P_{01}, P_{02} ($P_{01} \neq P_{02}$) of two re-entrant corners with angles $\varphi_{01}, \varphi_{02}$, say $\pi < \varphi_{01} \leq \varphi_{02} < 2\pi$, then $\frac{1}{2} < \lambda_2 \leq \lambda_1 < 1$ ($\lambda_j = \frac{\pi}{\varphi_{0j}}$) holds. According to λ_1, λ_2, we employ meshes with grading parameters μ_1, μ_2. Estimate (6.24) holds now with

$$\kappa(h,\mu) \;=\; \begin{cases} h^{\delta} & \text{for } \delta < 1 \\ h\,|\ln h|^{\frac{1}{2}} & \text{for } \delta = 1 \\ h & \text{for } \delta > 1 \end{cases}, \qquad \text{where } \delta := \min_{1\leq j\leq 2} \frac{\lambda_j}{\mu_j}.$$

Remark 6.8 Under the assumption of Theorem 6.6 and for the error in the L_2–norm, the estimate

$$\|u - u_h\|_{0,\Omega} \;\leq\; c\kappa^2(h,\mu)\, \|f\|_{0,\Omega} \tag{6.25}$$

holds. In particular, we have the $O(h^2)$ convergence rate for meshes with appropriate grading. Estimate (6.25) is proved by the Nitsche trick with additional ingredients, e.g. include again some interpolant (cf. the proof of Lemma 5.1). For the proof in the conforming case see e.g. [14].

7 Numerical experiments

We shall give some illustration of the Nitsche type mortaring in presence of some corner singularity. In particular we investigate the rate of convergence when local mesh refinement is applied. Consider the BVP

$$-\Delta u \;=\; f \quad \text{in } \Omega\,, \quad u \;=\; 0 \quad \text{on } \partial\Omega\,,$$

where Ω is the L-shaped domain of Figure 5. The right-hand side f is chosen such that the exact solution u is of the form

$$u(x,y) \;=\; (a^2 - x^2)(b^2 - y^2)r^{\frac{2}{3}} \sin(\frac{2}{3}\varphi), \tag{7.1}$$

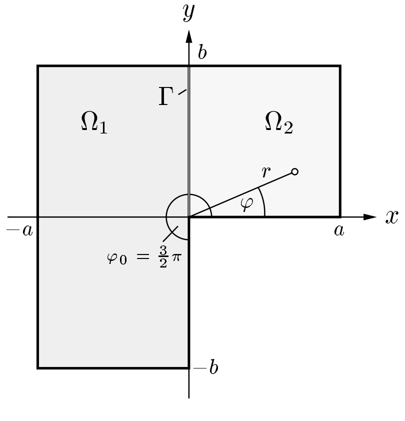

Figure 5: The L-shaped domain Ω.

where $r^2 = x^2 + y^2$, $0 \leq \varphi \leq \varphi_0$, $\varphi_0 = \frac{3}{2}\pi$. Clearly, $u|_{\partial\Omega} = 0$, $\lambda = \frac{\pi}{\varphi_0} = \frac{2}{3}$ and, therefore, $u \in H^{\frac{5}{3}-\varepsilon}(\Omega)$ is satisfied. We apply the Nitsche type mortaring method to this BVP and use initial meshes shown in Figure 6 and 7. The approximate solution u_h is visualized in Figure 9.

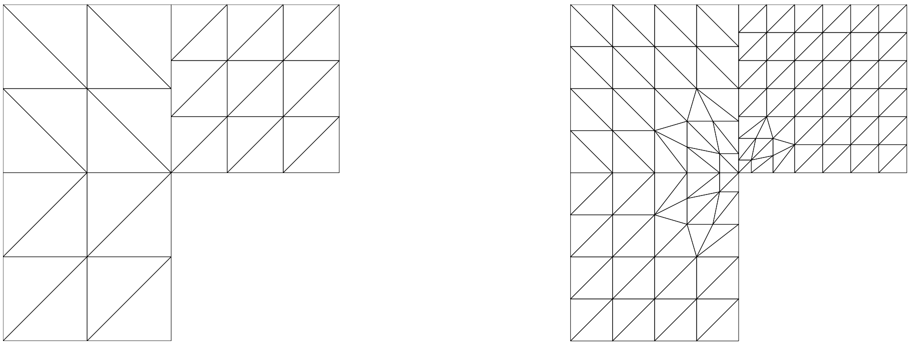

Figure 6: Triangulations with mesh ratio $2:3$, h_1–mesh (left) and h_2–mesh with refinement (right).

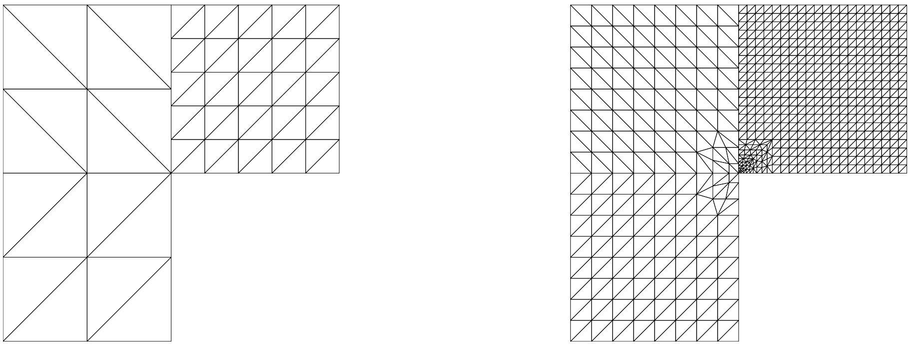

Figure 7: Triangulations with mesh ratio $2:5$, h_1–mesh (left) and h_3–mesh with refinement (right).

The initial mesh is refined globally by dividing each triangle into four equal triangles such that the mesh parameters form a sequence $\{h_1, h_2, h_3, \ldots\}$ given by $\{h, \frac{h}{2}, \frac{h}{4}, \ldots\}$. The ratio of the number of mesh segments on the mortar interface is given by $2:3$ (see Figure 6) and $2:5$ (see Figure 7). Furthermore, the values $\alpha_1 = 1$, $\alpha_2 = 0$ are chosen, i.e., the trace of the triangulation $\mathcal{T}_h^1$ of Ω_1 on the interface Γ forms the partition $\mathcal{E}_h$ (for Ω_1 cf. Figure 5). For the examples the choice $\gamma = 3$ was sufficient to ensure stability. (For numerical experiments with γ and also with regular solutions, cf. [18]). Moreover, we also apply local refinement by grading the mesh around the vertex P_0 of the corner, according to section 6. The parameter is chosen by $\mu = 0.7\lambda$.

Let u_h denote the finite element approximation according to (3.7) of the exact solution u from (7.1). Then the error estimate in the discrete norm $||\,.\,||_{1,h}$ is given by (6.24). We assume that h is sufficiently small such that

$$||u - u_h||_{1,h} \approx Ch^{\alpha} \tag{7.2}$$

holds with some constant C which is approximately the same for two consecutive levels of h, like $h, \frac{h}{2}$. Then $\alpha = \alpha_{obs}$ (observed value) is derived from (7.2) by $\alpha_{obs} := \log_2 q_h$, where $q_h := \|u - u_h\| / \left\| u - u_{\frac{h}{2}} \right\|$. The same is carried out for the L_2–norm, where $\|u - u_h\|_{0,\Omega} \approx Ch^{\beta}$ is supposed. The values of α and β are given in Table 1 and Table 2, respectively.

norm $\|\,.\,\|_{1,h}$	mesh ratio 2 : 3		mesh ratio 2 : 5		
	(h_3, h_4)-levels	(h_4, h_5)-levels	(h_3, h_4)-levels	(h_4, h_5)-levels	α (expected)
$\alpha_{\mu=1}$	0.6977	0.6676	0.7316	0.6798	0.6667
$\alpha_{\mu=0.7\lambda}$	1.1323	0.9784	1.0896	1.1749	1

Table 1: Observed convergence rates α_μ for different pairs $(h_i,\ h_{i+1})$ of h-levels, for $\mu = 1$ and for $\mu = 0.7\lambda$ $(\lambda = \frac{2}{3})$ in the norm $\|\,.\,\|_{1,h}$.

norm $\|\,.\,\|_{0,\Omega}$	mesh ratio 2 : 3		mesh ratio 2 : 5		
	(h_3, h_4)-levels	(h_4, h_5)-levels	(h_3, h_4)-levels	(h_4, h_5)-levels	β (expected)
$\beta_{\mu=1}$	1.2919	1.2971	1.3016	1.2991	1.3333
$\beta_{\mu=0.7\lambda}$	2.0093	2.0835	2.2252	2.0863	2

Table 2: Observed convergence rates β_μ for different pairs $(h_i,\ h_{i+1})$ of h-levels, for $\mu = 1$ and for $\mu = 0.7\lambda$ ($\lambda = \frac{3}{2}$) in the norm $\|\,.\,\|_{0,\Omega}$.

The numerical experiments show that the observed rates of convergence are approximately equal to the expected values. Furthermore, it can be seen that local mesh grading is suited to overcome the loss of accuracy (cf. Figure 9) and the diminishing of the rate of convergence on non-matching meshes caused by corner singularities.

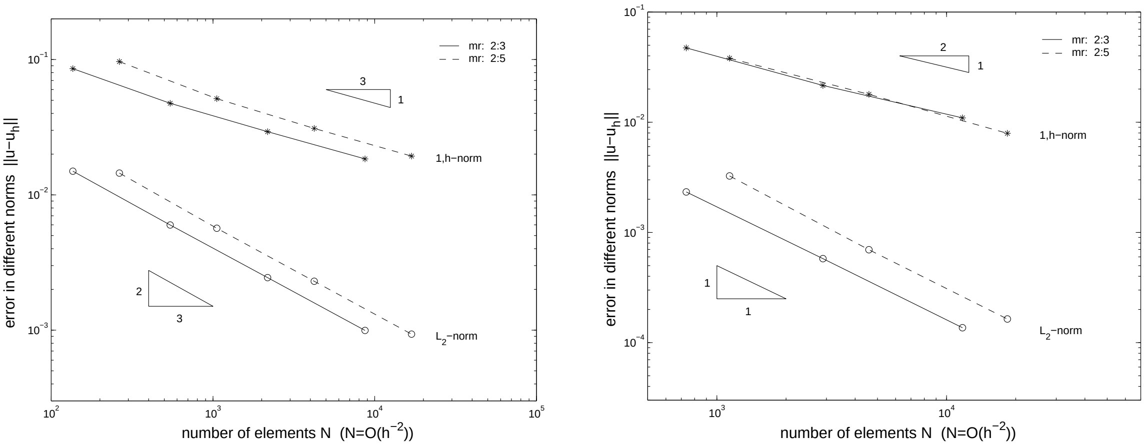

Figure 8: The error in different norms on quasi-uniform meshes (left) and on meshes with grading (right).

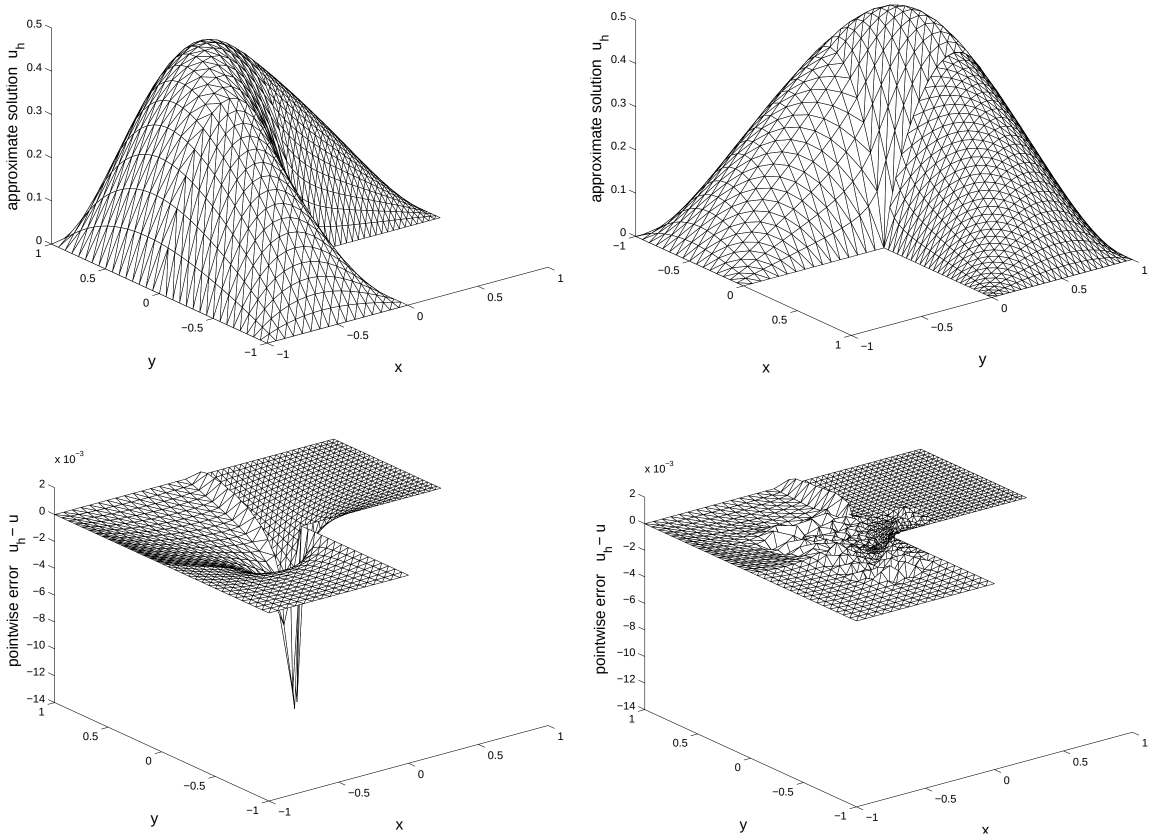

Figure 9: The approximate solution u_h in two different perspectives (top), the local pointwise error on the quasi-uniform mesh (bottom left) and the local pointwise error on the mesh with grading (bottom right).

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