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Preconditioned gradient iterative methods for nonlinear eigenvalue problems

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Abstract. The existence of eigenvalues of a finite-dimensional symmetric eigenvalue problem with nonlinear entrance of the spectral parameter is studied. Preconditioned gradient iterative methods are suggested for solving the problem. The convergence and the error of these methods for computing eigenvalues are investigated.

Keywords. symmetric eigenvalue problem, nonlinear eigenvalue problem, iterative method, preconditioner, gradient methods, steepest descent methods

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Introduction

Preconditioned gradient iterative methods have been studied for the symmetric eigenvalue problem $Au = \lambda Bu$ in an Euclidean space H, for example, in the papers [1–11]. These iterative methods for computing the smallest eigenvalue have the following form:

$$C\,\frac{\tilde{u}^{n+1}-u^n}{\tau_n} + (A-\mu^n B)u^n = 0,$$
$$u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B},$$
$$\mu^n = R(u^n), \quad n = 0, 1, \ldots,$$

where $R(v) = (Av, v)/(Bv, v)$, $v \in H \setminus \{0\}$, the symmetric operator C satisfies the condition: $\delta_0(Cv, v) \leq (Av, v) \leq \delta_1(Cv, v)$, $v \in H$, the iteration parameters τ_n can be chosen by the formula $\tau_n = \delta_1^{-1}$ or to maximize $R(u^{n+1})$. A survey of results on preconditioned iterative methods is presented in the paper [12].

Here, we generalize these methods for solving the symmetric *nonlinear* eigenvalue problem: $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, $A(\lambda)u = \lambda B(\lambda)u$. We propose preconditioned gradient iterative methods of the following kind:

$$C(\mu^n)\,\frac{\tilde{u}^{n+1}-u^n}{\tau_n} + (A(\mu^n) - \mu^n B(\mu^n))u^n = 0,$$
$$u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_{B(\mu^n)}},$$
$$\mu^n = R(\mu^n, u^n), \quad n = 0, 1, \ldots,$$

where $R(\mu, v) = (A(\mu)v, v)/(B(\mu), v, v)$, $v \in H \setminus \{0\}$, $\mu \in \Lambda$, Λ is an interval on the real axis, the symmetric operator $C(\mu)$ satisfies the condition: $\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v)$, $v \in H \setminus \{0\}$, $\mu \in \Lambda$, the iteration parameters τ_n are defined by the formula $\tau_n = \delta_1^{-1}(\mu^n)$ or to maximize $R(\mu^n, u^{n+1})$. In the paper [13], the preconditioned gradient subspace iteration method for computing a group of the smallest eigenvalues of finite–dimensional symmetric nonlinear eigenvalue problems was investigated. Nonlinear finite–dimensional eigenvalue problems arise after the discretization of infinite–dimensional nonlinear eigenvalue problems (see, for example, [14–32]).

In section 1 of the present paper, we give the statement of a symmetric eigenvalue problem in a finite–dimensional space with a nonlinear entrance of a spectral parameter. In section 2, results about existence and properties of the eigenvalues of the nonlinear eigenvalue problem are proved. Similar results were obtained earlier in the papers [14–16, 18–23]. In section 3, we describe auxiliary results obtained in the papers [2,6]. These results are used further for constructing and investigating the iterative methods. In sections 4, 5 and 6, we formulate the preconditioned gradient iterative methods for the nonlinear eigenvalue problem, and we investigate the convergence and the error of these methods for computing the smallest eigenvalue.

1. Formulation of the problem

Let H be an N–dimensional real Euclidean space with the scalar product $(.,.)$ and the norm $\|.\|$, and let Λ be an interval on the real axis $\mathbb{R}$, $\Lambda = (\alpha, \beta)$, $0 \le \alpha < \beta \le \infty$. Introduce the operators $A(\mu)$ and $B(\mu)$ that, for fixed $\mu \in \Lambda$, are symmetric linear operators from H to H satisfying the following conditions:

a) positive definiteness, i.e. there exist positive continuous functions $\alpha_1(\mu)$ and $\beta_1(\mu)$, $\mu \in \Lambda$, such that

$$(A(\mu)v, v) \ge \alpha_1(\mu)\|v\|^2, \quad (B(\mu)v, v) \ge \beta_1(\mu)\|v\|^2 \quad \forall v \in H, \mu \in \Lambda;$$

b) continuity with respect to the numerical argument, i.e.

$$\|A(\mu) - A(\eta)\| \to 0, \quad \|B(\mu) - B(\eta)\| \to 0,$$

as $\mu \to \eta$, $\mu, \eta \in \Lambda$. By $\|.\|$ also denote the norm of an operator from H to H.

Define the Rayleigh quotient by the formula:

$$R(\mu, v) = \frac{(A(\mu)v, v)}{(B(\mu)v, v)}, \quad v \in H \setminus \{0\}, \mu \in \Lambda.$$

Assume that the following additional conditions are fulfilled:

c) the Rayleigh quotient $R(\mu, v)$, $\mu \in \Lambda$, is, for fixed $v \in H$, a nonincreasing function of the numerical argument, i.e.

$$R(\mu, v) \ge R(\eta, v), \quad \mu < \eta, \mu, \eta \in \Lambda, v \in H \setminus \{0\};$$

d) there exists $\eta \in \Lambda$ such that

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) \le 0;$$

e) there exists $\eta \in \Lambda$ such that

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) \ge 0.$$

Consider the following eigenvalue problem: find $\lambda \in \Lambda$, $u \in H \setminus \{0\}$, such that

$$A(\lambda)u = \lambda B(\lambda)u. \tag{1}$$

The number λ that satisfies (1) is called an eigenvalue, and the element u is called an eigenelement of problem (1) corresponding to λ. The set $U(\lambda)$ that consists of the eigenelements corresponding to the eigenvalue λ and the zero element is a closed subspace in H, which is called the eigensubspace corresponding to the eigenvalue λ. The dimension of this subspace is called a multiplicity of the eigenvalue λ.

2. Existence of the eigenvalues

For fixed $\mu \in \Lambda$ we introduce the auxiliary linear eigenvalue problem: find $\gamma(\mu) \in \mathbb{R}$, $u = u(\mu) \in H \setminus \{0\}$, such that

$$A(\mu)u = \gamma(\mu)B(\mu)u. \tag{2}$$

For a symmetric positive definite linear operator A from H to H, denote by H_A the Euclidean space of elements from H with the scalar product $(u,v)_A = (Au, v)$ and the norm $\|v\|_A = (v,v)_A^{1/2}$, $u, v \in H_A$.

Lemma 1. *For fixed $\mu \in \Lambda$ problem (2) has N real positive eigenvalues $0 < \gamma_1(\mu) \leq \gamma_2(\mu) \leq \ldots \leq \gamma_N(\mu)$. The eigenelements $u_i = u_i(\mu)$, $i = 1, 2, \ldots, N$, correspond to these eigenvalues:*

$$(A(\mu)u_i, u_j) = \gamma_i(\mu)\delta_{ij}, \quad (B(\mu)u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, N.$$

The elements $u_i = u_i(\mu)$, $i = 1, 2, \ldots, N$, form an orthonormal basis of the space $H_{B(\mu)}$.

The proof is given, for example, in [33].

Lemma 2. *The formula of the minimax principle is valid:*

$$\gamma_i(\mu) = \min_{W_i \subset H} \max_{v \in W_i \setminus \{0\}} R(\mu, v), \quad i = 1, 2, \ldots, N,$$

where W_i is an i–dimensional subspace of the space H. In particular, the following relations hold:

$$\gamma_1(\mu) = \min_{v \in H \setminus \{0\}} R(\mu, v), \quad \gamma_N(\mu) = \max_{v \in H \setminus \{0\}} R(\mu, v).$$

The proof is given, for example, in [33].

For a fixed segment $[a, b]$ on Λ we set

$$\alpha_{1,min}(a,b) = \min_{\mu \in [a,b]} \alpha_1(\mu), \quad \beta_{1,min}(a,b) = \min_{\mu \in [a,b]} \beta_1(\mu).$$

Lemma 3. *Suppose that for $\mu, \eta \in [a, b]$ the following condition holds:*

$$\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a,b)} \leq \frac{1}{2}.$$

Then for $\mu, \eta \in [a, b]$ the following inequalities are valid:

$$\begin{aligned} |R(\mu, v) - R(\eta, v)| \ &\leq \ 2\left(\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a,b)} + \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a,b)}\right) R(a, v), \quad v \in H \setminus \{0\}, \\ |\gamma_i(\mu) - \gamma_i(\eta)| \ &\leq \ 2\left(\frac{\|A(\mu) - A(\eta)\|}{\alpha_{1,min}(a,b)} + \frac{\|B(\mu) - B(\eta)\|}{\beta_{1,min}(a,b)}\right) \gamma_i(a), \quad i = 1, 2, \ldots, N. \end{aligned}$$

The proof is given in [13].

Lemma 4. *The functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, are continuous nonincreasing functions with positive values.*

Proof. The continuity of the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, follows from Lemma 3 and condition b). Using the minimax principle of Lemma 2 and condition c), we obtain that the functions $\gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, are nonincreasing functions. Thus, the lemma is proved.

Lemma 5. *The functions $\mu - \gamma_i(\mu)$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

The proof is given in [13].

Lemma 6. *A number $\lambda \in \Lambda$ is an eigenvalue of problem* (1) *if and only if the number λ is a solution of an equation from the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

Proof. If λ is a solution of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, for some i, $1 \le i \le N$, then it follows from (1) and (2) that λ is an eigenvalue of problem (1). If λ is an eigenvalue of problem (1), then (1) and (2) imply $\lambda - \gamma_i(\lambda) = 0$ for some i, $1 \le i \le N$. This proves the lemma.

Theorem 1. *Problem* (1) *has N eigenvalues λ_i, $i = 1, 2, \ldots, N$, which are repeated according to their multiplicity: $\alpha < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N < \beta$. Each eigenvalue λ_i is a unique root of the equation $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$.*

Proof. By Lemma 5, each equation of the set $\mu - \gamma_i(\mu) = 0$, $\mu \in \Lambda$, $i = 1, 2, \ldots, N$, has a unique solution. Denote these solutions by λ_i, $i = 1, 2, \ldots, N$, i. e. $\lambda_i - \gamma_i(\lambda_i) = 0$, $i = 1, 2, \ldots, N$. To check that the numbers λ_i, $i = 1, 2, \ldots, N$, are put in an increasing order, let us assume the opposite, i. e. $\lambda_i > \lambda_{i+1}$. Then, according to Lemma 4, we obtain a contradiction, namely

$$\lambda_i = \gamma_i(\lambda_i) \le \gamma_i(\lambda_{i+1}) \le \gamma_{i+1}(\lambda_{i+1}) = \lambda_{i+1}.$$

By Lemma 6, the numbers λ_i, $i = 1, 2, \ldots, N$, are eigenvalues of problem (1). Thus, the theorem is proved.

Remark 1. If $\alpha = 0$, then condition d) follows from condition c).

Proof. Let us fix $\nu \in \Lambda$ and put $\eta = \min\{\gamma_1(\nu), \nu\}/2$. Taking into account condition c), Lemma 2, and the relations $\eta \le \gamma_1(\nu)/2$, $\eta \le \nu/2 < \nu$, we have

$$\eta - \min_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_1(\eta) \le \gamma_1(\nu)/2 - \gamma_1(\nu) = -\gamma_1(\nu)/2 < 0.$$

Thus, condition d) is satisfied for chosen $\eta \in \Lambda$.

Remark 2. If $\beta = \infty$, then condition e) follows from condition c).

Proof. For fixed $\nu \in \Lambda$ put $\eta = 2\max\{\gamma_N(\nu), \nu\}$. Since $\eta \geq 2\gamma_N(\nu)$ and $\eta \geq 2\nu > \nu$, according to condition c) and Lemma 2, we obtain the relations:

$$\eta - \max_{v \in H \setminus \{0\}} R(\eta, v) = \eta - \gamma_N(\eta) \geq 2\gamma_N(\nu) - \gamma_N(\nu) = \gamma_N(\nu) > 0,$$

which implies that condition e) is satisfied.

3. Auxiliary results

Assume that the symmetric positive definite linear operator $C(\mu)$ from H to H is given for fixed $\mu \in \Lambda$, and that there exist continuous functions $\delta_0(\mu)$, $\delta_1(\mu)$, $\mu \in \Lambda$, $0 < \delta_0(\mu) \leq \delta_1(\mu)$, $\mu \in \Lambda$, such that

$$\delta_0(\mu)(C(\mu)v, v) \leq (A(\mu)v, v) \leq \delta_1(\mu)(C(\mu)v, v), \quad v \in H, \quad \mu \in \Lambda.$$

For a given element $v \in H$, $\|v\|_{B(\mu)} = 1$, we define an element $w \in H$ and numbers ν^0 and ν^1 by the formulas:

$$C(\mu)\frac{\tilde{w} - v}{\tau(\mu)} + (A(\mu) - R(\mu, v)B(\mu))v = 0,$$
$$w = \frac{\tilde{w}}{\|\tilde{w}\|_{B(\mu)}}, \quad \nu^0 = R(\mu, v), \quad \nu^1 = R(\mu, w),$$

for fixed $\mu \in \Lambda$, $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$.

Lemma 7. *Let $\gamma_1(\mu)$ and $\gamma_2(\mu)$ be eigenvalues of problem (2) with $\mu \in \Lambda$ such that $\gamma_1(\mu) < \gamma_2(\mu)$. Assume that $\nu^0 < \gamma_2(\mu)$. Then $\gamma_1(\mu) \leq \nu^1 \leq \nu^0$, and the following estimate is valid:*

$$\nu^1 - \gamma_1(\mu) \leq \rho(\mu, \nu^0)(\nu^0 - \gamma_1(\mu)),$$

where $0 < \rho(\mu, \nu) < 1$,

$$\rho(\mu, \nu) = \frac{1 - \delta(\mu)(1 - \nu/\gamma_2(\mu))}{1 + \delta(\mu)(1 - \nu/\gamma_2(\mu))(\nu/\gamma_1(\mu) - 1)},$$
$$\delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \nu \in [\gamma_1(\mu), \gamma_2(\mu)), \quad \mu \in \Lambda.$$

The proof is given in [2], [6].

Remark 3. For fixed $\mu \in \Lambda$, the gradient of the Rayleight quotient $R(\mu, v)$ in the space $H_{C(\mu)}$ is defined by the formula:

$$\text{grad}_{C(\mu)} R(\mu, v) = 2\|v\|_{B(\mu)}^{-2} C^{-1}(\mu)(A(\mu) - R(\mu, v)B(\mu))v.$$

Hence

$$\begin{aligned}\tilde{w} &= v - \tau(\mu)C^{-1}(\mu)(A(\mu) - R(\mu,v)B(\mu))v = \\ &= v - \frac{1}{2}\tau(\mu)\|v\|^2_{B(\mu)}\mathrm{grad}_{C(\mu)}R(\mu,v) = \\ &= v - c_0\,\mathrm{grad}_{C(\mu)}R(\mu,v),\end{aligned}$$

where $c_0 = 0.5\,\tau(\mu)\|v\|^2_{B(\mu)}$.

4. Gradient iterative methods

Assume that the symmetric positive definite linear operator $C(\mu)$ from H to H is given for fixed $\mu \in \Lambda$, and that there exist continuous functions $\delta_0(\mu)$, $\delta_1(\mu)$, $\mu \in \Lambda$, $0 < \delta_0(\mu) \le \delta_1(\mu)$, $\mu \in \Lambda$, such that

$$\delta_0(\mu)(C(\mu)v,v) \le (A(\mu)v,v) \le \delta_1(\mu)(C(\mu)v,v), \quad v \in H, \quad \mu \in \Lambda.$$

Consider the following iterative method:

$$\begin{gathered} C(\mu^n)\frac{\tilde{u}^{n+1} - u^n}{\tau(\mu^n)} + (A(\mu^n) - \mu^n B(\mu^n))u^n = 0, \\ u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_{B(\mu^n)}}, \quad n = 0,1,\ldots, \end{gathered} \tag{3}$$

where the number μ^n is defined as a solution of the equation:

$$\mu - \varphi_n(\mu) = 0, \quad \mu \in \Lambda, \tag{4}$$

for $n = 0,1,\ldots$ Here u^0 is a given element of H, $\|u^0\|_{B(\mu^0)} = 1$, $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$, the functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0,1,\ldots$, are defined by the formulas:

$$\varphi_n(\mu) = R(\mu,u^n), \quad \mu \in \Lambda,$$

for $n = 0,1,\ldots$

Remark 4. The following formula holds:

$$\tilde{u}^{n+1} = u^n - c_n\mathrm{grad}_{C(\mu^n)}R(\mu^n,u^n),$$

where $c_n = 0.5\,\tau(\mu^n)\|u^n\|^2_{B(\mu^n)}$. This formula follows from Remark 3.

Lemma 8. *The functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous nonincreasing functions with positive values. In addition, the following inequalities are valid: $\varphi_n(\mu) \geq \gamma_1(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$*

The proof follows from Lemmas 2 and 3.

Lemma 9. *The functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

The proof is similar to the proof of Lemma 5 (see [13]).

Let λ_1 and λ_2 be eigenvalues of problem (1) such that $\lambda_1 < \lambda_2$. Put

$$\rho(\nu) = \frac{1 - d(1 - \nu/\lambda_2)}{1 + d(1 - \nu/\lambda_2)(\nu/\lambda_1 - 1)}, \quad \nu \in [\lambda_1, \lambda_2),$$
$$d = \min_{\mu \in [\lambda_1, \lambda_2]} \delta(\mu), \quad \delta(\mu) = \delta_0(\mu)/\delta_1(\mu), \quad \mu \in \Lambda.$$

Note that $0 < d \leq 1$, $0 < \rho(\nu) < 1$ for $\nu \in [\lambda_1, \lambda_2)$.

Lemma 10. *The half-open interval $[\lambda_1, \lambda_2)$ is contained in the half-open interval $[\gamma_1(\mu), \gamma_2(\mu))$ for any $\mu \in [\lambda_1, \lambda_2)$.*

Proof. Taking into account Lemma 4, we get $\gamma_1(\mu) \leq \lambda_1$ and $\gamma_2(\mu) \geq \lambda_2$ for $\mu \in [\lambda_1, \lambda_2)$. These inequalities prove the lemma.

Lemma 11. *The following inequality holds: $\rho(\mu, \nu) \leq \rho(\nu)$ for $\mu, \nu \in [\lambda_1, \lambda_2)$.*

Proof. By Lemma 10, if $\nu \in [\lambda_1, \lambda_2)$ and $\lambda_1 < \lambda_2$, then $\nu \in [\gamma_1(\mu), \gamma_2(\mu))$ and $\gamma_1(\mu) < \gamma_2(\mu)$ for $\mu \in [\lambda_1, \lambda_2)$. Now relations $\gamma_1(\mu) \leq \lambda_1$, $\gamma_2(\mu) \geq \lambda_2$, $\mu \in [\lambda_1, \lambda_2)$, imply the desired inequality:

$$\begin{aligned} \rho(\mu, \nu) &= \frac{1 - \delta(\mu)(1 - \nu/\gamma_2(\mu))}{1 + \delta(\mu)(1 - \nu/\gamma_2(\mu))(\nu/\gamma_1(\mu) - 1)} \leq \\ &\leq \frac{1 - d(1 - \nu/\lambda_2)}{1 + d(1 - \nu/\lambda_2)(\nu/\lambda_1 - 1)} = \rho(\nu) \end{aligned}$$

for $\mu, \nu \in [\lambda_1, \lambda_2) \subset [\gamma_1(\mu), \gamma_2(\mu))$. Thus, the lemma is proved.

Theorem 2. *Let λ_1 and λ_2 be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Suppose that the sequence μ^n, $n = 0, 1, \ldots$ is calculated by the formulas* (3), (4), *$\mu^0 < \lambda_2$. Then $\mu^n \to \lambda_1$ as $n \to \infty$, and the following inequalities are valid*

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1.$$

Moreover, the following estimate holds:

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \le (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_1, \lambda_2)$, $n = 0, 1, \dots$

Proof. Let us show that the solutions μ^n, $n = 0, 1, \dots$ of the equations $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, $n = 0, 1, \dots$ satisfy the following inequalities:

$$\lambda_2 > \mu^0 \ge \mu^1 \ge \dots \ge \mu^n \ge \dots \ge \lambda_1.$$

Assume that the equation $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, has the solution μ^n such that

$$\lambda_2 > \mu^0 \ge \mu^1 \ge \dots \ge \mu^n \ge \lambda_1, \quad n \ge 0.$$

Hence we obtain

$$\nu^0 = \varphi_n(\mu^n) = \mu^n < \lambda_2 = \gamma_2(\lambda_2) \le \gamma_2(\mu^n).$$

Consequently, by Lemma 7, we have

$$\nu^1 = \varphi_{n+1}(\mu^n) \le \nu^0 = \varphi_n(\mu^n) = \mu^n.$$

It follows from Lemmas 8 and 9 that the equation $\mu - \varphi_{n+1}(\mu) = 0$, $\mu \in \Lambda$, has the unique solution μ^{n+1} and

$$\lambda_2 > \mu^0 \ge \mu^1 \ge \dots \ge \mu^n \ge \mu^{n+1} \ge \lambda_1.$$

Let us prove that $\mu^n \to \lambda_1$ as $n \to \infty$. Taking into account Lemma 7, 10, 11, we obtain the following relations:

$$\begin{aligned}
\mu^{n+1} - \gamma_1(\mu^{n+1}) &= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_1(\mu^{n+1})) \le \\
&\le (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\varphi_{n+1}(\mu^n) - \gamma_1(\mu^n)) = \\
&= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\nu^1 - \gamma_1(\mu^n)) \le \\
&\le (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n, \nu^0)(\nu^0 - \gamma_1(\mu^n)) \le \\
&\le (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),
\end{aligned}$$

where $\nu^0 = \varphi_n(\mu^n) = \mu^n$, $\nu^1 = \varphi_{n+1}(\mu^n)$.

Since $\lambda_2 > \mu^0 \ge \mu^1 \ge \dots \ge \mu^n \ge \dots \ge \lambda_1$, there exists $\xi \in [\lambda_1, \lambda_2)$ such that $\mu^n \to \xi$ as $n \to \infty$.

By condition a) and the relations $\|u^n\|_{B(\mu^n)} = 1, n = 0, 1, \dots$, we obtain that there exists a constant $c > 0$ such that

$$\|u^n\| \le \frac{\|u^n\|_{B(\mu^n)}}{\sqrt{\beta_1(\mu^n)}} = \frac{1}{\sqrt{\beta_1(\mu^n)}} \le c, \quad n = 0, 1, \dots,$$

$$c = \max_{\mu \in [\lambda_1, \lambda_2]} \frac{1}{\sqrt{\beta_1(\mu)}}.$$

Hence there exists an element $w \in H$ and a subsequence u^{n_i+1}, $i = 1, 2, \ldots$, such that $u^{n_i+1} \to w$ as $i \to \infty$.

Let us prove that $\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) \to 0$ as $i \to \infty$. We have

$$0 \le \mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i}) = R(\mu^{n_i+1}, u^{n_i+1}) - R(\mu^{n_i}, u^{n_i+1}) \to 0$$

as $i \to \infty$. Here, we have taken into account that

$$R(\mu^{n_i+1}, u^{n_i+1}) \to R(\xi, w), \quad R(\mu^{n_i}, u^{n_i+1}) \to R(\xi, w),$$

as $i \to \infty$.

Using the relations

$$0 \le \mu^{n_i+1} - \gamma_1(\mu^{n_i+1}) \le (\mu^{n_i+1} - \varphi_{n_i+1}(\mu^{n_i})) + \rho(\mu^{n_i})(\mu^{n_i} - \gamma_1(\mu^{n_i}))$$

as $i \to \infty$, we get

$$0 \le \xi - \gamma_1(\xi) \le \rho(\xi)(\xi - \gamma_1(\xi)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_1, \lambda_2)$. Hence the number $\xi \in [\lambda_1, \lambda_2)$ satisfies the equation $\xi - \gamma_1(\xi) = 0$, i. e. $\xi = \lambda_1$ is an eigenvalue of problem (1) and $\mu^n \to \lambda_1$ as $n \to \infty$. This completes the proof of the theorem.

5. Error estimates of the gradient iterative methods

Assume that there exist positive continuous functions $\alpha_0(\mu, \eta)$ and $\beta_0(\mu, \eta)$, $\mu, \eta \in \Lambda$, such that

$$\|A(\mu) - A(\eta)\| \le \alpha_0(\mu, \eta)|\mu - \eta|, \quad \|B(\mu) - B(\eta)\| \le \beta_0(\mu, \eta)|\mu - \eta|,$$

for $\mu, \eta \in \Lambda$.

For a fixed segment $[a, b]$ on Λ we set

$$\alpha_{0,max}(a, b) = \max_{\mu,\eta \in [a,b]} \alpha_0(\mu, \eta), \quad \beta_{0,max}(a, b) = \max_{\mu,\eta \in [a,b]} \beta_0(\mu, \eta).$$

Lemma 12. *Assume that the following inequality holds:*

$$\frac{\alpha_{0,max}(a, b)}{\alpha_{1,min}(a, b)} (b - a) \le \frac{1}{2},$$

for a fixed segment $[a, b]$ on Λ. Then the following estimate is valid:

$$|R(\mu, v) - R(\eta, v)| \le r(a, b, v)\, |\mu - \eta|, \quad \mu, \eta \in [a, b], \quad v \in H \setminus \{0\},$$

where

$$r(a, b, v) = 2 \left(\frac{\alpha_{0,max}(a, b)}{\alpha_{1,min}(a, b)} + \frac{\beta_{0,max}(a, b)}{\beta_{1,min}(a, b)} \right) R(a, v).$$

The proof follows from Lemma 3.

Put

$$q(\mu) = \max\{\rho(\lambda_1), \rho(\mu)\}, \quad \mu \in [\lambda_1, \lambda_2),$$
$$\omega = \frac{\lambda_2 \sqrt{1-d}}{1+\sqrt{1-d}}.$$

Note that $0 < q(\mu) < 1$ for $\mu \in [\lambda_1, \lambda_2)$.

Lemma 13. *The following equality is valid:*

$$\max_{\mu \in [\lambda_1, \mu^0]} \rho(\mu) = q(\mu^0)$$

for $\mu^0 \in [\lambda_1, \lambda_2)$*. If* $0 \le \omega \le \lambda_1$*, then* $q(\mu^0) = \rho(\mu^0)$*. If* $\lambda_1 \le \omega < \lambda_2$ *and* $\lambda_1 \le \mu^0 \le \omega$*, then* $q(\mu^0) = \rho(\lambda_1)$*.*

Proof. It is not difficult to make sure (see also [6]) that $\rho'(\omega) = 0$, $\rho'(\mu) < 0$ for $\mu \in (0, \omega)$, $\rho'(\mu) > 0$ for $\mu \in (\omega, \lambda_2)$. These relations imply desired results. Thus, the lemma is proved.

Theorem 3. *Let* λ_1 *and* λ_2 *be eigenvalues of problem* (1) *such that* $\lambda_1 < \lambda_2$*. Assume that the sequence* μ^n, $n = 0, 1, \ldots$ *is calculated by the formulas* (3), (4), $\mu^0 < \lambda_2$*, and that numbers* $n_0 \ge 0$ *and* $\varepsilon > 0$ *such that* $\lambda_1 \le \mu^{n+1} \le \mu^n \le \lambda_1 + \varepsilon < \lambda_2$ *and*

$$\frac{\alpha_{0,max}(\lambda_1, \lambda_1+\varepsilon)}{\alpha_{1,min}(\lambda_1, \lambda_1+\varepsilon)} \varepsilon \le \frac{1}{2}$$

for $n \ge n_0$*. Then the following estimate is valid:*

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \le q_n(\mu^n - \gamma_1(\mu^n)),$$

where $q_n = r(\lambda_1, \lambda_1+\varepsilon, u^{n+1}) + \rho(\mu^n)$, $n \ge n_0$*.*
Suppose $r(\lambda_1, \lambda_1+\varepsilon, u^{n+1}) \le \sigma$, $n \ge n_0$*. Then*

$$\begin{aligned} \mu^{n+1} - \gamma_1(\mu^{n+1}) &\le q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \\ \mu^{n+1} - \lambda_1 &\le q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \end{aligned}$$

for $q_0 = \sigma + q(\mu^0)$, $n \ge n_0$*.*

Proof. According to Lemma 12, for $n \ge n_0$ we obtain the following relation:

$$\begin{aligned} \mu^{n+1} - \varphi_{n+1}(\mu^n) &= \varphi_{n+1}(\mu^{n+1}) - \varphi_{n+1}(\mu^n) = \\ &= R(\mu^{n+1}, u^{n+1}) - R(\mu^n, u^{n+1}) \le \\ &\le r(\lambda_1, \lambda_1+\varepsilon, u^{n+1})(\mu^n - \mu^{n+1}) \le \\ &\le r(\lambda_1, \lambda_1+\varepsilon, u^{n+1})(\mu^n - \gamma_1(\mu^n)), \end{aligned}$$

in which we have taken into account that

$$\gamma_1(\mu^n) \le \varphi_{n+1}(\mu^n) \le \varphi_{n+1}(\mu^{n+1}) = \mu^{n+1}.$$

Now, by Theorem 2 and Lemma 13, we obtain desired estimates. Thus, the theorem is proved.

Remark 5. Assume that the operators $A(\mu) = A$, $B(\mu) = B$, $C(\mu) = C$, do not depend on $\mu \in \mathbb{R}$, and that the following relations are valid:

$$\delta_0(Cv, v) \le (Av, v) \le \delta_1(Cv, v), \quad v \in H,$$

for given constants δ_0 and δ_1, $0 < \delta_0 \le \delta_1$. In this case, the iterative method (3) and (4) has the following form:

$$\begin{aligned} &C\,\frac{\tilde{u}^{n+1} - u^n}{\tau} + (A - \mu^n B)u^n = 0, \\ &u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^n = R(u^n) \quad n = 0, 1, \dots, \end{aligned}$$

where $R(v) = (Av, v)/(Bv, v)$, $v \in H \setminus \{0\}$, $\tau = \delta_1^{-1}$, u^0 is a given element of H, $\|u^0\|_B = 1$.

Then the error estimates of Theorem 3 are transformed to the estimates:

$$\begin{aligned} \mu^{n+1} - \lambda_1 &\le \rho(\mu^n)(\mu^n - \lambda_1), \\ \mu^{n+1} - \lambda_1 &\le q_0^{n+1}(\mu^0 - \lambda_1), \end{aligned}$$

for $n = 0, 1, \dots$, where $0 < \rho(\mu) < 1$ for $\mu \in [\lambda_1, \lambda_2)$, $q(\mu) = \max\{\rho(\lambda_1), \rho(\mu)\}$, $\mu \in [\lambda_1, \lambda_2)$, $0 < q_0 = q(\mu^0) < 1$,

$$\rho(\nu) = \frac{1 - \delta(1 - \nu/\lambda_2)}{1 + \delta(1 - \nu/\lambda_2)(\nu/\lambda_1 - 1)}, \quad \delta = \delta_0/\delta_1, \quad \nu \in [\lambda_1, \lambda_2).$$

These error estimates are identical with known results (see, for example, [2], [6]).

6. Steepest descent methods

Now we shall investigate the preconditioned gradient iterative method which, unlike method (3), (4), does not use the function $\tau(\mu) = \delta_1^{-1}(\mu)$, $\mu \in \Lambda$, introduced in section 3:

$$\begin{aligned} &C(\mu^n)\frac{\tilde{u}^{n+1} - u^n}{\tau_n} + (A(\mu^n) - \mu^n B(\mu^n))u^n = 0, \\ &u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_{B(\mu^n)}}, \quad n = 0, 1, \dots, \end{aligned} \tag{5}$$

where the number μ^n is defined as a solution of the equation:

$$\mu - \varphi_n(\mu) = 0, \quad \mu \in \Lambda, \tag{6}$$

for $n = 0, 1, \ldots$ Here u^0 is a given element of H, $\|u^0\|_{B(\mu^0)} = 1$, the functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are defined by the formulas:

$$\varphi_n(\mu) = R(\mu, u^n), \quad \mu \in \Lambda, \quad n = 0, 1, \ldots$$

The number τ_n in (5) is calculated by the formulas:

$$\tau_n = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\begin{aligned} a &= a_2 b_1 - a_1 b_2, \\ b &= a_1 - \mu^n a_2, \\ c &= (v^n, w^n), \end{aligned}$$

$$a_1 = (A(\mu^n)w^n, w^n), \quad b_1 = (A(\mu^n)u^n, w^n),$$
$$a_2 = (B(\mu^n)w^n, w^n), \quad b_2 = (B(\mu^n)u^n, w^n),$$

$$\begin{aligned} v^n &= (A(\mu^n) - \mu^n B(\mu^n))u^n, \\ w^n &= C^{-1}(\mu^n)v^n, \end{aligned}$$

for $n = 0, 1, \ldots$

Remark 6. The formulas for calculating the number τ_n were obtained from the following condition:

$$R(\mu^n, u^n - \tau_n w^n) = \min_{\tau \in [0, \infty)} R(\mu^n, u^n - \tau w^n).$$

Remark 7. The following formula holds:

$$\tilde{u}^{n+1} = u^n - \tau_n w^n = u^n - c_n \operatorname{grad}_{C(\mu^n)} R(\mu^n, u^n),$$

where

$$c_n = \frac{1}{2}\tau_n \|u^n\|^2_{B(\mu^n)}.$$

This formula follows from Remark 3.

Lemma 14. *The functions $\varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous nonincreasing functions with positive values. In addition, the following inequalities are valid: $\varphi_n(\mu) \geq \gamma_1(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$*

The proof follows from Lemmas 2 and 3.

Lemma 15. *The functions $\mu - \varphi_n(\mu)$, $\mu \in \Lambda$, $n = 0, 1, \ldots$, are continuous and strictly increasing functions with negative and positive values in the neighbourhoods of the points α and β, respectively.*

The proof is similar to the proof of Lemma 5 (see [13]).

Theorem 4. *Let λ_1 and λ_2 be eigenvalues of problem (1) such that $\lambda_1 < \lambda_2$. Assume that the sequence μ^n, $n = 0, 1, \ldots$ is calculated by the formulas (5), (6), $\mu^0 < \lambda_2$. Then $\mu^n \to \lambda_1$ as $n \to \infty$, and the following inequalities are valid*

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1.$$

Moreover, the following estimate holds:

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),$$

where $0 < \rho(\mu) < 1$, $\mu \in [\lambda_1, \lambda_2)$, $n = 0, 1, \ldots$

Proof. The solutions μ^n, $n = 0, 1, \ldots$ of the equations $\mu - \varphi_n(\mu) = 0$, $\mu \in \Lambda$, $n = 0, 1, \ldots$ satisfy the following inequalities:

$$\lambda_2 > \mu^0 \geq \mu^1 \geq \ldots \geq \mu^n \geq \ldots \geq \lambda_1.$$

This is proving by analogy with Theorem 2 according to Lemmas 14 and 15.

Let us prove that $\mu^n \to \lambda_1$ as $n \to \infty$. Taking into account Lemma 7, 10, 11, we obtain the following relations:

$$\begin{aligned}
\mu^{n+1} - \gamma_1(\mu^{n+1}) &= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (R(\mu^n, u^{n+1}) - \gamma_1(\mu^{n+1})) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (R(\mu^n, w) - \gamma_1(\mu^n)) = \\
&= (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + (\nu^1 - \gamma_1(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n, \nu^0)(\nu^0 - \gamma_1(\mu^n)) \leq \\
&\leq (\mu^{n+1} - \varphi_{n+1}(\mu^n)) + \rho(\mu^n)(\mu^n - \gamma_1(\mu^n)),
\end{aligned}$$

where $\nu^0 = \varphi_n(\mu^n) = \mu^n$, $\nu^1 = \varphi_{n+1}(\mu^n)$, the element $w \in H$ is defined by the formulas:

$$\begin{aligned}
&C(\mu^n)\frac{\tilde{w} - u^n}{\tau(\mu^n)} + (A(\mu^n) - \mu^n B(\mu^n))u^n = 0, \\
&w = \frac{\tilde{w}}{\|\tilde{w}\|_{B(\mu^n)}}.
\end{aligned}$$

Note that from this formula and Remark 6, we have

$$\begin{aligned}
R(\mu^n, u^{n+1}) &= R(\mu^n, u^n - \tau_n w^n) = \min_{\tau \in [0,\infty)} R(\mu^n, u^n - \tau w^n) \leq \\
&\leq R(\mu^n, u^n - \tau(\mu^n) w^n) = R(\mu^n, w).
\end{aligned}$$

Now the proof is completed similarly to the proof of Theorem 2. Thus, the theorem is proved.

Theorem 5. *Let λ_1 and λ_2 be eigenvalues of problem* (1) *such that $\lambda_1 < \lambda_2$. Assume that the sequence μ^n, $n = 0, 1, \ldots$ is calculated by the formulas* (5), (6), *$\mu^0 < \lambda_2$, and that numbers $n_0 \geq 0$ and $\varepsilon > 0$ such that $\lambda_1 \leq \mu^{n+1} \leq \mu^n \leq \lambda_1 + \varepsilon < \lambda_2$ and*

$$\frac{\alpha_{0,max}(\lambda_1, \lambda_1 + \varepsilon)}{\alpha_{1,min}(\lambda_1, \lambda_1 + \varepsilon)} \varepsilon \leq \frac{1}{2}$$

for $n \geq n_0$. Then the following estimate is valid:

$$\mu^{n+1} - \gamma_1(\mu^{n+1}) \leq q_n(\mu^n - \gamma_1(\mu^n)),$$

where $q_n = r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1}) + \rho(\mu^n)$, $n \geq n_0$.

Suppose $r(\lambda_1, \lambda_1 + \varepsilon, u^{n+1}) \leq \sigma$, $n \geq n_0$. Then

$$\begin{aligned} \mu^{n+1} - \gamma_1(\mu^{n+1}) &\leq q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \\ \mu^{n+1} - \lambda_1 &\leq q_0^{n+1}(\mu^0 - \gamma_1(\mu^0)), \end{aligned}$$

for $q_0 = \sigma + q(\mu^0)$, $n \geq n_0$.

The proof is similar to the proof of Theorem 3.

Remark 8. Assume that the operators $A(\mu) = A$, $B(\mu) = B$, $C(\mu) = C$, do not depend on $\mu \in \mathbb{R}$, and that the following relations are valid:

$$\delta_0(Cv, v) \leq (Av, v) \leq \delta_1(Cv, v), \quad v \in H,$$

for given constants δ_0 and δ_1, $0 < \delta_0 \leq \delta_1$. In this case, the iterative method (5) and (6) has the following form:

$$\begin{aligned} &C\,\frac{\tilde{u}^{n+1} - u^n}{\tau_n} + (A - \mu^n B)u^n = 0, \\ &u^{n+1} = \frac{\tilde{u}^{n+1}}{\|\tilde{u}^{n+1}\|_B}, \quad \mu^n = R(u^n) \quad n = 0, 1, \ldots, \end{aligned}$$

where $R(v) = (Av, v)/(Bv, v)$, $v \in H \setminus \{0\}$, u^0 is a given element of H, $\|u^0\|_B = 1$. The number τ_n is defined by the formulas:

$$\tau_n = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$\begin{aligned} a &= a_2 b_1 - a_1 b_2, \\ b &= a_1 - \mu^n a_2, \\ c &= (v^n, w^n), \end{aligned}$$

$$a_1 = (Aw^n, w^n), \quad b_1 = (Au^n, w^n),$$
$$a_2 = (Bw^n, w^n), \quad b_2 = (Bu^n, w^n),$$

$$\begin{aligned} v^n &= (A - \mu^n B)u^n, \\ w^n &= C^{-1}v^n, \end{aligned}$$

for $n = 0, 1, \ldots$

Then the error estimates of Theorem 5 are transformed to the estimates:

$$\begin{aligned} &\mu^{n+1} - \lambda_1 \leq \rho(\mu^n)(\mu^n - \lambda_1), \\ &\mu^{n+1} - \lambda_1 \leq q_0^{n+1}(\mu^0 - \lambda_1), \end{aligned}$$

for $n = 0, 1, \ldots$, where $0 < \rho(\mu) < 1$ for $\mu \in [\lambda_1, \lambda_2)$, $q(\mu) = \max\{\rho(\lambda_1), \rho(\mu)\}$, $\mu \in [\lambda_1, \lambda_2)$, $0 < q_0 = q(\mu^0) < 1$,

$$\rho(\nu) = \frac{1 - \delta(1 - \nu/\lambda_2)}{1 + \delta(1 - \nu/\lambda_2)(\nu/\lambda_1 - 1)}, \quad \delta = \delta_0/\delta_1, \quad \nu \in [\lambda_1, \lambda_2).$$

These error estimates are identical with known results (see, for example, [2], [6]).

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