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Numerische Simulation auf massiv parallelen Rechnern

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On a Quadrature Algorithm for the Piecewise Linear Wavelet Collocation Applied to Boundary Integral Equations

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Abstract

In this paper we consider a piecewise linear collocation method for the solution of a pseudo-differential equation of order $\mathbf{r} = 0, -1$ over a closed and smooth boundary manifold. The trial space is the space of all continuous and piecewise linear functions defined over a uniform triangular grid and the collocation points are the grid points. For the wavelet basis in the trial space we choose the three-point hierarchical basis together with a slight modification near the boundary points of the global patches of parametrization. We choose linear combinations of Dirac delta functionals as wavelet basis in the space of test functionals. For the corresponding wavelet algorithm, we show that the parametrization can be approximated by low order piecewise polynomial interpolation and that the integrals in the stiffness matrix can be computed by quadrature, where the quadrature rules are composite rules of simple low order quadratures. The whole algorithm for the assembling of the matrix requires no more than $O(N[\log N]^3)$ arithmetic operations, and the error of the collocation approximation, including the compression, the approximative parametrization, and the quadratures, is less than $O(N^{-(2-\mathbf{r})/2})$. Note that, in contrast to well-known algorithms by v.Petersdorff, Schwab, and Schneider, only a finite degree of smoothness is required. In contrast to an algorithm of Ehrich and Rathsfeld, no multiplicative splitting of the kernel function is required. Beside the usual mapping properties of the integral operator in low order Sobolev spaces, estimates of Calderón-Zygmund type are the only assumptions on the kernel function.

1 Introduction

It is a well-known fact that usual finite element discretizations of linear integral equations (e.g. of boundary integral equations) lead to systems of linear equations with fully populated matrices. Thus, even an iterative solution method requires a huge number of arithmetic operations and a large storage capacity. In order to improve these finite element approaches for integral equations, several algorithms have been developed. One of these consists in employing wavelet bases of the finite element spaces. The basic idea goes back to Beylkin, Coifman, and Rokhlin [3], and has been thoroughly investigated by Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [11, 12, 30, 29, 28, 40] (cf. also the contributions by Alpert, Harten, Yad-Shalom, Ehrich, and Rathsfeld [1, 21, 36, 18]). In the present paper, we shall apply the wavelet technique to the piecewise linear collocation of two-dimensional boundary integral equations of order $\mathbf{r} = 0$ and $\mathbf{r} = -1$ corresponding to three-dimensional boundary value problems.

First we shall recall the definition of a simple biorthogonal wavelet basis analyzed in [38] (cf. the familiar constructions in [22, 42, 24] and compare [13, 14, 15, 5, 6, 7, 16]). The grids will be supposed to be uniform refinements of a coarse initial triangulation, and the basis will be the system of three-point hierarchical basis functions, i.e. each basis function will be a linear combination of no more than three finite element functions defined over the corresponding level of a grid hierarchy. In comparison to other bases of continuous wavelet functions our basis functions will have a rather small support, and we believe that this property is essential for the wavelet algorithm. Indeed, small supports lead to better compression rates, especially, for higher levels and to faster quadrature algorithms for the assembling of the stiffness matrix.

For the basis in the test space spanned by Dirac delta functionals, we shall take the usual test functionals which can be considered at as scaled versions of difference formulas (cf. the wavelet collocation methods by Dahmen, Prößdorf, Schneider, Harten, Yad-Shalom, and Rathsfeld [12, 21, 36, 35, 37]). Applying the wavelet basis functions of the trial and test space, we shall obtain the well-known compression results for trial wavelets with vanishing moments due to Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [12, 30, 40]. The compression for trial functions without vanishing moments is the same as in [35] (cf. also the univariate analogue for the Galerkin method treated in [30, 4]). Note that we have to assume that the derivatives of the kernel function up to a finite order satisfy the Calderón-Zygmund estimate. This is the fundamental relation not only for the wavelet compression but also for the fast assembling of the stiffness matrix via quadrature.

In general, the stiffness matrix cannot be computed exactly. This is the case, for instance, if the boundary manifold is given by a discrete set of points or if no analytic formula is available to integrate the kernel and trial functions. Therefore, we shall consider an algorithm for the approximation of the boundary surface and for the quadrature of the integrals. We emphasize that this is the most time consuming and the most difficult part of the wavelet method. To set up the stiffness matrix, we shall proceed as follows. We shall replace the parametrization of the boundary manifold by a low order piecewise polynomial interpolation over the finest grid. Depending on the test functional and on the trial function, we shall define an appropriate partition of the supports of the trial basis functions. We shall apply a low order composite quadrature rule over this partition. This way, we shall arrive at a fully discretized wavelet algorithm with $O(N[\log N]^3)$ arithmetic operations to compute the $O(N[\log N])$ entries of the compressed stiffness matrix. If $\mathbf{r} = -1$, then even $O(N[\log N]^2)$ arithmetic operations are sufficient. Here N stands for the number of degrees of freedom. Assuming that the collocation without wavelet algorithm is stable, the asymptotic error of the exact collocation solution is known to be less than $O(N^{-(2-\mathbf{r})/2})$ which is optimal for piecewise linear trial spaces. The fully discrete wavelet algorithm will be shown to be stable and convergent with an optimal error less than $O(N^{-(2-\mathbf{r})/2})$.

Notice that alternative quadrature algorithms have been considered by Beylkin, Coifman, Rokhlin [3] for integral operators with smooth kernels and by v.Petersdorff, Schwab, and Schneider [30, 40] (cf. also the numerical implementation by Lage and Schwab [23]) for boundary integral operators with analytic Green kernels over piecewise analytic boundaries. Another quadrature algorithm due to Ehrich and Rathsfeld [19] applies to product kernels, where one factor has a finite degree of smoothness and no singularity whereas the second factor can be singular but must be analytic outside the singularity. In contrast to these, the fully discrete algorithm of the present paper applies to boundary integral equations over surfaces with finite degree of smoothness and including kernel functions with finite degree of smoothness. In fact, the required degree of smoothness for the geometry will be equal to $2[2-\mathbf{r}]+1$, i.e. to the doubled order of convergence increased by one. Moreover, the kernel function of the integral operator will be assumed to have continuous mixed derivatives up to order $2[2-\mathbf{r}]$ outside the diagonal. In the proof of the corresponding error estimates, we shall show that the techniques developed for the compression algorithm apply to the analysis of the discretization as well. The only thing to do is to replace the decay properties in the matrix entries due to the vanishing moments of the trial functions and the norm estimates due to the smoothness of the solution by error estimates of the approximate parameter mappings and by estimates of the quadrature rules, respectively.

The plan of the paper is as follows. In Sect. 2 we shall describe the boundary manifold, the integral equation, and the conventional piecewise linear collocation method. We shall introduce the three-point hierarchical wavelet functions of the piecewise linear trial space, the test wavelet functionals, and the corresponding compression algorithm in Sect. 3. Sect. 4 will be devoted to the description of the interpolation of the parameter mappings and to the quadrature algorithm. All proofs will be deferred to Sects. 5 and 6. In particular, in Sect. 5 we shall recall some technical results from the compression estimates, and the discretization including the approximation of the parametrizations and of the integration will be analyzed in Sect. 6.

Finally, we remark that our algorithm applies in particular to the double layer potential equation (cf. the examples in Sect. 2.2). However, though the double layer operator is a pseudodifferential operator of order zero, the kernel function is the kernel of a pseudodifferential operator of order minus one. Moreover, the constant functions are eigen functions corresponding to the eigen value one. Using these additional properties and the technique of the present paper, a rate of convergence $O(N)$ multiplied by logarithmic factors can be derived for a modified algorithm, which applies to thrice continuously differentiable manifolds, which replaces the exact parametrization by a piecewise quadratic interpolation, and which is based on composite quadrature rules of convergence order two.

2 The Piecewise Linear Collocation Method

2.1 The Manifold

We suppose that the integral equation to be solved is given on a closed boundary manifold $\Gamma \subset \mathbb{R}^3$ with finite degree of smoothness. More exactly, we assume that Γ is the union of m_Γ triangular patches Γ_m, i.e.

$$\begin{aligned} \Gamma &= \cup_{m=1}^{m_\Gamma} \Gamma_m, \quad \Gamma_m := \kappa_m(T), \\ T &:= \Big\{(s,t) \in \mathbb{R}^2 :\ 0 \le s \le 1,\ 0 \le t \le \min\{s, 1-s\}\Big\}. \end{aligned} \tag{2.1}$$

Here the κ_m denote parametrization mappings from the standard triangle T to the manifold Γ. We assume that the κ_m extend to mappings from the larger triangle

$$T^e := \Big\{(s,t) \in \mathbb{R}^2 :\ -3 \le s \le 5,\ -1 \le t \le \min\{s+2, 4-s\}\Big\}$$

to Γ and that these extensions are d_Γ times continuously differentiable. Here d_Γ is an integer which is assumed to be greater or equal to three when dealing with zero order operators and greater or equal to four when dealing with operators of order $\mathbf{r} = -1$. This degree of smoothness is sufficient for the usual convergence estimates of the linear collocation and for an almost optimal compression algorithm. For the quadrature, however, we need more smoothness. We assume that d_Γ is greater or equal to five when dealing with zero order operators and greater or equal to seven when dealing with operators of order $\mathbf{r} = -1$.

Further we suppose that the intersection of two patches Γ_m and $\Gamma_{m'}$ is either empty or a corner point for both patches or a whole side for Γ_m and $\Gamma_{m'}$. In the last case we assume that the representations

$$\begin{aligned}\Gamma_m \cap \Gamma_{m'} &= \Big\{\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) : \; 0 \le \lambda \le 1\Big\}, \\ \Gamma_m \cap \Gamma_{m'} &= \Big\{\kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big) : \; 0 \le \lambda \le 1\Big\}\end{aligned}$$

satisfy the condition

$$\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) \;=\; \kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big), \quad 0 \le \lambda \le 1. \tag{2.2}$$

Note that, for the numerical method, the parameter mappings κ_m need not to be given for all points of T. We shall use only the values of κ_m at the points of a uniform grid over the triangle T.

To secure stability of the so constructed basis (cf. [38]), we even need two further assumptions. In connection with the numbering we suppose that, if the corner P of a patch Γ_m is contained in the union $\cup_{m'=1}^{m-1}\Gamma_{m'}$ of the preceding patches, then at least one of the sides of Γ_m ending at P is contained in $\cup_{m'=1}^{m-1}\Gamma_{m'}$. It is not hard to see that, for a boundary manifold Γ homeomorphic to the sphere and for any fixed triangulation, there always exists a numbering of the triangular patches which fulfills the assumption. Finally, for the parametrizations, we suppose the following assumption. For any $m = 2, \ldots, m_\Gamma - 1$, we suppose that, if one of the two "shorter" sides $\kappa_m(\{(s,s) : \; 0 \le s \le 0.5\})$ and $\kappa_m(\{(s, 1-s) : \; 0.5 \le s \le 1\})$ is contained in $\cup_{m'=1}^{m-1}\Gamma_m$, then the other must also be contained in $\cup_{m'=1}^{m-1}\Gamma_m$. This last assumption can always be satisfied if the parameter mappings κ_m are replaced by a composition of κ_m with a suitable affine automorphism of T.

Since the manifold is at least continuously differentiable, for each $Q \in \Gamma$, there exists a unit vector n_Q normal to Γ at Q and pointing into the exterior domain bounded by Γ. The Sobolev spaces $H^s(\Gamma)$ over Γ can be defined in the usual way. We define the space $H^s(\Gamma_m)$ over Γ_m as the image of the Sobolev space over T, i.e.

$$H^s(\Gamma_m) \;:=\; \{f : \; f \circ \kappa_m \in H^s(T)\}.$$

Consequently, we get

$$H^s(\Gamma) \quad = \quad \left\{(f_m)_{m=1}^{m_\Gamma} \in \bigoplus_{m=1}^{m_\Gamma} H^s(\Gamma_m) : \; f_m|_{\Gamma_m \cap \Gamma_{m'}} = f_{m'}|_{\Gamma_m \cap \Gamma_{m'}}\right\}, \quad \frac{1}{2} < s < \frac{3}{2},$$

$$
\begin{aligned}
H^s(\Gamma) &= \bigoplus_{m=1}^{m_\Gamma} H^s(\Gamma_m), \quad -\frac{1}{2} < s < \frac{1}{2}, && (2.3)\\
\|f\|_{H^s(\Gamma)} &\sim \sqrt{\sum_{m=1}^{m_\Gamma} \|f|_{\Gamma_m}\|^2_{H^s(\Gamma_m)}}\,, \quad f \in H^s(\Gamma),\ -\frac{1}{2} < s < \frac{3}{2}.
\end{aligned}
$$

2.2 The Integral Equation

Over Γ we consider a pseudo-differential operator A of order $\mathbf{r} = 0$ or $\mathbf{r} = -1$ mapping $H^{\mathbf{r}/2}$ into $H^{-\mathbf{r}/2}$. We suppose that A is an integral operator of the form $A = K$ for $\mathbf{r} = -1$ and $A = aI + K$ for $\mathbf{r} = 0$, where aI stands for the operator of multiplication by a function a which may be zero, and the integral operator K is defined by

$$
Ku(P) \ := \ \int_\Gamma k(P,Q)u(Q)\,\mathrm{d}_Q\Gamma, \quad k(P,Q) := k(P,Q,n_Q). \tag{2.4}
$$

The function k depends on the points $P,Q \in \Gamma$, and k and a are supposed to have a finite degree of smoothness, i.e. the function a and the kernel k are supposed to be d_k times continuously differentiable. More precisely, for any d_k-th order derivative ∂_P^α, $|\alpha| = d_k$ taken with respect to variable $P \in \Gamma$ and for any d_k-th order derivative ∂_Q^β, $|\beta| = d_k$ taken with respect to the variables $Q \in \Gamma$, we require that $\partial_P^\alpha \partial_Q^\beta k(P,Q)$ is continuous if $P \neq Q$. The degree of smoothness d_k is supposed to be greater or equal to two for $\mathbf{r} = 0$ and to three for $\mathbf{r} = -1$. Moreover, we assume the so-called Calderón-Zygmund estimate, i.e. the existence of a constant $C > 0$ such that, for any multiindices α and β with $|\alpha|, |\beta| \le d_k$,

$$
\left|\partial_P^\alpha \partial_Q^\beta k(P,Q)\right| \ \le \ C_{k,\alpha,\beta}|P-Q|^{-2-\mathbf{r}-|\alpha|-|\beta|}. \tag{2.5}
$$

The function k need not to be a restriction to $\Gamma \times \Gamma$ of a function defined on the space $I\!R^3 \times I\!R^3$. It may depend for instance on the unit normals n_P and n_Q pointing into the exterior or on any different kind of differentiable vector field over Γ. To specify the notation, we assume a special dependence and take $k = k(P,Q) = k(P,Q,n_Q)$ with k defined on at least a neighbourhood of $\{(P,Q,n) :\ P,Q \in \Gamma,\ n = n_Q\} \subset \Gamma \times \Gamma \times I\!R^3$. If $\mathbf{r} = 0$, then the integrand in (2.4) can be strongly singular and the integral is to be understood in the sense of a Cauchy principal value. To ensure the existence of this principal value, we assume the Mikhlin-Gireaud property (parity property)

$$
k\Big(P, P + (Q-P)\Big) \ = \ -k\Big(P, P - (Q-P)\Big) \ + \ \mathcal{O}\Big(|Q-P|^{-1}\Big).
$$

For the operator A including the just defined integral operator K, we assume the continuity of the mapping

$$
A :\ H^{s+\mathbf{r}}(\Gamma) \longrightarrow H^s(\Gamma) \tag{2.6}
$$

with $s = 0$ and $s = 1.1$ (or $s = 1.1$ replaced by a different s with $1 < s < 1.5$) and the invertibility of (2.6) with $s = 0$. For an operator A which satisfies all these assumptions, we shall solve the operator equation $Au = v$ with known right-hand side v and unknown u. To get error estimates with optimal order $2 - \mathbf{r}$, we finally assume $u \in H^2(\Gamma)$. Unfortunately, the smoothness of the kernel is not sufficient for the quadrature algorithm. To get a convergence order $2 - \mathbf{r}$ even with wavelet compression and adapted quadrature approximation, we need $d_k = 2[2 - \mathbf{r}]$. Furthermore, we suppose that Ap is $2[2 + \mathbf{r}]$ times continuously differentiable for all functions p which are linear polynomials with respect to the parametrization. Note that this higher differentiability and this higher d_k is needed for the quadrature in the Sects. 4.2.2, 4.3.2, and 4.3.3. The compression in Theorem 3.1 and the quadrature in the Sects. 4.1, 4.2.1, and 4.3.1 can easily be modified such that a degree of smoothness d_k equal to $2 - \mathbf{r}$ is sufficient. Of course, there would arise additional logarithmic factors in the estimates of the modified method.

Let us consider some examples. For instance, single and double layer potential equations belong to our class of operator equations. Indeed, for the single layer case $A = A_s$ corresponding to Laplace's equation, the order $\mathbf{r}_s$ is -1, and

$$k_s(P,Q) := \frac{1}{4\pi} \frac{1}{|P - Q|}.$$

In case of the double layer operator $A = A_d$ we get the order $\mathbf{r}_d = 0$, and the multiplication function $a_d \equiv 0.5$ is constant. The integral operator K_d is defined by

$$k_d(P,Q,n_Q) \quad = \quad -\frac{1}{4\pi} \frac{n_Q \cdot (P - Q)}{|P - Q|^3}.$$

Note that the operator K_d without aI is a pseudo-differential operator of order -1. Boundary integral operators for the Stokes system or for Lamè's system can be represented in a similar fashion (cf. [25]).

To get a further example, we take the adjoint operator K_d^* and replace the normal vector field n_Q by an oblique field o_Q. We arrive at a strongly singular boundary integral operator $A = A_o$ which corresponds to the oblique derivative boundary value problem for Laplace's equation. In this case, $a_o := -0.5\, n_P \cdot o_P$ and K_o is given by

$$k_o(P,Q,o_P) \quad = \quad -\frac{1}{4\pi} \frac{o_P \cdot (P - Q)}{|P - Q|^3}.$$

2.3 Grid and Collocation Points

Let us introduce a hierarchy of uniform grids over the standard triangle T. For the step sizes 2^{-l}, $l = 0, \ldots, L$, we set

$$\begin{aligned} \triangle_l^T &:= {}^1\!\triangle_l^T \cup {}^2\!\triangle_l^T, \\ {}^1\!\triangle_l^T &:= \left\{ (i2^{-l}, j2^{-l}) : \ 0 \le i \le 2^l,\ 0 \le j \le \min\{2^l - i, i\} \right\}, \\ {}^2\!\triangle_l^T &:= \left\{ (2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}) : \ 0 \le i < 2^l,\ 0 \le j < \min\{2^l - i, i+1\} \right\} \end{aligned}$$

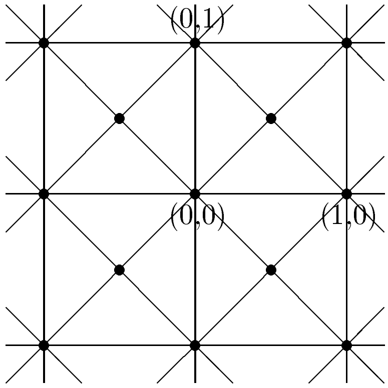

Figure 1: Grid $\triangle_0^{\mathbb{R}^2}$.

and denote the grid points by $\tau = (s,t) \in \triangle_l^T$. The grid $\triangle_l^T$ is the restriction of the grid (cf. Figure 1)

$$\triangle_l^{\mathbb{R}^2} := \left\{ (i2^{-l}, j2^{-l}) : \; i,j \in \mathbb{Z}^2 \right\} \cup \left\{ (2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}) : \; i,j \in \mathbb{Z}^2 \right\}$$

to the triangle T. Using the parametrizations, we arrive at a grid hierarchy on Γ.

$$\triangle_l^\Gamma \;\; := \;\; \left\{ \kappa_m(\tau) : \; m = 1, \ldots, m_\Gamma, \; \tau \in \triangle_l^T \right\}.$$

Clearly, a grid point $P = \kappa_m(\tau)$ may have more than one representation. If P is in the interior of a side of the triangular patch Γ_m which is a common side with $\Gamma_{m'}$, then there are exactly two representations $P = \kappa_m(\tau)$ and $P = \kappa_{m'}(\tau')$. If P is a corner point of a patch, then there exist $k > 2$ representations $P = \kappa_{m_1}(\tau_1) = \kappa_{m_2}(\tau_2) = \ldots = \kappa_{m_k}(\tau_k)$. We introduce ${}^i\triangle_l^\Gamma$ as the set of those $P \in \triangle_l^\Gamma$ whose representation $P = \kappa_m(\tau)$ with the smallest m satisfies $\tau \in {}^i\triangle_l^T$, i.e.,

$${}^i\triangle_l^\Gamma \;\; := \;\; \cup_{m=1}^{m_\Gamma} \left\{ \kappa_m(\tau) : \; \tau \in {}^i\triangle_l^T, \; \kappa_m(\tau) \notin \cup_{m'=1}^{m-1} \kappa_{m'}(\triangle_l^T) \right\},$$

and arrive at $\triangle_l^\Gamma = {}^1\triangle_l^\Gamma \cup {}^2\triangle_l^\Gamma$. The points of $\triangle_l^\Gamma$ will be denoted by upper capital letters like P and Q.

To each grid $\triangle_l^\Gamma$ there corresponds a partition of Γ into triangular pieces. Indeed, let us introduce the sets of centroids

$$\begin{aligned}
\square_0^{\mathbb{R}^2} \;\; &:= \;\; \left\{ \left(\frac{1}{2}, \frac{1}{6}\right) + k, \; \left(\frac{1}{2}, \frac{5}{6}\right) + k, \; \left(\frac{1}{6}, \frac{1}{2}\right) + k, \; \left(\frac{5}{6}, \frac{1}{2}\right) + k : \; k \in \mathbb{Z}^2 \right\}, \\
\square_l^{\mathbb{R}^2} \;\; &:= \;\; \left\{ 2^{-l}\tau : \; \tau \in \square_0^{\mathbb{R}^2} \right\}, \quad \square_l^T \; := \; T \cap \square_l^{\mathbb{R}^2}, \\
\square_l^\Gamma \;\; &:= \;\; \left\{ \kappa_m(\tau) : \; \tau \in \square_l^T, \; m = 1, 2, \ldots, m_\Gamma \right\}.
\end{aligned}$$

For each point $\tau \in \square_l^T$, there exist three uniquely defined neighbour points τ_1, τ_2, and τ_3 such that $\tau_1, \tau_2, \tau_3 \in \triangle_l^T$, that the triangle T_τ spanned by the three corners τ_1, τ_2, and τ_3 is of square measure $2^{-2l}/4$, and that τ is the centroid of T_τ. We arrive at the triangulation $\{T_\tau : \tau \in \square_l^T\}$ of T. Note that, for $l' > l$, the centroids in $\square_l^T$ are located at the boundaries of the smaller triangles $T_{\tau'}$ with $\tau' \in \square_{l'}^T$. Hence there is a one to one correspondence between the triangles T_τ over several levels and the centroids in $\cup_{l=0}^L \square_l^T$. Similarly to the triangulation over T, we define the triangulation $\{T_\tau : \tau \in \square_l^{I\!R^2}\}$ of $I\!R^2$. For Γ and a point $Q = \kappa_m(\tau) \in \square_l^\Gamma$, we set $\Gamma_Q := \{\kappa_m(\sigma) : \sigma \in T_\tau\}$ and arrive at the triangulation $\{\Gamma_Q : Q \in \square_l^\Gamma\}$. Further, we denote the level l of the points $Q \in \square_l^\Gamma$ by $l(Q)$. Notice that each partition triangle Γ_Q, $Q \in \square_l^\Gamma$, of the generation l splits into four subtriangles of the generation $l+1$.

Beside the grids $\triangle_l^\Gamma$ we introduce the difference grids

$$\nabla_l^\Gamma := \begin{cases} \triangle_0^\Gamma & \text{if } l = -1 \\ \triangle_{l+1}^\Gamma \setminus \triangle_l^\Gamma & \text{if } l = 0, \ldots, L-1, \end{cases}$$

and obtain $\triangle_L^\Gamma = \bigcup_{l=-1}^{L-1} \nabla_l^\Gamma$. For $P \in \triangle_L^\Gamma$, we denote the unique level l for which $P \in \nabla_l^\Gamma$ by $l(P)$. Analogously to ∇_l^Γ, we define the difference grids and the point levels over T and $I\!R^2$ and get $\triangle_L^T = \bigcup_{l=-1}^{L-1} \nabla_l^T$ as well as $\triangle_L^{I\!R^2} = \bigcup_{l=-1}^{L-1} \nabla_l^{I\!R^2}$. Finally, in accordance to the splitting $\triangle_l^T = {}^1\triangle_l^T \cup {}^2\triangle_l^T$, we introduce ${}^i\nabla_l^T = \nabla_l^T \cap {}^i\triangle_{l+1}^T$ for $i = 1, 2$ and get $\nabla_l^T = {}^1\nabla_l^T \cup {}^2\nabla_l^T$ as well as ${}^2\nabla_l^T = {}^2\triangle_{l+1}^T$. Similarly, we define ${}^i\nabla_l^{I\!R^2}$ and ${}^i\nabla_l^\Gamma$.

Now the set of collocation points will be the grid $\triangle_L^\Gamma$, i.e. the test functionals of the collocation scheme are the Dirac delta functionals δ_P with $P \in \triangle_L^\Gamma$. The test space Dir_L^Γ is the span of all these δ_P.

2.4 The Trial Functions

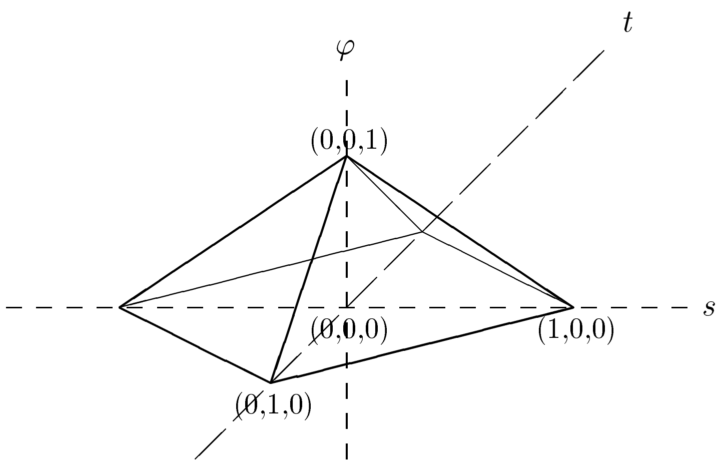

Figure 2: Hat function $(s,t) \mapsto {}^1\varphi(s,t)$.

To prepare the introduction of linear spaces, we first define two-dimensional hat functions for the grid $\triangle_0^{\mathbb{R}^2}$.

$$\begin{aligned} {}^1\varphi\,(s,t) &:= \max\Big\{0, 1-\max\{|s-t|,|s+t|\}\Big\}, \\ {}^2\varphi\,(s,t) &:= \max\Big\{0, 1-2\max\{|s|,|t|\}\Big\}. \end{aligned}$$

Clearly, the function ${}^1\varphi$ and the function ${}^2\varphi$ shifted to the point $(0.5, 0.5)$ are piecewise linear functions subordinate to the triangulation $\{T_\tau\,:\ \tau \in \square_0^{\mathbb{R}^2}\}$ (cf. the grid in Figure 1, the graph of ${}^1\varphi$ in Figure 2, and the graph of ${}^2\varphi$ shifted to the point $(0.5, 0.5)$ in Figure 3).

Now we get piecewise linear basis functions by dilating and shifting ${}^1\varphi$ and ${}^2\varphi$ to each grid point. More precisely, for each grid point on T, we set

$$\varphi_\tau^l(\sigma) \;:=\; {}^i\varphi\left(2^l(\sigma-\tau)\right), \quad \tau \in {}^i\!\triangle_l^T.$$

With the help of the parametrizations we introduce the piecewise linear (with respect to the parametrization) hat functions over Γ. For each grid point $P \in \triangle_l^\Gamma$, we set

$$\varphi_P^l(Q) \;:=\; \begin{cases} \varphi_\tau^l(\sigma) & \text{if there exist } m,\tau,\sigma \text{ s.t. } Q=\kappa_m(\sigma),\ P=\kappa_m(\tau) \\ 0 & \text{else.} \end{cases} \tag{2.7}$$

Due to the assumptions on the parametrizations (cf. (2.2)) the basis functions are well defined. Note that if $P \in \triangle_l^\Gamma$ is in the interior of the parametrization patch Γ_m, then the support $\operatorname{supp}\varphi_P^l$ of φ_P^l is contained in Γ_m. If $P = \kappa_m(\tau) = \kappa_{m'}(\tau)$ is in the interior of a side, then $\operatorname{supp}\varphi_P^l \subseteq \Gamma_m \cup \Gamma_{m'}$. For corner points $P = \kappa_{m_1}(\tau_1) = \kappa_{m_2}(\tau_2) = \ldots = \kappa_{m_k}(\tau_k)$ of the triangular parametrization patches we get $\operatorname{supp}\varphi_P^l \subseteq \cup_{n=1}^k \Gamma_{m_n}$. We denote the span of the functions φ_P^l, $P \in \triangle_l^\Gamma$ by Lin_l^Γ. Obviously, this is the space of all continuous and piecewise linear functions over the partition $\{\Gamma_Q\,:\ \ Q \in \square_l^\Gamma\}$ corresponding to the grid $\triangle_l^\Gamma$. Here linearity is understood with respect to the parametrization. The space Lin_L^Γ will be the set of trial functions for the collocation.

2.5 The Collocation Scheme

Now the collocation method seeks an approximate solution u_L for the exact solution u of $Au = v$. This is sought in the trial space Lin_L^Γ by solving

$$Au_L(P) = v(P), \quad P \in \triangle_L^\Gamma. \tag{2.8}$$

Using the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \xi_P \varphi_P^L$, the collocation equation can be written in form of a matrix equation $A_L\xi = \eta$, where we set

$$\xi := (\xi_P)_{P\in\triangle_L^\Gamma}, \quad \eta := (\eta_P)_{P\in\triangle_L^\Gamma}, \quad \eta_P := v(P).$$

The matrix of the linear system is the so called stiffness matrix given by

$$A_L := (a_{P',P})_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P} := (A\varphi_P^L)(P').$$

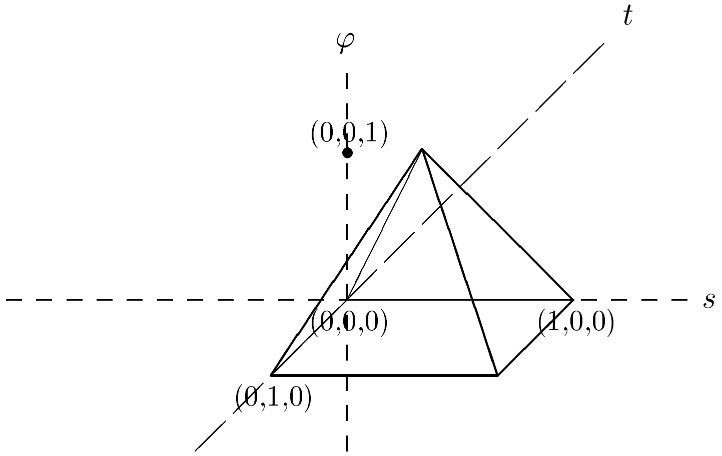

Figure 3: Hat function $(s,t) \mapsto {}^2\varphi(s-0.5,t-0.5)$.

Moreover, using the interpolation projection R_L defined by $R_L f := \sum_{P\in\triangle_L^\Gamma} f(P)\varphi_P^L$, the collocation can be treated as a projection equation of the form $R_L A u_L = R_L v$.

Throughout this paper we shall assume that the collocation method applied to the operator equation $Au = v$ is stable. For the exact definition of stability and some remarks we refer to Sect. 5.3. If the collocation is stable, if the exact solution u is in $H^2(\Gamma)$, and if $h \sim 2^{-L}$ denotes the step size of the discretization, then the approximate solution u_L satisfies the well-known optimal convergence estimates

$$\|u-u_L\|_{L^2(\Gamma)} \leq Ch^2, \quad \mathbf{r}=0,-1, \tag{2.9}$$

$$\|u-u_L\|_{H^{-1}(\Gamma)} \leq Ch^3, \quad \mathbf{r}=-1. \tag{2.10}$$

3 The Wavelet Algorithm

3.1 The Wavelet Basis of the Trial space

Now we introduce a simple wavelet basis for the piecewise linear space. These functions have been considered first for the case of different grids in the plane $\mathbb{R}^2$ (cf. [22, 42, 24]) and are called three-point hierarchical basis functions. More precisely, for the plane and for any point $\tau \in \triangle_L^{\mathbb{R}^2}$, we set (cf. Figure 5 for the supports of such functions)

$$\psi_\tau := \begin{cases} \varphi_\tau^0 & \text{if } \tau \in \nabla_{-1}^{\mathbb{R}^2} \\ \varphi_\tau^{l+1} - \frac{1}{2}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^1\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\} \\ \varphi_\tau^{l+1} - \frac{1}{4}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^2\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\}. \end{cases} \tag{3.1}$$

Here τ_1 and τ_2 denote the uniquely defined neighbours of τ on $\triangle_{l+1}^{\mathbb{R}^2}$ (cf. Figure 4). Indeed any difference grid point $\tau \in {}^2\nabla_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$ has exactly two neighbour points τ_1 and τ_2 at

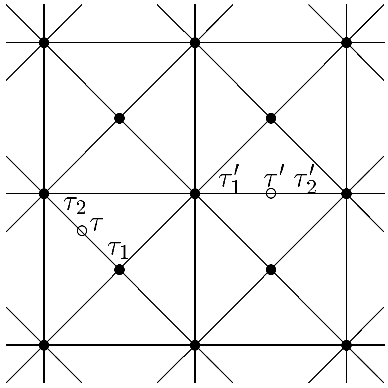

Figure 4: Neighbours τ_1 and τ_2.

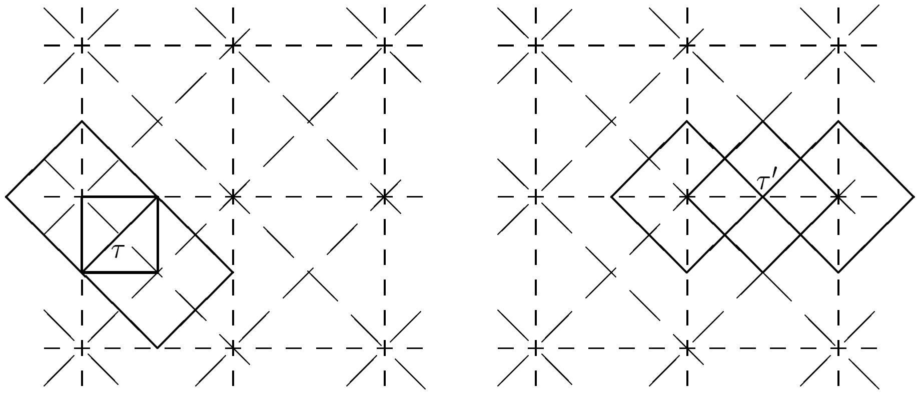

Figure 5: Supports of wavelets ψ_τ and $\psi_{\tau'}$.

minimal distance which belong to $\triangle_l^{I\!R^2} \subset \triangle_{l+1}^{I\!R^2}$. Any difference grid point $\tau' \in {}^1\nabla_l^{I\!R^2} \subset \triangle_{l+1}^{I\!R^2}$ has exactly two neighbour points τ_1' and τ_2' at minimal distance which belong to ${}^1\triangle_l^{I\!R^2} \subset \triangle_{l+1}^{I\!R^2}$. The functions ψ_τ with $\tau \in \nabla_l^{I\!R^2}$, $l = 0, \ldots, L-1$ have two vanishing moments, i.e. they are orthogonal to all constant and linear functions.

The wavelet functions ψ_τ on the manifold Γ are slight modifications of (3.1). The definition is not very difficult. However, to motivate this definition, we shortly explain the construction:

- We start with the first parametrization patch Γ_1 and the definition of functions ψ_P such that $P \in \triangle_L^\Gamma \cap \Gamma_1$. First we restrict the functions ψ_τ from (3.1) to T. If these restrictions intersect the boundary of T, then we modify them adding restrictions of three-point basis functions $\psi_{\tau'}$ with τ' outside of T. The resulting basis functions $\psi_\tau^\&$ are restrictions of functions which are symmetric (even) with respect to the boundary of T. For $P = \kappa_1(\tau)$, we take the composition $\psi_P = \psi_\tau^\& \circ \kappa_1^{-1}$ to arrive at functions over the parametrization patch Γ_1. To get continuous trial functions over

Γ, we extend the ψ_P with $P \in \nabla_l^\Gamma \cap \Gamma_1,\ l = -1, 0, \ldots, L-1$ from Γ_1 to Γ such that the extensions are piecewise linear on the partition $\{\Gamma_Q :\ Q \in \Box_{l+1}^\Gamma\}$ corresponding to the grid $\triangle_{l+1}^\Gamma$ and vanish at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_1$.

- Next we define the functions ψ_P such that $P \in \triangle_L^\Gamma \cap \{\Gamma_2 \setminus \Gamma_1\}$. We start again with the restrictions of (3.1) to T. Since we have already basis functions over the boundary $\Gamma_1 \cap \Gamma_2$, we need basis functions on Γ_2 vanishing over $\Gamma_1 \cap \Gamma_2$, i.e. basis functions on T vanishing on the side S' for which $\kappa_2(S') = \Gamma_2 \cap \Gamma_1$. Therefore, we modify the functions on T such that they are restrictions of functions antisymmetric (odd) with respect to the side S' and symmetric (even) with respect to the sides S of T with $\kappa_2(S) \not\subset \Gamma_1$. Clearly all these functions vanish on S'. We take the composition with κ_2^{-1} to arrive at functions over the parametrization patch Γ_2 which vanish over $\Gamma_2 \cap \Gamma_1$. To get continuous trial functions, we extend these functions ψ_P with $P \in \nabla_l^\Gamma \cap \{\Gamma_2 \setminus \Gamma_1\},\ l = -1, 0, \ldots, L-1$ from Γ_2 to Γ such that the extensions are piecewise linear on the partition $\{\Gamma_Q :\ Q \in \Box_{l+1}^\Gamma\}$ corresponding to the grid $\triangle_{l+1}^\Gamma$ and vanish at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_2$.

- Analogously to the previous step, we define the functions ψ_P such that the point P is in $\triangle_L^\Gamma \cap \{\Gamma_3 \setminus (\Gamma_1 \cup \Gamma_2)\}$. Then we construct the functions ψ_P with point P in $\triangle_L^\Gamma \cap \{\Gamma_4 \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)\}$ and so on. Finally, we define ψ_P with point P in $\triangle_L^\Gamma \cap \{\Gamma_{m_\Gamma} \setminus \cup_{m=1}^{m_\Gamma - 1} \Gamma_m\}$.

For more details and the properties of the basis we refer to [38] and Sect. 5.1. The final definition of the three-point hierarchical wavelet functions over the manifold Γ is

$$\psi_P := \begin{cases} \varphi_P^0 & \text{if } P \in \nabla_{-1}^\Gamma \\ \varphi_P^{l+1} - \frac{1}{2}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^1\nabla_l^\Gamma \text{ with } l \in \{0, \ldots, L-1\} \\ \varphi_P^{l+1} - \frac{1}{4}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^2\nabla_l^\Gamma \text{ with } l \in \{0, \ldots, L-1\}, \end{cases} \tag{3.2}$$

where P_1 and P_2 are the uniquely defined neighbours on $\triangle_{l+1}^\Gamma$ of $P \in \nabla_l^\Gamma$, i.e. $P_1 = \kappa_m(\tau_1)$ and $P_2 = \kappa_m(\tau_2)$ if $P = \kappa_m(\tau)$ is the representation with the minimal $m \in \{1, \ldots, m_\Gamma\}$ and if $\tau_1,\ \tau_2$ are the neighbours of τ. The coefficients $\varepsilon^{P,P'}$ are equal to one in almost all cases. Only if the point $P' = P_1, P_2$ is at the boundary of a parametrization patch, then a value $\varepsilon^{P,P'}$ different from one is needed. More precisely, the coefficients $\varepsilon^{P,P'}$ are given by (cf. Sect. 2.3 for the definition of ${}^{i}\triangle_L^\Gamma$)

$$
\varepsilon^{P,P'} := \left\{ \begin{array}{ll}
1 & \text{if} \quad \text{there is a parametrization patch } \Gamma_m \text{ such that } P \text{ and } P' \text{ belong} \\
 & \qquad \text{to the interior of the triangle } \Gamma_m \\
 & \quad \text{or there exists a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch such} \\
 & \qquad \text{that } P \text{ and } P' \text{ belong to the interior of the side } \Gamma_m \cap \Gamma_{m'} \\
2 & \text{if} \quad \text{there exists a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch such} \\
 & \qquad \text{that } m < m', \text{ that } P \text{ is an interior point of } \Gamma_m, \text{ and that } P' \\
 & \qquad \text{belongs to the interior of the side } \Gamma_m \cap \Gamma_{m'} \\
 & \quad \text{or } P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^{2}\triangle_0^\Gamma, \\
 & \qquad \text{the point } P \text{ is an interior point of a side } \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and} \\
 & \qquad m_1 < m_i, \; i = 2, \ldots, k \\
4 & \text{if} \quad P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^{1}\triangle_0^\Gamma, \\
 & \qquad \text{the point } P \text{ is an interior point of a side } \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and} \\
 & \qquad m_1 < m_i, \; i = 2, \ldots, k \\
 & \quad \text{or } P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^{2}\triangle_0^\Gamma, \\
 & \qquad \text{the point } P \text{ is an interior point of the face } \Gamma_{m_1}, \text{ and} \\
 & \qquad m_1 < m_i, \; i = 2, \ldots, k \\
0 & \text{else.}
\end{array} \right. \tag{3.3}
$$

Clearly, the support of ψ_P is contained in the union of all those Γ_m in which P or at least one of the neighbour points P_1 or P_2 is located. The basis $\{\psi_P : \; P \in \triangle_L^\Gamma\}$ spans the trial space Lin_L^Γ since the system is linearly independent (cf. (5.1)). Moreover, it represents a hierarchical basis, i.e.

$$
\begin{aligned}
&\left\{\psi_P : \; P \in \triangle_L^\Gamma\right\} = \bigcup_{l=-1}^{L-1} \left\{\psi_P : \; P \in \nabla_l^\Gamma\right\}, \\
&Lin_0^\Gamma \subset Lin_1^\Gamma \subset \ldots \subset Lin_L^\Gamma, \\
&Lin_{l'}^\Gamma = \operatorname{span} \bigcup_{l=-1}^{l'-1} \left\{\psi_P : \; P \in \nabla_l^\Gamma\right\}.
\end{aligned}
$$

The function ψ_P with $P \in \nabla_l^\Gamma$, $l = 0, \ldots, L-1$ and with $\operatorname{supp} \psi_P$ contained in the interior of only one parametrization patch has two vanishing moments, i.e. it is orthogonal to the set of all functions that are constant or linear with respect to the parametrization. Orthogonality means here orthogonality with respect to the L^2 scalar product in the parameter domain.

3.2 The Wavelet Basis of the Test space

Let us retain the definition of neighbour points $P_1, P_2 \in \triangle_l^\Gamma$ of $P \in \nabla_l^\Gamma$, $l = 0, \ldots, L-1$ from the last subsection, and recall that δ_P stands for the Dirac delta functional at point P. With this notation, we introduce the functionals

$$
\vartheta_P := \left\{ \begin{array}{ll}
\delta_P & \text{if } P \in \nabla_{-1}^\Gamma \\
\delta_P - \frac{1}{2}\left\{\delta_{P_1} + \delta_{P_2}\right\} & \text{if } P \in \nabla_l^\Gamma \text{ with } l = l(P) \in \{0, \ldots, L-1\}.
\end{array} \right. \tag{3.4}
$$

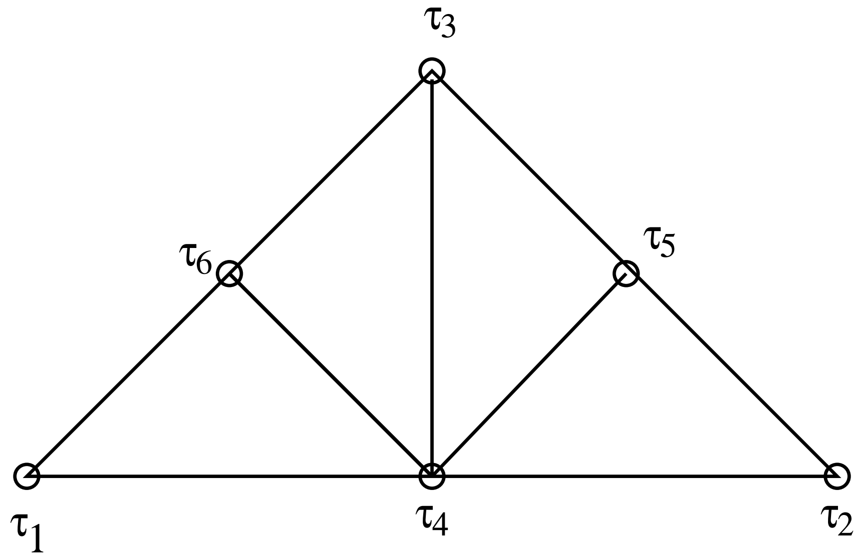

Figure 6: Interpolation points for $m_\vartheta = 3$.

Clearly, the support $\operatorname{supp} \vartheta_P$ is contained in Γ_m if P belongs to Γ_m. In particular, $\operatorname{supp} \vartheta_P$ is on the side of a parametrization patch if P is on this side. If P is a corner of a parametrization patch, then $\operatorname{supp} \vartheta_P = \{P\}$. The set $\{\vartheta_P : \ P \in \triangle_L^\Gamma\}$ is a hierarchical basis of the test space Dir_L^Γ (cf. the Sects. 2.3 and 5.2). For any $P \in \nabla_l^\Gamma$, $l = 0, \ldots, L-1$, the functional ϑ_P has two vanishing moments, i.e. it vanishes over the set of all functions that are constant or linear with respect to the parametrization. To simplify the notation, some times we shall write $f(\vartheta_P)$ for $\vartheta_P(f)$.

The basis $\{\vartheta_P\}$ will be suitable for the compression applied to operators of order $\mathbf{r} = 0$. For $\mathbf{r} = -1$ and for the quadrature estimates, a basis with more vanishing moments is needed (cf. [12, 40]). Thus we have to generalize the construction of the test functional basis to get a system $\{\vartheta_P\}$ with m_ϑ vanishing moments, where $m_\vartheta \geq 2$ is an arbitrarily prescribed positive integer. To this end we follow the ideas of Harten and Yad-Shalom [21]. We choose the integer l_ϑ such that $2^{l_\vartheta - 2} < m_\vartheta - 1 \leq 2^{l_\vartheta - 1}$. Moreover, for each $\Gamma_Q = \kappa_m(T_\tau) \in \square_l^\Gamma$ with the three corner points $\kappa_m(\tau_1)$, $\kappa_m(\tau_1)$, and $\kappa_m(\tau_1)$, we introduce a system $\{P_{Q,i} : \ i = 1, 2, \ldots, m_\vartheta(m_\vartheta + 1)/2\}$ of interpolation points on Γ_Q such that the first three points are the corner points, such that each side of Γ_Q contains exactly m_ϑ of the points, and such that all points are from the grid $\Gamma_Q \cap \triangle_{l+l_\vartheta-1}^\Gamma$. If $m_\vartheta = 2$, then $\{P_{Q,i}\}$ is exactly the set of corner points. For $m_\vartheta = 3$, $m_\vartheta = 4$, and $m_\vartheta = 5$, we choose the points $P_{Q,i} = \kappa_m(\tau_i)$ according to the figures 6, 7, and 8. By $l_{Q,i}$, $i = 1, \ldots, m_\vartheta(m_\vartheta + 1)/2$ we denote the interpolation basis (Lagrange basis) of the space of polynomials with total degree less than m_ϑ defined by

$$l_{Q,i}\left(P_{Q,j}\right) \ = \ \delta_{i,j}, \quad i, j = 1, 2, \ldots, \frac{m_\vartheta(m_\vartheta + 1)}{2}.$$

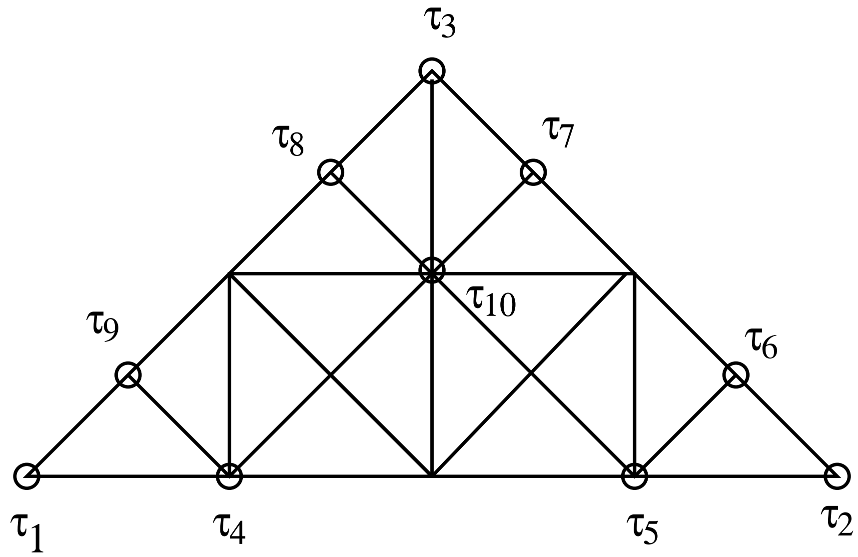

Figure 7: Interpolation points for $m_\vartheta = 4$.

Finally, the generalized test functional ϑ_P is given by

$$\vartheta_P(f) := \begin{cases} f(P) & \text{if } P \in \nabla_l^\Gamma,\ l = 1, \ldots, l_\vartheta - 2 \\ f(P) - \sum_{i=1}^{m_\vartheta(m_\vartheta+1)/2} l_{Q,i}(P)\, f(P_{Q,i}) & \text{if } P \in \nabla_{l+l_\vartheta-1}^\Gamma\ l = 0, \ldots, L - l_\vartheta, \\ & \quad P \in \Gamma_Q,\ Q \in \square_l^\Gamma. \end{cases} \tag{3.5}$$

Note that this definition is independent of the choice of Γ_Q if P is contained in more than one triangle Γ_Q, i.e. for $P \in \Gamma_Q \cap \Gamma_{Q'},\ Q, Q' \in \square_l^\Gamma$ and $P \in \nabla_{l+l_\vartheta-1}^\Gamma$, we get

$$\vartheta_P(f) := f(P) - \sum_{\substack{i=1,\ldots,m_\vartheta(m_\vartheta+1)/2 \\ P_{Q,i} \in \Gamma_Q \cap \Gamma_{Q'}}} l_{Q,i}(P)\, f(P_{Q,j}).$$

Clearly, if f is a polynomial of degree less than m_ϑ with respect to the parametrization κ_m, then the interpolation polynomial $R \mapsto \sum l_{Q,i}(R) f(P_{Q,i})$ coincides with f, and we get $\vartheta_P(f) = 0$. In other words, ϑ_P has m_ϑ vanishing moments if $l(P) \geq l_\vartheta - 1$. If $m_\vartheta = 3$ and $P \in \Gamma_Q = \kappa_m(T_\tau)$ with T_τ as in figure 9, then we get

$$\begin{aligned}
\vartheta_{\kappa_m(\sigma_1)}(f) &= f\Big(\kappa_m(\sigma_1)\Big) - \frac{3}{4} f\Big(\kappa_m(\tau_4)\Big) - \frac{3}{8} f\Big(\kappa_m(\tau_1)\Big) + \frac{1}{8} f\Big(\kappa_m(\tau_2)\Big), \\
\vartheta_{\kappa_m(\sigma_2)}(f) &= f\Big(\kappa_m(\sigma_2)\Big) - \frac{3}{4} f\Big(\kappa_m(\tau_4)\Big) - \frac{3}{8} f\Big(\kappa_m(\tau_2)\Big) + \frac{1}{8} f\Big(\kappa_m(\tau_1)\Big), \\
\vartheta_{\kappa_m(\sigma_3)}(f) &= f\Big(\kappa_m(\sigma_3)\Big) - \frac{3}{4} f\Big(\kappa_m(\tau_6)\Big) - \frac{3}{8} f\Big(\kappa_m(\tau_1)\Big) + \frac{1}{8} f\Big(\kappa_m(\tau_3)\Big), \\
\vartheta_{\kappa_m(\sigma_8)}(f) &= f\Big(\kappa_m(\sigma_8)\Big) - \frac{3}{4} f\Big(\kappa_m(\tau_6)\Big) - \frac{3}{8} f\Big(\kappa_m(\tau_3)\Big) + \frac{1}{8} f\Big(\kappa_m(\tau_1)\Big),
\end{aligned}$$

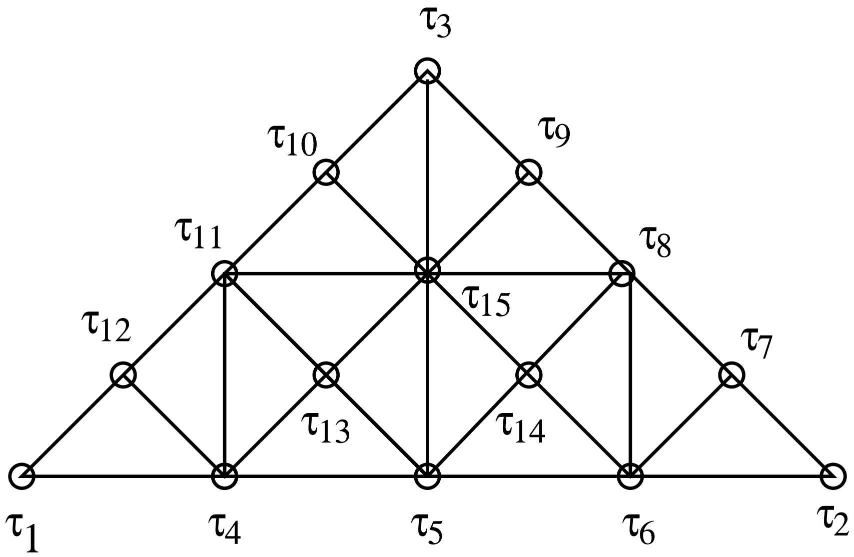

Figure 8: Interpolation points for $m_\vartheta = 5$.

$$
\begin{aligned}
\vartheta_{\kappa_m(\sigma_6)}(f) &= f\Big(\kappa_m(\sigma_6)\Big) - \frac{3}{4}f\Big(\kappa_m(\tau_5)\Big) - \frac{3}{8}f\Big(\kappa_m(\tau_2)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_3)\Big),\\
\vartheta_{\kappa_m(\sigma_9)}(f) &= f\Big(\kappa_m(\sigma_9)\Big) - \frac{3}{4}f\Big(\kappa_m(\tau_5)\Big) - \frac{3}{8}f\Big(\kappa_m(\tau_3)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_2)\Big),\\
\vartheta_{\kappa_m(\sigma_4)}(f) &= f\Big(\kappa_m(\sigma_4)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_2)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_3)\Big) - \frac{1}{4}f\Big(\kappa_m(\tau_5)\Big)\\
&\qquad -\frac{1}{2}f\Big(\kappa_m(\tau_6)\Big) - \frac{1}{2}f\Big(\kappa_m(\tau_4)\Big),\\
\vartheta_{\kappa_m(\sigma_5)}(f) &= f\Big(\kappa_m(\sigma_5)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_1)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_3)\Big) - \frac{1}{4}f\Big(\kappa_m(\tau_6)\Big)\\
&\qquad -\frac{1}{2}f\Big(\kappa_m(\tau_5)\Big) - \frac{1}{2}f\Big(\kappa_m(\tau_4)\Big),\\
\vartheta_{\kappa_m(\sigma_7)}(f) &= f\Big(\kappa_m(\sigma_7)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_1)\Big) + \frac{1}{8}f\Big(\kappa_m(\tau_2)\Big) - \frac{1}{4}f\Big(\kappa_m(\tau_4)\Big)\\
&\qquad -\frac{1}{2}f\Big(\kappa_m(\tau_5)\Big) - \frac{1}{2}f\Big(\kappa_m(\tau_6)\Big).
\end{aligned}
$$

3.3 Wavelet Transforms

For the trial space Lin_L^Γ we have two different systems of basis functions $\{\varphi_P^L\}$ and $\{\psi_P\}$ at our disposal. We denote the basis transform by $\mathcal{T}_A$ (lower index A stands for ansatz), i.e. the matrix $\mathcal{T}_A$ maps the coefficient vector $\xi^L := (\xi_P^L)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \xi_P^L \varphi_P^L$ into the coefficient vector $\beta := (\beta_P)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \beta_P \psi_P$. This transform can be determined by a pyramid type algorithm which

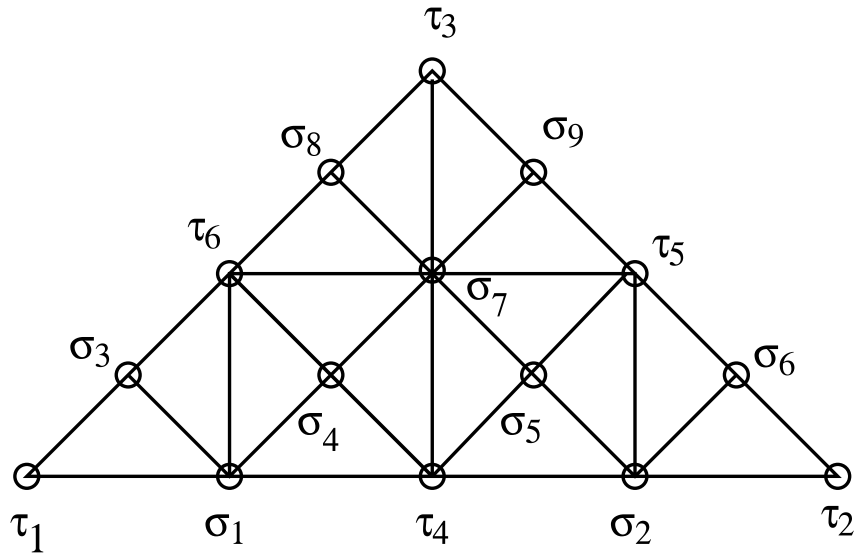

Figure 9: Points for test functional if $m_\vartheta = 3$.

is called fast wavelet transform (cf. e.g. [17]). Similarly, the inverse transform $\mathcal{T}_A^{-1}$ can be realized by such a pyramid scheme. Analogously to the trial space, we have two different bases in the test space. By $\mathcal{T}_T$ (lower index T stands for test space) we denote the linear transform which maps the vector $\gamma = (\gamma_P)_{P\in\triangle_L^\Gamma} := (\vartheta_P(f))_{P\in\triangle_L^\Gamma}$ of functionals applied to a function f into the vector of function values $\eta = (\eta_P)_{P\in\triangle_L^\Gamma} := (\delta_P(f))_{P\in\triangle_L^\Gamma} = (f(P))_{P\in\triangle_L^\Gamma}$. Again, the transform can be realized by a fast wavelet algorithm. The inverse $\mathcal{T}_T^{-1}$ is simply a multiplication by a sparse matrix.

3.4 Wavelet Algorithm

Analogously to the stiffness matrix A_L in Sect. 2.5 we can set up a matrix with respect to the wavelet basis. We introduce A_L^w by

$$A_L^w := \left(a_{P',P}^w\right)_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P}^w := \vartheta_{P'}(A\psi_P). \tag{3.6}$$

Note that $A_L = \mathcal{T}_T A_L^w \mathcal{T}_A$. It will turn out that most of the entries $a_{P',P}^w$ are so small that they can be neglected. Thus in the next subsection we will give an a priori matrix pattern $\mathcal{P} \subset \triangle_L^\Gamma \times \triangle_L^\Gamma$ with no more than $O(2^{2L}L)$ elements. We will replace A_L^w by the sparse matrix obtained by the compression

$$A_L^{w,c} := \left(a_{P',P}^{w,c}\right)_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P}^{w,c} := \vartheta_{P'}(a\psi_P) + \begin{cases} \vartheta_{P'}(K\psi_P) & \text{if } (P',P)\in\mathcal{P} \\ 0 & \text{else.} \end{cases} \tag{3.7}$$

In the numerical computation the entries have to be computed by approximating the parametrization and by quadrature. We denote the approximate value for $a_{P',P}^{w,c}$ by $a_{P',P}^{w,c,q}$

and set

$$A_L^{w,c,q} := \left(a_{P',P}^{w,c,q}\right)_{P',P\in\triangle_L^\Gamma}, \quad A_L^c := \mathcal{T}_T A_L^{w,c} \mathcal{T}_A, \quad A_L^{c,q} := \mathcal{T}_T A_L^{w,c,q} \mathcal{T}_A. \tag{3.8}$$

With this notation we can describe two variants of the wavelet algorithm which differ in the iterative solution of the discretized linear systems. The first is designed for integral operators of arbitrary order $\mathbf{r}$ and requires the application of one transform $\mathcal{T}_A^{-1}$ and one transform $\mathcal{T}_T^{-1}$ during the whole algorithm.

First Wavelet Algorithm

$$\begin{array}{ll} \text{i)} & \text{compute the right-hand side } \gamma := (\vartheta_P(v))_P = \mathcal{T}_T^{-1}(v(P))_P \\ \text{ii)} & \text{compute the sparsity pattern } \mathcal{P} \\ \text{iii)} & \text{assemble } A_L^{w,c,q} \text{ by a quadrature algorithm} \\ \text{iv)} & \text{solve } A_L^{w,c,q}\beta = \gamma \text{ iteratively, e.g. by the diagonally preconditioned} \\ & \quad \text{GMRes method} \\ \text{v)} & \text{compute } \xi = \mathcal{T}_A^{-1}\beta \\ \text{vi)} & \text{post processing of the values } u(P) \approx \xi_P, \text{ e.g. computation} \\ & \quad \text{of linear functionals of the solution } u \end{array} \tag{3.9}$$

The second is designed for operators of order $\mathbf{r} = 0$. Though an application of the two wavelet transforms $\mathcal{T}_A$ and $\mathcal{T}_T$ is required in each iteration, the corresponding number of all iterations is often much smaller, and the second algorithm is faster.

Second Wavelet Algorithm

$$\begin{array}{ll} \text{i)} & \text{compute the right-hand side } \eta := (v(P))_P \\ \text{ii)} & \text{compute the sparsity pattern } \mathcal{P} \\ \text{iii)} & \text{assemble } A_L^{w,c,q} \text{ by a quadrature algorithm} \\ \text{iv)} & \text{solve } A_L\xi = \eta \text{ iteratively, e.g. by the GMRes method,} \\ & \quad \text{whenever a multiplication by matrix } A_L \text{ is required, then} \\ & \quad \text{multiply by } \mathcal{T}_A, \text{ by } A_L^{w,c,q}, \text{ and by } \mathcal{T}_T \\ \text{v)} & \text{post processing of the values } u(P) \approx \xi_P, \text{ e.g. computation} \\ & \quad \text{of linear functionals of the solution } u \end{array} \tag{3.10}$$

The GMRes algorithm is described in [39], and the diagonal preconditioner for the algorithm (3.9) will be derived in Sect. 5.3 (cf. (5.14)).

3.5 The Compression Algorithm

From now on we suppose that the number $\mathbf{m}_\vartheta$ of vanishing moments of the test functionals is equal to $4-\mathbf{r}$. We note, however, that for the compression and for most of the quadrature algorithm the choice $\mathbf{m}_\vartheta = 2 - \mathbf{r}$ would be sufficient. Only for the quadrature in Sect. 4.3.3 the choice $\mathbf{m}_\vartheta = 4 - \mathbf{r}$ is crucial. In order to introduce the compression pattern $\mathcal{P}$, we need some notation. Let us retain the definition of ∇_l^Γ and $\triangle_L^\Gamma$ from Sect. 2.3. For

$P \in \triangle_L^\Gamma$, recall that $l(P)$ is the level of P (cf. the end of Sect. 2.3). By Ψ_P we denote the support of the function ψ_P and by Θ_P the convex hull of the support of the test functional ϑ_P, i.e., $\vartheta_P := \kappa_m(\mathrm{conv}(\kappa_m^{-1}(\mathrm{supp}\,\vartheta_P)))$. Now we take a constant $d \geq 1$ and define the set $\mathcal{P}$ as the set of all $(P', P) \in \triangle_L^\Gamma \times \triangle_L^\Gamma$ such that Ψ_P is completely contained in the interior of a single parameter patch Γ_m and

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \leq \max\left\{2^{-l(P)}, 2^{-l(P')}, d2^{0.6\,L-0.7\,l(P)-0.9\,l(P')}\right\} \tag{3.11}$$

or such that Ψ_P contains points of at least two parameter patches and

$$\mathrm{dist}\,(\Psi_P, \Theta_{P'}) \leq \max\left\{2^{-l(P)}, 2^{-l(P')}, d2^{L-0.7\,l(P)-1.3\,l(P')}\right\}. \tag{3.12}$$

In numerical computations the compression parameter d should be determined by experiments. However, to get an asymptotically optimal compression result which is asymptotically optimal up to logarithmic factors and which is convenient for the subsequent quadrature scheme, it is sufficient to choose d sufficiently large. The well-known proof techniques of [12, 29, 40, 35] yield

Theorem 3.1 *For the pattern $\mathcal{P}$, the number of non-zero entries $N_{\mathcal{P}}$ is less than $CL2^{2L} \sim N \log N$, where $N \sim 2^{2L}$ is the number of degrees of freedom. If the piecewise linear collocation is stable, then the collocation method with compression is stable, too. The error estimates (2.9) and (2.10) remain valid if $u_L = \sum \xi_P \psi_P$ is the solution of the compressed matrix equation $A_L^{w,c}(\xi_P)_P = (\vartheta_P(v))_P$.*

Clearly the number of necessary arithmetic operations of all steps in the algorithms (3.9) and (3.10) except the steps iii) and iv) is less than $C\,N_{\mathcal{P}}$. Step iv) requires $C\,N_{\mathcal{P}} \log N$ operations. However, if we solve the systems successively over the grids $\triangle_l^\Gamma$, $l = 0, \ldots, L$ and if the initial solution for the grid $\triangle_{l+1}^\Gamma$ is the final solution from the coarser grid $\triangle_l^\Gamma$, then the number of necessary iterations is uniformly bounded. This cascadic iteration method requires no more than $C\,N_{\mathcal{P}}$ operations. The key point for a fast algorithm, however, is the implementation of step iii). Usually, this is the most time consuming part of the numerical computation. For its realization and complexity, we refer to the results in Sect. 4 and the proofs in Sect. 6. Further details for the implementation of the wavelet algorithm can be found in [23, 34].

4 Approximation of the Parametrization Mappings and Quadrature

4.1 Parametrization and Quadrature for the Far Field

Now we consider the computation of the matrix entries $a_{P',P}^{w,c,q}$ (cf. Sect. 3.4). Obviously, the terms $\vartheta_{P'}(a\psi_P)$ (cf. (3.7)) can be computed without difficulty, and the corresponding number of arithmetic operations is less than $O(N \log N)$. Therefore, we only have to deal with the computation of $\vartheta_{P'}(K\psi_P)$ corresponding to the integral operator K. In this subsection we shall indicate the assembling of those entries for which $\mathrm{dist}(\Psi_P, \Theta_{P'})$ is larger or equal to $\max\{2^{-l(P)}, 2^{-l(P')}\}$. These entries will be called the far field entries.

For the quadrature over Ψ_P, we shall apply a composite quadrature rule with a fixed basis rule of convergence order three or four. Thus we have to start with the introduction of the partition for the composite rule. Clearly, Ψ_P is the union of a finite number of triangles Γ_Q with $l(Q) = l(P)+1$ where the trial basis function ψ_P is linear with respect to the parametrization parameter. In general, however, this first partition is not sufficiently fine. Instead we split Ψ_P into the union of all Γ_Q with level $l(Q) = l(P, P') + l_0$, where

$$l(P, P') \;:=\; \begin{cases} l(P)+1 & \text{if } \mathrm{dist}(\Psi, \Theta_{P'})^{1.1} \geq 2^{0.9\,L-l(P)-l(P')} \\ l+1 & \text{if } \mathrm{dist}(\Psi, \Theta_{P'})^{1.1} < 2^{0.9\,L-l(P)-l(P')} \text{ and if} \\ & \quad 2^{0.9\,L-l-l(P')} \leq \mathrm{dist}(\Psi, \Theta_{P'})^{1.1} < 2^{0.9\,L-(l-1)-l(P')}\;. \end{cases} \tag{4.1}$$

and where l_0 is a fixed integer which is supposed to be sufficiently large. This constant l_0 is introduced to enforce stability. For practical computations, however, we expect that the choice $l_0 = 0$ is acceptable. In accordance with (3.7) and (2.4), we shall introduce quadrature approximations $a^{w,c,q}_{P',P,Q}$ for

$$\vartheta_{P'} \left(\int_{\Gamma_Q} k(\cdot, R, n_R) \psi_P(R) \, \mathrm{d}_R\Gamma \right). \tag{4.2}$$

Here the functional $\vartheta_{P'}$ is applied to the function in brackets depending on the variable indicated by a dot. Using these $a^{w,c,q}_{P',P,Q}$, we define the entries $a^{w,c,q}_{P',P}$ by

$$a^{w,c,q}_{P',P} \;:=\; \vartheta_{P'}(a\psi_P) + \begin{cases} 0 & \text{if } (P', P) \notin \mathcal{P} \\ \sum_{Q \in \Box^\Gamma_{l(P)+1}:\, \Gamma_Q \subset \mathrm{supp}\, \psi_P} a^{w,c,q}_{P',P,Q} & \text{if } (P', P) \in \mathcal{P}. \end{cases} \tag{4.3}$$

We shall defer the definition of the near field terms $a^{w,c,q}_{P',P,Q}$, i.e. the terms with the property $\mathrm{dist}(\Psi_P, \Theta_{P'}) < \max\{2^{-l(P)}, 2^{-l(P')}\}$ to Sects. 4.2 - 4.3. In this subsection we introduce the far field terms $a^{w,c,q}_{P',P,Q}$.

Let us fix a far field subdomain Γ_Q with $Q = \kappa_m(\tau) \in \Box^\Gamma_l$ and $l = l(P, P')$. Using the parametrization κ_m over $T_\tau = \kappa_m^{-1}(\Gamma_Q)$, we write the integral of (4.2) in the form

$$\vartheta_{P'} \left(\int_{T_\tau} k(\cdot, \kappa_m(\sigma), n_{\kappa_m(\sigma)}) \tilde{\psi}_P(\sigma) \mathcal{J}_m(\sigma) \, \mathrm{d}\sigma \right), \tag{4.4}$$

where $\mathcal{J}_m(\sigma) := |\partial_{\sigma_1}\kappa_m(\sigma) \times \partial_{\sigma_2}\kappa_m(\sigma)|$ is the Jacobian determinant of the transformation κ_m at $\sigma = (\sigma_1, \sigma_2) \in T_\tau$ and where $\tilde{\psi}_P(\sigma)$ stands for the factor $\psi_P(R) = \psi_P(\kappa_m(\sigma))$ which is independent of the parametrization κ_m (cf. (3.2) and (2.7)). We derive the approximation $a^{w,c,q}_{P',P,Q}$ for (4.4) in two steps.

In the **first step**, we replace the parametrization κ_m over T_τ by a piecewise polynomial interpolation κ'_m. For a fixed $\sigma \in T_\tau$, the polynomial interpolant is defined over the level L triangle $T_{\tau'}$ determined by $\tau' \in \Box^T_L$ and $\sigma \in T_{\tau'} \subseteq T_\tau$. The polynomial interpolation κ'_m is chosen to be of degree $\mathbf{m}_p := 3 - \mathbf{r}$ which is greater than the optimal order of convergence

$\mathbf{m} := 2 - \mathbf{r}$. In particular, for $\mathbf{m}_p = 3$ a cubic interpolation with ten interpolation knots can be chosen. For $\mathbf{m}_p = 2$, which unfortunately is less than $3 - \mathbf{r}$ and which leads to suboptimal rates of convergence, a quadratic interpolation with six knots would be possible. This quadratic interpolation is defined as in [2]. Denoting by $\tau_i,\ i = 1,2,3$ the three corner points of the triangle $T_{\tau'} \subseteq \kappa_m^{-1}(\Gamma_Q)$, respectively, and by $\tau_i,\ i = 4,5,6$ the mid-points

$$\tau_4 = \frac{1}{2}(\tau_2 + \tau_3), \quad \tau_5 = \frac{1}{2}(\tau_1 + \tau_2), \quad \tau_6 = \frac{1}{2}(\tau_1 + \tau_3),$$

of the three sides of the triangle, we set

$$\begin{aligned}
\kappa'_m(\sigma) &= \sum_{i=1}^{6} \kappa_m(\tau_i)\mathcal{L}_i(\sigma), \qquad (4.5)\\
\mathcal{L}_1\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= s[2s-1],\\
\mathcal{L}_2\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= t[2t-1],\\
\mathcal{L}_3\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= (1-s-t)[2(1-s-t)-1],\\
\mathcal{L}_4\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= 4t(1-s-t),\\
\mathcal{L}_5\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= 4st,\\
\mathcal{L}_6\Big(\tau_3 + s(\tau_1 - \tau_3) + t(\tau_2 - \tau_3)\Big) &:= 4s(1-s-t).
\end{aligned}$$

In any case, we approximate (4.4) by

$$\vartheta_{P'}\left(\int_{T_\tau} k(\cdot, \kappa'_m(\sigma), n'_{\kappa'_m(\sigma)})\tilde{\psi}_P(\sigma)\mathcal{J}'_m(\sigma)\,\mathrm{d}\sigma\right), \qquad (4.6)$$

where $\mathcal{J}'_m(\sigma) := |\partial_{\sigma_1}\kappa'_m(\sigma) \times \partial_{\sigma_2}\kappa'_m(\sigma)|$ is the Jacobian determinant of the transformation κ'_m at $\sigma = (\sigma_1, \sigma_2) \in T_\tau$. The symbol $n'_{\kappa'_m(\sigma)}$ in the last formula stands for the unit vector at the point $\kappa'_m(\sigma)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In the **second step**, we split the integrand of (4.6) into the product $f(\sigma)\tilde{\varrho}(\sigma)$

$$\begin{aligned}
f(\sigma) &:= k(\cdot, \kappa'_m(\sigma), n_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma),\\
\tilde{\varrho}(\sigma) &:= \varrho\Big(\kappa'_m(\sigma)\Big) = \tilde{\psi}_P(\sigma).
\end{aligned}$$

Note that $\tilde{\varrho}$ is linear with respect to σ. We apply a product quadrature with weight $\tilde{\varrho}$ and of order $\mathbf{q} := 3 - \mathbf{r}$ to the integral in (4.6). In general, for all following approximations, we always assume that the order of convergence of the quadrature rule $\mathbf{q}$ coincides with the degree $\mathbf{m}_p$ of the approximate piecewise polynomial parametrization. If $\mathbf{q} = 3$, then we choose the six point rule based upon quadratic interpolation which has been used for

(4.5). In case $\mathbf{q} = 2$, which unfortunately is less than $3 - \mathbf{r}$ and leads to suboptimal rates of convergence, we take the three point rule. The product quadrature rule takes the form

$$\int_{T_\tau} f(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma \;\approx\; \sum_{v=1}^{3} f(\tau_v) b^{w,c,q}_{P,Q,v}, \quad b^{w,c,q}_{P,Q,v} \;:=\; \int_{T_\tau} \tilde{\phi}_{Q,v}(\sigma)\tilde{\psi}_P(\sigma)\,\mathrm{d}\sigma, \tag{4.7}$$

where $\tilde{\phi}_{Q,v}$ is the linear function on T_τ defined by $\tilde{\phi}_{Q,v}(\tau_{v'}) = \delta_{v,v'}$. Similar rules including more knots τ_v and higher order Lagrange interpolation polynomials $\tilde{\phi}_{Q,v}$ can be defined for arbitrary $\mathbf{q}$. An easier but equivalent choice for $\mathbf{q} = 2$ only is to replace the three corner points τ_v by the three mid-points of the sides of triangle T_τ. If the quadrature weights are one third of the measure of T_τ, then the resulting quadrature is known to be exact for quadratic functions and we get

$$\int_{T_\tau} f(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma \;\approx\; \sum_{v=1}^{3} f(\tau_v) b^{w,c,q}_{P,Q,v}, \quad b^{w,c,q}_{P,Q,v} \;:=\; \frac{1}{3}|T_\tau|\tilde{\psi}_P(\tau_v). \tag{4.8}$$

In any case, the integral (4.6) is approximated by

$$a^{w,c,q}_{P',P,Q} \;:=\; \vartheta_{P'}\left(\sum_v k(\cdot, Q'_v, n'_{Q'_v})\mathcal{J}'_m(\tau_v) b^{w,c,q}_{P,Q,v}\right), \tag{4.9}$$

where $Q'_v := \kappa'_m(\tau_v)$ denote the corner points and, possibly, some additional quadrature knots of the triangles $\kappa'_m(T_\tau)$, respectively. The symbol $n'_{Q'_v}$ in the last formula stands for the unit vector at the point $Q'_v = \kappa'_m(\tau_v)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In Sect. 6.1 we shall prove that the additional error due to the far field quadrature is, roughly speaking, less than the error of the exact collocation. Analogous error estimates are true also for the approximation of the near field and the singular integrals in the Sects. 4.2 - 4.3. More precisely, we get

Theorem 4.1 *Consider the wavelet collocation and the matrix compressed according to the pattern $\mathcal{P}$ of Theorem 3.1 and suppose the integer constant l_0 is sufficiently large. If the exact collocation described in Sect. 2.5 is stable, then the compressed collocation with approximation of the boundary and with the quadrature of Sects. 4.1 - 4.3 is stable, too. The error for the collocation solution u_L, including compression, approximation of the parameter mappings, and quadrature, satisfies (2.9) and (2.10), respectively. The number of quadrature knots and the number of necessary arithmetic operations for the computation of the stiffness matrix $A^{w,c,q}_L$ is less than $C\,N[\log N]^3$ if $\mathbf{r} = 0$ and less than $C\,N[\log N]^2$ if $\mathbf{r} = -1$.*

Proof. Due to Sect. 5.3, the stability and the error estimates will be a consequence of the Lemmata 6.1, 6.3, and 6.5. The complexity bound will be shown in the Lemmata 6.2, 6.4, and 6.6. ∎

4.2 Parametrization and Quadrature for the First Part of the Near Field

4.2.0. Let us fix $\vartheta_{P'}$ and ψ_P with $0 < \operatorname{dist}(\Psi_P, \Theta_{P'}) < \max\{2^{-l(P)}, 2^{-l(P')}\}$, and let us consider the integral (4.2) for which we seek the partition of $\operatorname{supp}\psi_P$ into triangles Γ_Q and the corresponding quadratures $a^{w,c,q}_{P',P,Q}$. The near field part with $\operatorname{dist}(\Psi_P, \Theta_{P'}) = 0$ will be treated in Sect. 4.3. In particular, the computation of the singular integrals will be discussed in Sect. 4.3. For the first part of the near field, we shall distinguish two cases in this subsection.

4.2.1. We start with the case determined by $l(P) \geq l(P')$ and $0 < \operatorname{dist}(\Psi_P, \Theta_{P'})$. In view of the near field condition, we have $0 < \operatorname{dist}(\Psi_P, \Theta_{P'}) \leq 2^{-l(P')}$. Moreover, there is a constant $c_\Gamma > 0$ such that $c_\Gamma 2^{-l(P)} < \operatorname{dist}(\Psi_P, \Theta_{P'}) \leq 2^{-l(P')}$. Indeed, suppose c'_Γ is the reciprocal Lipschitz constant of the inverse parametrization mappings, i.e. for $m = 1, \ldots, m_\Gamma$ and for any pair of points $\tau_1, \tau_2 \in T$, there holds

$$c'_\Gamma\, |\tau_1 - \tau_2| \quad \leq \quad |\kappa_m(\tau_1) - \kappa_m(\tau_2)|\,.$$

Set $c_\Gamma := c'_\Gamma/2$. Then the distance of a point τ_1 of the level l grid to a triangle T_{τ_2} of the level l triangulation not containing τ_1 is at least $0.5\, 2^{-l}$. Hence, the distance of a point $P_1 := \kappa_m(\tau_1)$ of the level l grid over Γ to a triangle $\Gamma_Q := \kappa_m(T_{\tau_2})$ of the level l triangulation not containing τ_1 is at least $c_\Gamma\, 2^{-l}$. Since the points of $\vartheta_{P'}$ are on the grid of level $l(P') + 1$ and Ψ_P consists of triangles of level $l(P) + 1$, the lower estimate $c_\Gamma 2^{-l(P)} < \operatorname{dist}(\Psi_P, \Theta_{P'})$ follows.

We introduce the integer $l(P, P')$ just as in (4.1) but with $\operatorname{dist}(\Psi_P, \Theta_{P'})$ replaced by $\operatorname{dist}(\Psi_P, \operatorname{supp}\vartheta_{P'})$, i.e. this time the distance is measured to the single points in $\operatorname{supp}\vartheta_{P'}$ and not to their convex hull $\Theta_{P'}$. The partition of $\Psi_P = \operatorname{supp}\psi_P$ is obtained like in the far field case in Sect. 4.1 as the union of all Γ_Q of level $l(P, P') + l_0$ contained in Ψ_P. Retaining the definition $\mathbf{q} = \mathbf{m}_p := 3 - \mathbf{r}$ and using the definition (4.3), we get the corresponding quadrature approximation.

4.2.2. Next we consider the case determined by $l(P) < l(P')$ and $0 < \operatorname{dist}(\Psi_P, \Theta_{P'})$. In view of the near field condition and the fact that ψ_P resp. $\vartheta_{P'}$ are defined on the grids of level $l(P)$ resp. $l(P')$, we have $c_\Gamma 2^{-l(P')} < \operatorname{dist}(\Psi_P, \Theta_{P'}) \leq 2^{-l(P)}$. Proceeding similarly to Sect. 4.1, we set $\mathrm{Dist} := \operatorname{dist}(\Psi_P, \Theta_{P'})$ and introduce $l(P, P')$ by

$$l(P,P') \quad := \quad \begin{cases} l(P)+1 & \text{if } \mathrm{Dist}^{0.55} \geq 2^{0.95\,L - 1.1\,l(P') - 0.4\,l(P)} \\ l+1 & \text{if } \mathrm{Dist}^{0.55} < 2^{0.95\,L-1.1\,l(P')-0.4\,l(P)} \text{ and if} \\ & \quad 2^{0.95\,L-1.1\,l(P')-l+0.6\,l(P)} \leq \mathrm{Dist}^{0.55}, \\ & \quad \mathrm{Dist}^{0.55} < 2^{0.95\,L-1.1\,l(P')-(l-1)+0.6\,l(P)}\,. \end{cases} \tag{4.10}$$

The partition of $\Psi_P = \operatorname{supp}\psi_P$ is obtained in three steps.

i) We split Ψ_P into the triangles of level $l(P) + 1$.

ii) We introduce the dyadic partition of each of these triangles into a minimal number of triangles from $\{\Gamma_{Q'},\ Q' \in \triangle^\Gamma_L\}$ such that the distance of these triangles to $\Theta_{P'}$ is greater or equal to $2^{-l(Q')-1}$. This is obtained as follows. We start with the level

$l(P)+1$ triangles of step i) and let the level l run from $l = l(P)+1$ to L. For each level l, we have a certain number of level l triangles. We check if the distance of these triangles to $\Theta_{P'}$ is greater or equal to 2^{-l-1}. If yes, then we keep these triangles. If not, then we split these triangles into the four subtriangles of level $l+1$ and replace the level l triangles by the new level $l+1$ triangles. The procedure ends, if no triangle of level $l+1$ is produced. Obviously, the number of all these triangles $\Gamma_{Q'}$ is less than a constant times L.

iii) Now we split each of the triangles $\Gamma_{Q'}$ from the previous step ii) uniformly into higher level triangles. Note that in Sect. 4.1 each $l(P)+1$ level triangle of Ψ_P is split into the $l(P,P')+l_0$ level subtriangles, i.e. the partition is refined over $[l(P,P')+l_0-(l(P)+1)]$ levels. Analogously, we refine the partition of step ii) over $[l(P,P')+l_0-(l(P)+1)]$ levels. In other words, each triangle $\Gamma_{Q'}$ of ii) is split into the triangles Γ_Q with $\Gamma_Q \subseteq \Gamma_{Q'}$ and $Q \in \square_{\tilde{l}}^{\Gamma}$, $\tilde{l} := l(Q') + [l(P,P')+l_0-(l(P)+1)]$.

We denote the resulting partition of Ψ_P by $\{\Gamma_Q : \; Q \in \square_{P',P}^{\Gamma}\}$. Using this partition and proceeding analogously to Section 4.1, we arrive at the quadrature approximation defined by

$$a_{P',P}^{w,c,q} \quad := \quad \vartheta_{P'}(a\psi_P) + \sum_{Q\in\square_{P',P}^{\Gamma}} a_{P',P,Q}^{w,c,q}, \tag{4.11}$$

where the terms $a_{P',P,Q}^{w,c,q}$ are given by (4.9), where the product rule (4.7) is replaced by the analogous product rule of order $\mathbf{q} := 4$, and where a piecewise polynomial interpolation κ_m' of degree $\mathbf{m}_p := \mathbf{q}$ is employed.

4.3 Parametrization and Quadrature for the Second Part of the Near Field

4.3.1.0. Throughout the present section we suppose $\mathrm{dist}(\Psi_P, \Theta_{P'}) = 0$. First we consider the case $l(P) \geq l(P')$. By definition, the functional $\vartheta_{P'}$ is a linear combination of point evaluation functionals

$$\vartheta_{P'}(f) := \sum_{\lambda=1}^{\lambda_{P'}} c_\lambda f(P_\lambda)$$

and $\Psi := \operatorname{supp}\psi$ is the union of level $l(P)+1$ triangles Γ_{Q_μ} for $\mu = 1,\ldots,\mu_P$. According to this splitting, we get

$$\begin{aligned} a_{P',P}^{w,c,q} \quad &:= \quad \vartheta_{P'}(a\psi_P) + \sum_{\lambda=1}^{\lambda_{P'}} c_\lambda \sum_{\mu=1}^{\mu_P} a_{P',\lambda,P,\mu}^{w,c,q} \\ a_{P',\lambda,P,\mu}^{w,c,q} \quad &\sim \quad \int_{\Gamma_{Q_\mu}} k(P_\lambda, R, n_R)\psi_P(R)\,\mathrm{d}_R\Gamma. \end{aligned} \tag{4.12}$$

In the following we compute $a_{P',\lambda,P,\mu}^{w,c,q}$ analogously to $a_{P',P}^{w,c,q}$ in Sect. 4.2.1.

4.3.1.1. If $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda) > 0$, then $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda) > c_\Gamma 2^{-l(P)}$. We introduce $l(P,P') = l(P,\lambda,P',\mu)$ just as in (4.1) but with $\mathrm{dist}(\Psi,\Theta_{P'})$ replaced by $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda)$. The partition $\{\Gamma_Q : Q \in \square^\Gamma_{Q_\mu,P_\lambda}\}$ of Γ_{Q_μ} for the quadrature is obtained like that of Ψ_P in the far field case in Sect. 4.1 as the union of all Γ_Q of level $l(Q)$ contained in Γ_{Q_μ} with

$$l(Q) \ := \ l(P,P') + l_0 + \begin{cases} 0 & \text{if } \mathbf{r} = -1 \\ \left[\frac{1}{3-\mathbf{r}}\, {}^2\!\log L\right] & \text{if } \mathbf{r} = 0. \end{cases} \tag{4.13}$$

Using the definition (4.12) and

$$a^{w,c,q}_{P',\lambda,P,\mu} \ := \ \sum_{Q\in\square^\Gamma_{Q_\mu,P_\lambda}} \sum_v k(P_\lambda, Q'_v, n'_{Q'_v}) \mathcal{J}'_m(\tau_v) b^{w,c,q}_{P,Q,v} \tag{4.14}$$

with the quadrature weights $b^{w,c,q}_{P,Q,v}$ of a quadrature rule of order $\mathbf{q} := 3 - \mathbf{r}$ (cf. (4.7) and (4.8)) with an approximate piecewise polynomial interpolation κ'_m of degree $\mathbf{m}_p := \mathbf{q}$, we get the corresponding quadrature approximation.

4.3.1.2.1. If $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda) = 0$, we introduce $l(P,P')$ just as in (4.1) but with $\mathrm{dist}(\Psi,\Theta_{P'})$ replaced by $2^{-l(P)}$. Additionally we assume that P_λ is contained in the interior of exactly one parametrization patch Γ_m or that $\mathbf{r} = -1$. The partition of Γ_{Q_μ} is obtained in two steps (compare the three steps in Sect. 4.2.2).

i) We subtract the triangles Γ_{Q_*} of level $\mathbf{m}L$, defined by $P_\lambda \in \Gamma_{Q_*} \subseteq \Gamma_{Q_\mu}$, from Γ_{Q_μ}. Then we introduce the dyadic partition of $\Gamma_{Q_\mu} \setminus \cup\Gamma_{Q_*}$ into a minimal number of triangles $\Gamma_{Q'}$ with levels $l(Q')$ between $l(Q_\mu)+1$ and $\mathbf{m}L$ such that the distance of these triangles to P' is greater or equal to $2^{-l(Q')-1}$. Obviously, the number of all these triangles is less than a constant times L.

ii) Now we split each of the triangles $\Gamma_{Q'}$ from the previous step i) uniformly into higher level triangles. Each triangle $\Gamma_{Q'}$ of i) is split into the triangles Γ_Q with $\Gamma_Q \subseteq \Gamma_{Q'}$ and $Q \in \square^\Gamma_{\tilde{l}}$ such that

$$\tilde{l} \ := \ l(Q') + [l(P,P') + l_0 - (l(P)+1)] + \begin{cases} 0 & \text{if } \mathbf{r} = -1 \\ \left[\frac{1}{3-\mathbf{r}}\, {}^2\!\log L\right] & \text{if } \mathbf{r} = 0. \end{cases} \tag{4.15}$$

We denote the resulting partition of Γ_{Q_μ} by $\{\Gamma_Q : Q \in \square^\Gamma_{P',\lambda,P,\mu}\}$. Using this partition and the formulae (4.12) and (4.14) with quadrature order $\mathbf{q} = 3 - \mathbf{r}$ and with a piecewise polynomial interpolation κ'_m of degree $\mathbf{m}_p := \mathbf{q}$, we obtain the quadrature approximation.

4.3.1.2.2. If $\mathbf{r} = 0$ and if P_λ is at the boundary of a parametrization patch and thus contained in at least two parametrization patches Γ_m, then we have to modify the triangles Γ_{Q^*} in the partition. This is necessary to get the right value of the integral in accordance with Cauchy's finite part definition (cf. [26]). More precisely, suppose $P_\lambda = \kappa_{m_i}(\tau_i)$, $i = 0,\ldots,i_\lambda$ and denote the level $\mathbf{m}L$ triangles of the parametrization patch Γ_{m_0} containing P_λ by $\Gamma^j_0 := \kappa_{m_0}(T^j_0)$, $j = 0,\ldots,j_\lambda$. The subtriangles in Γ_{Q_μ} which we neglect are now the triangles

$$\Gamma^j_i \ := \ \kappa_{m_i}(T^j_i), \quad T^j_i \ := \ \left\{\nabla\left[[\kappa_{m_i}]^{-1} \circ \kappa_{m_0}\right](\tau_0)\left(T^j_0\right)\right\} \cap T,$$

where $\nabla[[\kappa_{m_i}]^{-1} \circ \kappa_{m_0}](\tau_0)$ stands for the Fréchet derivative of the mapping $[\kappa_{m_i}]^{-1} \circ \kappa_{m_0}$ taken at the point τ_0. To get the right quadrature formula we have to replace step i) by the following i').

i') We introduce the dyadic partition of $\Gamma_{Q_\mu} \setminus \cup\Gamma_i^j$ into a minimal number of triangles $\Gamma_{Q'}$ with levels $l(Q')$ between $l(Q_\mu)+1$ and $\mathbf{m}L$ such that the distance of these triangles to P' is greater or equal to $2^{-l(Q')-1}$. This is obtained as follows. We start with the level $l(P)+1$ triangles Γ_{Q_μ} and let the level l run from $l = l(P)+1$ to L. For each level l, we have a certain number of level l triangles. We check if the distance of these triangles to P' is greater or equal to 2^{-l-1}. If yes, then we keep these triangles. If not, then we split these triangles into the four subtriangles of level $l+1$ and replace the level l triangles by the new level $l+1$ triangles. The procedure ends, if no new triangle is produced. To get a full partition of $\Gamma_{Q_\mu} \setminus \cup\Gamma_i^j$, we replace the level $\mathbf{m}L$ triangles intersecting $\cup\Gamma_i^j$ by a few number of triangles contained in $\Gamma_{Q_\mu} \setminus \cup\Gamma_i^j$. Obviously, the number of all these triangles is less than a constant times L.

4.3.2.0. Next we consider the case $\mathrm{dist}(\Psi_P, \Theta_{P'}) = 0$ and $l(P) < l(P')$. Again we split Ψ_P into the union of the Γ_{Q_μ}. According to this splitting, we get

$$\begin{aligned} a^{w,c,q}_{P',P} &:= \vartheta_{P'}(a\psi_P) + \sum_{\mu=1}^{\mu_P} a^{w,c,q}_{P',P,\mu} \\ a^{w,c,q}_{P',P,\mu} &\sim \vartheta_{P'}\left(\int_{\Gamma_{Q_\mu}} k(\cdot,R,n_R)\psi_P(R)\,\mathrm{d}_R\Gamma\right). \end{aligned} \tag{4.16}$$

Further, we denote the boundary of Γ_{Q_μ} considered as a topological subset of Γ by $\partial\Gamma_{Q_\mu}$.

4.3.2.1. If $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) > 0$, then we even get $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) > c_\Gamma 2^{-l(P')}$. Setting $\mathrm{Dist} := \mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'})$ and $\Gamma^e_m := \kappa_m(T^e)$ (cf. Sect. 2.1) and supposing $\Gamma_{Q_\mu} \subseteq \Gamma_m$, we get

$$\begin{aligned} a^{w,c,q}_{P',P,\mu} &\sim \vartheta_{P'}\left(\int_{\Gamma_{Q_\mu}} k(\cdot,R,n_R)\psi_P(R)\,\mathrm{d}_R\Gamma\right) \\ &= \vartheta_{P'}\left(\int_{\Gamma^e_m} k(\cdot,R,n_R)\psi_P(R)\,\mathrm{d}_R\Gamma\right) \\ &\quad -\vartheta_{P'}\left(\int_{\Gamma^e_m\setminus\Gamma_{Q_\mu}} k(\cdot,R,n_R)\psi_P(R)\,\mathrm{d}_R\Gamma\right) \sim a^{w,c,q}_{P',\Gamma^e_m} - a^{w,c,q}_{P',\Gamma^e_m\setminus\Gamma_{Q_\mu}}. \end{aligned}$$

We shall define the approximation $a^{w,c,q}_{P',\Gamma^e_m}$ for the integral over Γ^e_m in Sect. 4.3.3. The approximation $a^{w,c,q}_{P',\Gamma^e_m\setminus\Gamma_{Q_\mu}}$ for the integral over $\Gamma^e_m \setminus \Gamma_{Q_\mu}$ can be computed analogously to the approximation $a^{w,c,q}_{P',P}$ in Sect. 4.2.2. More precisely, we set $\mathrm{Dist} := \mathrm{dist}(\Theta_{P'}, \partial\Gamma_{Q_\mu})$ and define $l(P,P') := l(P,\mu,P')$ by (4.10). The partition $\{\Gamma_Q : \; Q \in \square^\Gamma_{P',P,\mu}\}$ of $\Gamma^e_m \setminus \Gamma_{Q_\mu}$ is obtained in the two following steps.

i) We introduce the dyadic partition of $\Gamma^e_m \setminus \Gamma_{Q_\mu}$ into a minimal number of triangles from $\{\Gamma_{Q'},\ Q' \in \nabla^\Gamma_l,\ l = l(P)+1,\dots,\mathbf{m}L\}$ such that the distance of these triangles to $\Theta_{P'}$ is greater or equal to $2^{-l(Q')-1}$. Obviously, the number of all these triangles is less than a constant times L.

ii) Now we split each of the triangles from the previous step i) uniformly into the triangles of level $\tilde{l} = l(Q') + [l(P,\mu,P') + l_0 - (l(P)+1)]$.

Using this partition, applying the product rule of order $\mathbf{q} = 4$ (compare (4.7)) and employing a piecewise polynomial interpolation κ'_m of degree $\mathbf{m}_p := \mathbf{q}$, we arrive at

$$a^{w,c,q}_{P',\Gamma^e_m \setminus \Gamma_{Q_\mu}} \;:=\; \vartheta_{P'} \left(\sum_{Q \in \Box^\Gamma_{P',P,\mu}} \sum_v k(\cdot, Q'_v, n'_{Q'_v}) \mathcal{J}'_m(\tau_v) b^{w,c,q}_{P,Q,v} \right). \tag{4.17}$$

4.3.2.2. If $\operatorname{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) = 0$, then we have to split $\Theta_{P'}$ as we did in Sect. 4.3.1. Instead of the $a^{w,c,q}_{P',P,\mu}$ from Sect. 4.3.2.1 we have to determine the $a^{w,c,q}_{P',\lambda,P,\mu}$. However, these $a^{w,c,q}_{P',\lambda,P,\mu}$ can be computed similarly to the case $\operatorname{dist}(\Gamma_{Q_\mu}, P_\lambda) = 0$ in Sect. 4.3.1. More precisely, we introduce $l(P,P')$ just as in (4.10) but with Dist replaced by $2^{-l(P')}$. The partition $\{\Gamma_Q : Q \in \Box^\Gamma_{P',\lambda,P,\mu}\}$ of Γ_{Q_μ} is obtained in the following three steps.

i') We proceed from level $l = l(P)+1$ to level $l = l(P')$ and construct partitions of Γ_{Q_μ}. For $l = l(P)+1$, we simply take Γ_{Q_μ}. If level l is finished and level $l+1$ is considered, then we check whether the $\Gamma_{Q'}$ of the level l partition have a distance $\operatorname{dist}(\Gamma_{Q'}, \Theta_{P'})$ greater than $2^{-l(Q')-1}$. If yes, then we keep these triangles. If not, then we replace the $\Gamma_{Q'}$ by the four level $l+1$ subtriangles contained in $\Gamma_{Q'}$.

ii') We proceed from level $l = l(P')+1$ to at most $l = \mathbf{m}L$ and construct further partitions of Γ_{Q_μ}. The starting partition is taken from the last step. If level l is finished and level $l+1$ is considered, then we check whether the $\Gamma_{Q'}$ of the level l partition have a distance $\operatorname{dist}(\Gamma_{Q'}, P_\lambda)$ greater than $2^{-l(Q')-1}$. If yes, then we keep these triangles in our partition. If not, then we replace $\Gamma_{Q'}$ by the four level $l+1$ subtriangles contained in $\Gamma_{Q'}$. If there are level $\mathbf{m}L$ triangles in the last partition containing the point P_λ, then, for $\mathbf{r} = -1$, we throw these triangles away and, for the construction of a full partition of $\Gamma_{Q_\mu} \setminus \cup\Gamma^j_i$ in the case $\mathbf{r} = 0$, we replace the level $\mathbf{m}L$ triangles with distance to P' less than $2^{-\mathbf{m}L-1}$ by a few number of triangles contained in $\Gamma_{Q_\mu} \setminus \cup\Gamma^j_i$.

iii') Now we split each of the triangles $\Gamma_{Q'}$ from the previous step ii) uniformly into the triangles Γ_Q with $\Gamma_Q \subseteq \Gamma_{Q'}$ and $Q \in \Box^\Gamma_{\tilde{l}}$ such that $\tilde{l}$ is defined by (4.15) but with $\frac{1}{3-\mathbf{r}}$ replaced by $\frac{1}{4}$.

Using this partition and the formulae (4.12) and (4.14) based on the quadrature weights $b^{w,c,q}_{P,Q,v}$ of the product rule of order $\mathbf{q} = 4$ and on a piecewise polynomial interpolation κ'_m of degree $\mathbf{m}_p := \mathbf{q}$, we get the corresponding quadrature approximation.

4.3.3.0. We fix the index m and the point P_λ in the support $\Theta_{P'}$ of the test functional $\vartheta_{P'}$ with $\Theta_{P'} \subseteq \Gamma_m$. We have $P_\lambda := \kappa_m(\tau_\lambda)$ and consider a linear function $p(\kappa_m(\tau)) = \tilde{p}(\tau)$ defined on Γ^e_m which is either constant or equal to one of the two components of the vector function $\kappa_m(\tau) \mapsto \tau - \tau_\lambda$. To get the approximations

$$a^{w,c,q}_{P',\lambda,\Gamma^e_m,p} \;\sim\; \int_{\Gamma^e_m} k(P_\lambda, R, n_R) p(R) \,\mathrm{d}_R\Gamma$$

$$a^{w,c,q}_{P',\Gamma^e_m,p} \;:=\; \sum_\lambda c_\lambda a^{w,c,q}_{P',\lambda,\Gamma^e_m,p} \;\sim\; \vartheta_{P'} \left(\int_{\Gamma^e_m} k(\cdot, R, n_R) p(R) \,\mathrm{d}_R\Gamma \right) \tag{4.18}$$

and, as a linear combination of these, the values $a^{w,c,q}_{P',\Gamma^e_m}$, we distinguish two cases.

4.3.3.1. If $l(P') \geq \frac{\mathbf{m}}{\mathbf{m}_\vartheta} L = \frac{2-\mathbf{r}}{4-\mathbf{r}} L$, then we can choose $a^{w,c,q}_{P',\Gamma^e_m,p} := 0$. Indeed, the definition (4.18) of $a^{w,c,q}_{P',\Gamma^e_m,p}$ involves the functional $\vartheta_{P'}$ with $\mathbf{m}_\vartheta = 4-\mathbf{r}$ vanishing moments. In view of this fact $a^{w,c,q}_{P',\lambda,\Gamma^e_m,p}$ can be neglected for higher levels $l(P')$ (cf. the "second" compression in [40]).

4.3.3.2. If $l(P') < \frac{\mathbf{m}}{\mathbf{m}_\vartheta} L$, then we compute $a^{w,c,q}_{P',\lambda,\Gamma^e_m,p}$ by the composite product quadratures which we have applied before. The partition $\{\Gamma_Q : Q \in \square^\Gamma_{P',\lambda,\Gamma^e_m,p}\}$ of Γ^e_m is obtained in the following three steps.

i) We proceed from level $l = -3$ to level $l = l(P')$ and construct partitions of Γ^e_m. For $l = -3$, we simply take Γ^e_m. If level l is finished and level $l+1$ is considered, then we check whether the $\Gamma_{Q'}$ of the level l partition have a distance $\operatorname{dist}(\Gamma_{Q'}, \Theta_{P'})$ greater than $2^{-l(Q')-1}$. If yes, then we keep these triangles in our partition. If not, then we replace $\Gamma_{Q'}$ by the four level $l+1$ subtriangles contained in $\Gamma_{Q'}$.

ii) We proceed from level $l = l(P') + 1$ to at most $l = \mathbf{m}L$ and construct further partitions of Γ^e_m. The starting partition is taken from the last step. If level l is finished and level $l+1$ is considered, then we check whether the $\Gamma_{Q'}$ of the level l partition have a distance $\operatorname{dist}(\Gamma_{Q'}, P_\lambda)$ greater than $2^{-l(Q')-1}$. If yes, then we keep these triangles in our partition. If not, then we replace $\Gamma_{Q'}$ by the four level $l+1$ subtriangles contained in $\Gamma_{Q'}$. If there are level $\mathbf{m}L$ triangles in the last partition containing the point P_λ, then we throw these triangles away.

iii) Now we split each of the triangles $\Gamma_{Q'}$ from the previous step ii) uniformly into the triangles Γ_Q with $\Gamma_Q \subseteq \Gamma_{Q'}$ and $Q \in \square^\Gamma_{\tilde{l}}$, where

$$\tilde{l} \;:=\; l(Q') + \zeta L - \zeta' l(P') + \begin{cases} 0 & \text{if } \mathbf{r} = -1 \\ \left[\frac{1}{4-2\mathbf{r}}\, {}^2\!\log L\right] & \text{if } \mathbf{r} = 0. \end{cases} \tag{4.19}$$

and where $\zeta := 3/\mathbf{m}_\vartheta$ and $\zeta' := 1/\mathbf{m}$.

Using this partition, applying the product quadrature of order $\mathbf{q} = 2\mathbf{m}$ (compare (4.7)), and employing a piecewise polynomial interpolation κ'_m of degree $\mathbf{m}_p := \mathbf{q}$, we obtain

$$a^{w,c,q}_{P',\lambda,\Gamma^e_m,p} \;:=\; \sum_{Q \in \square^\Gamma_{P',\lambda,\Gamma^e_m}} \sum_{v} k(P_\lambda, Q'_v, n'_{Q'_v}) \mathcal{J}'_m(\tau_v) b^{w,c,q}_{p,Q,v}, \quad b^{w,c,q}_{p,Q,v} \;:=\; \int_{T_\tau} \tilde{\phi}_{Q,v}(\sigma)\tilde{p}(\sigma)\,\mathrm{d}\sigma.$$

5 Preliminary Results from the Analysis of the Compression

5.1 The Properties of the Three-Point Hierarchical Basis

Retain the notation of the basis from 3.1. From now on C stands for a generic constant the value of which varies from instance to instance. For two expressions E_1 and E_2, we write $E_1 \sim E_2$ if there is a constant independent of the parameters involved in E_1 and E_2 such that $E_1/C \leq E_2 \leq C\, E_1$. We infer the following two lemmata from [38].

Lemma 5.1 *i) For $-0.5 < s < 1.5$, the basis $\{\psi_P : P \in \cup_{L=0}^{\infty} \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any L and for any vector of real numbers $(\xi_P)_P$, we get*

$$\left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P \right\|_{H^s(\Gamma)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2}. \tag{5.1}$$

ii) For the Sobolev space orders $s \le t \le 2,\ s < 1.5$, the functions from Lin_L^Γ fulfill the approximation property (Jackson type theorem)

$$\inf_{u_L \in Lin_L^\Gamma} \|u - u_L\|_{H^s(\Gamma)} \le C 2^{-L(t-s)} \|u\|_{H^t(\Gamma)}. \tag{5.2}$$

iii) For the interpolation projection R_L defined in Sect. 2.5, for $u \in H^t(\Gamma)$, and for the Sobolev space orders $0 \le s \le t \le 2,\ s < 1.5,\ t > 1$, we get

$$\|u - R_L u\|_{H^s(\Gamma)} \le C 2^{-L(t-s)} \|u\|_{\oplus_{m=1}^{m_\Gamma} H^t(\Gamma_m)}. \tag{5.3}$$

iv) For the $L^2(\Gamma)$ orthogonal projection P_L and for the Sobolev space orders $-2 \le s \le t \le 2,\ s < 1.5,\ t > -1.5$, we get

$$\|u - P_L u\|_{H^s(\Gamma)} \le C 2^{-L(t-s)} \|u\|_{H^t(\Gamma)}. \tag{5.4}$$

v) For the Sobolev space orders $s \le t < 1.5$, the functions u_L from Lin_L^Γ fulfill the inverse property (Bernstein inequality)

$$\|u_L\|_{H^t(\Gamma)} \le C 2^{L(t-s)} \|u_L\|_{H^s(\Gamma)}. \tag{5.5}$$

Lemma 5.2 *Suppose the continuous function u belongs to $\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)$ for an s with $-0.5 < s \le 2$ and suppose $\sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P$ is the representation of the orthogonal projection $P_L u$. Then*

$$\sqrt{\sum_{P \in \nabla_l^\Gamma} 2^{2l(s-1)} |\xi_P|^2} \le C \, \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)}, \tag{5.6}$$

$$\sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2} \le C \, \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)} \cdot \begin{cases} 1 & \text{if } -0.5 < s < 1.5 \\ \sqrt{L} & \text{if } 1.5 \le s \le 2. \end{cases} \tag{5.7}$$

5.2 The Properties of the Wavelet Basis in the Test Space

The properties of the basis of test wavelets introduced in Sect. 3.2 can be described using the predual basis. If the number of vanishing moments $\mathbf{m}_\vartheta$ is equal to two, then we simply define the classical hierarchical basis by $\chi_P := \varphi_P^{l+1}$ for $P \in \nabla_l^\Gamma$ and observe

$$\langle \vartheta_P, \chi_{P'} \rangle := \vartheta_P(\chi_{P'}) = \delta_{P,P'} \tag{5.8}$$

as well as span$\{\chi_P : \ P \in \triangle_L^\Gamma\} = Lin_L^\Gamma$. The interpolation projection can be represented as

$$R_L u \;=\; \sum_{P\in\triangle_L^\Gamma} u(P)\varphi_P^L \;=\; \sum_{P\in\triangle_L^\Gamma} \langle \vartheta_P, u\rangle\, \chi_P. \tag{5.9}$$

If $\mathbf{m}_\vartheta \geq 2$, then we introduce the space X_L^Γ of piecewise polynomials as the set of all $f \in C(\Gamma)$ such that $f|_{\Gamma_Q} \circ \kappa_m$ is a polynomial of degree less than $\mathbf{m}_\vartheta$ for any triangle $\Gamma_Q \subseteq \Gamma_m$ of level L, i.e. for any Γ_Q with $Q \in \square_l^\Gamma \cap \Gamma_m$. Retaining the definition of l_ϑ from Sect. 3.2, we can define the spaces X_l^Γ of piecewise polynomials of level $l \geq l_\vartheta - 1$ in the same manner. We get the hierarchy $X_{l_\vartheta-1}^\Gamma \subset X_{l_\vartheta}^\Gamma \subset \ldots \subset X_L^\Gamma$ and we can define the hierarchical basis $\{\chi_P\}$ as follows. If $P \in \nabla_l^\Gamma,\ l \geq l_\vartheta - 1$ and $R, P \in \Gamma_Q$ with $Q \in \square_{l+1}^\Gamma$ and $P = \kappa_m(\tau_i)$ (cf. Sect. 3.2), then we set $\chi_P(R) := l_{Q,i}(R)$. For $P \in \nabla_l^\Gamma,\ l < l_\vartheta - 1$, we set $\chi_P := \varphi_P^{l_\vartheta-1}$. With the so defined basis, we conclude (5.8) as well as span$\{\chi_P : \ P \in \triangle_L^\Gamma\} = X_L^\Gamma$. Again, the interpolation projection R_L can be represented by (5.9). If $\mathbf{m}_\vartheta = 2$, then $X_L^\Gamma = Lin_L^\Gamma$ and the functions χ_P coincide with $\varphi_P^{l(P)+1}$. The following properties are straightforward generalizations of well-known results for the classical hierarchical basis.

Lemma 5.3 *i) For $1 < s < 1.5$, the basis $\{\chi_P : \ P \in \cup_{L=0}^\infty \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any L and for any vector of real numbers $(\xi_P)_P$, we get*

$$\Big\| \sum_{P\in\triangle_L^\Gamma} \xi_P \chi_P \Big\|_{H^s(\Gamma)} \;\sim\; \sqrt{\sum_{P\in\triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2}. \tag{5.10}$$

ii) The approximation and inverse properties for the space predual to the test functionals are the same as those formulated in Lemma 5.1 ii)-v). The upper bound 2 and the lower bound -2, however, can be replaced by $\mathbf{m}_\vartheta$ and $-\mathbf{m}_\vartheta$, respectively.

iii) The finite element basis $\varphi_P^L,\ P \in \triangle_L^\Gamma$ satisfies the discrete norm equivalence

$$\Big\| \sum_{\tilde P\in\triangle_L^\Gamma} \xi_{\tilde P} \varphi_{\tilde P}^L \Big\|_{L^2(\Gamma)} \;\sim\; \frac{1}{2^L}\sqrt{\sum_{\tilde P\in\triangle_L^\Gamma} |\xi_{\tilde P}|^2}.$$

In particular, we get

$$\Big\| \sum_{P\in\triangle_L^\Gamma} \xi_P \chi_P \Big\|_{L^2(\Gamma)} \;\sim\; \frac{1}{2^L}\sqrt{\sum_{\tilde P\in\triangle_L^\Gamma} \big| \sum_{P\in\triangle_L^\Gamma} \xi_P \chi_P(\tilde P)\big|^2}. \tag{5.11}$$

5.3 General Error Estimates for the Numerical Solution and Preconditioning

In this subsection we recall well-known error estimates for stable numerical methods. We formulate results on the stability and derive necessary conditions which ensure that the numerical methods, perturbed by compression and by boundary and quadrature approximation, admit the same asymptotic orders of convergence as the unperturbed methods. Moreover, we give necessary conditions which ensure the existence of diagonal preconditioners for the matrix $A^{w,c,q}$ of the compressed and approximated collocation method.

The collocation method for the equation $Au = v$ defines an approximate solution $u_L \in Lin_L^\Gamma$ by $R_L A u_L = R_L v$ (cf. Sect. 2.5). This method is called stable in the space $H^s(\Gamma)$ if the approximate operators $R_L A : \ Lin_L^\Gamma \longrightarrow Lin_L^\Gamma$ are invertible for sufficiently large L and if their inverses are bounded, i.e.,

$$\left\| \left(R_L A|_{Lin_L^\Gamma} \right)^{-1} w_L \right\|_{H^{s+\mathbf{r}}(\Gamma)} \ \leq \ C \, \|w_L\|_{H^s(\Gamma)}, \quad w_L \in Lin_L^\Gamma.$$

We suppose that the collocation method is stable for $s = 0$. Additionally, if $\mathbf{r} = -1$ or if the algorithm (3.9) is applied to an operator A of order $\mathbf{r} = 0$, then we suppose stability also for $s = 1.1$ (or for an arbitrary s with $1 < s < 1.5$ instead of 1.1). Note that stability is well known for second kind integral operators including compact integral operators. In particular this is true for double layer operators over smooth boundaries (cf. e.g. [2]). For first kind operators and operators involving strongly singular integral operators, the question of stability is not solved yet. A first step toward the solution is done in [31, 32, 8, 11]. Note that, since our trial space Lin_L^Γ is generated by two scaling functions, the stability is needed for a multiwavelet space (cf. the univariate multiwavelet paper [33]). Though a rigorous proof of stability is missing engineers frequently use collocation methods without observing instabilities.

To simplify the notation, let us denote the operator $R_L A|_{Lin_L^\Gamma}$ by A_L, i.e., by the same symbol as for its matrix with respect to the basis $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ (cf. Sect. 2.5). Similarly, we denote by A_L^c and $A_L^{c,q}$ the operators in Lin_L^Γ the matrix of which with respect to $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ is A_L^c and $A_L^{c,q}$, respectively (cf. (3.8)). Using the L^2 orthogonal projection P_L, we represent the error $u - u_L$ of the fully discretized and compressed method $A_L^{c,q} u_L = R_L v$ as

$$\begin{aligned} u - u_L &= u - P_L u - (A_L^{c,q})^{-1} \left\{ R_L A u - A_L^{c,q} P_L u \right\} \\ &= u - P_L u - (A_L^{c,q})^{-1} \left\{ [A_L - A_L^{c,q}] P_L u + A(I - P_L)u - (I - R_L)A(I - P_L)u \right\}. \end{aligned}$$

We apply the boundedness assumption on A (cf. Sect. 2.2), assume the stability of $A_L^{c,q}$ for Sobolev index $s = 0$, and use Lemma 5.1 to get

$$\begin{aligned} \|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} &\leq \|u - P_L u\|_{H^{\mathbf{r}}(\Gamma)} + C \Big\{ \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} + \\ &\qquad \|(I - P_L)u\|_{H^{\mathbf{r}}(\Gamma)} + 2^{-1.1L} \, \|A(I - P_L)u\|_{H^{1.1}(\Gamma)} \Big\} \\ &\leq C 2^{-(2-\mathbf{r})L} \, \|u\|_{H^2(\Gamma)} + C \, \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} . \end{aligned}$$

In other words, to ensure the optimal convergence order $\mathbf{m} = 2 - \mathbf{r}$, we need the estimate

$$\| \, [A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} \ \leq \ C_u 2^{-(s-\mathbf{r})L} \tag{5.12}$$

for $s = 2$ and the stability of $A_L^{c,q}$. Since A_L is stable by assumption and since $A_L^{c,q} = A_L\{I + A_L^{-1}[A_L^{c,q} - A_L]\}$, for the stability of $A_L^{c,q}$, it will be sufficient to require

$$\|A_L - A_L^{c,q}\|_{H^0(\Gamma)\leftarrow H^{\mathbf{r}}(\Gamma)} \leq \frac{1}{2}\left[\sup_{L'=L_0,L_0+1,\dots} \|A_{L'}^{-1}\|_{H^{\mathbf{r}}(\Gamma)\leftarrow H^0(\Gamma)}\right]^{-1}.$$

The last condition is a consequence of (5.12) with $s = 0$ if we can show that $C_u \leq C\|u\|_{L^2}$ for a constant C which can be made smaller than any prescribed positive threshold. Moreover, due to the inverse property, it suffices to show (5.12) with $s = 1.1$ and a small constant $C_u \leq C\|u\|_{H^{1.1}}$. The usual compression estimates prove the error estimate in (5.12) but with the difference $A_L - A_L^{c,q}$ replaced by $A_L - A_L^c$. We refer the reader to [12, 29, 40, 35] for the details. In the present paper it will be our task to prove the estimates (5.12) for $s = 2$ and for $s = 1.1$ with $A_L - A_L^{c,q}$ replaced by $A_L^c - A_L^{c,q}$.

The issue of wavelet preconditioners has been addressed by many authors (cf. e.g. [10, 12, 24, 43]) and we will follow the same ideas. In the case $\mathbf{r} = 0$ the stability of $A_L^{c,q}$ implies that the matrix $A_L^{c,q}$ has a condition number which is already uniformly bounded with respect to L. Thus, for the algorithm (3.10), no preconditioning is needed, and we can restrict our consideration to algorithm (3.9). Unfortunately, the wavelet transform $\mathcal{T}_T^{-1}$ (cf. Sect. 3.3) does not have a uniformly bounded condition number with respect to Euclidean matrix norm. Therefore, preconditioning is needed even for $\mathbf{r} = 0$, and the preconditioner is to be derived from the stability for a different Sobolev index. We choose e.g. $s = 1.1$.

Let us consider an operator A of order $\mathbf{r} = 0, -1$ and suppose the stability of A_L in the Sobolev space $H^{1.1}(\Gamma)$. If we could prove

$$\|A_L - A_L^{c,q}\|_{H^{1.1}(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} \leq \frac{1}{2}\left[\sup_{L'=L_0,L_0+1,\dots} \|A_{L'}^{-1}\|_{H^{1.1+\mathbf{r}}(\Gamma)\leftarrow H^{1.1}(\Gamma)}\right]^{-1}, \quad (5.13)$$

then $A_L^{c,q}$ is stable in $H^{1.1}(\Gamma)$, too. From Sects. 3.1 and 5.2, we recall that $A_L^{w,c,q}$ is the matrix of the operator $A_L^{c,q}$ with respect to the bases $\{\psi_P :\ P \in \triangle_L^\Gamma\}$ and $\{\chi_P :\ P \in \triangle_L^\Gamma\}$. Under assumption (5.13), the assertions i) of the Lemmata 5.3 and 5.1 imply that the matrices

$$\left(\delta_{P,P'}2^{l(P')(1.1-1)}\right)_{P,P'\in\triangle_L^\Gamma} A_L^{w,c,q} \left(\delta_{P,P'}2^{-l(P)(\mathbf{r}+1.1-1)}\right)_{P,P'\in\triangle_L^\Gamma} \quad (5.14)$$

have condition numbers which are uniformly bounded with respect to L, i.e. the matrix $A_L^{w,c,q}$ admits a diagonal preconditioning. The boundedness of the condition number ensures the fast convergence of the iterative solver in the wavelet algorithm (3.9). In other words, for the fast iterative solution of the linear systems $A_L^{w,c,q}\beta = \gamma$ (cf. part iv) of (3.9)) using preconditioning, we only have to prove (5.13). This is well known for the difference $A_L - A_L^{c,q}$ replaced by $A_L - A_L^c$ (cf. [12, 29, 40, 35]). The estimate (5.13) with $A_L - A_L^{c,q}$ replaced by $A_L^c - A_L^{c,q}$, however, follows from (5.12) with $s = 1.1 + \mathbf{r}$ and the inverse property v) of Lemma 5.1. All together, we have

Remark 5.1 *For almost optimal rates of convergence, for stability, and for preconditioning, we have to prove*

$$\|[A_L^c - A_L^{c,q}]P_L u\|_{H^0(\Gamma)} \leq 2^{-\mathbf{q}l_0} \begin{cases} C_u 2^{-(2-\mathbf{r})L} & \textit{if } u \in H^2(\Gamma) \\ C\|u\|_{H^{1.1}} 2^{-(1.1-\mathbf{r})L} & \textit{if } u \in H^{1.1}(\Gamma), \end{cases} \quad (5.15)$$

where $\mathbf{q} > 0$ *and where* l_0 *is chosen such that* $C2^{-\mathbf{q}l_0}$ *is sufficiently small.*

To derive an estimate like (5.15), we shall use the following well-known Schur lemma.

Lemma 5.4 *Denote the entries of the compressed matrix of quadrature errors* $[A_L^c - A_L^{c,q}]$ *with respect to the wavelet bases* $\{\chi_{P'}\}$ *and* $\{\psi_P\}$ *by* $a_{P',P} := a_{P',P}^{w,c} - a_{P',P}^{w,c,q}$. *Suppose* x *is a fixed real parameter which can be arbitrary. Usually* x *is equal to zero if it is not given explicitly. Then the left-hand side of (5.15) can be estimated as*

$$\left\|[A_L^c - A_L^{c,q}]P_L u\right\|_{L^2(\Gamma)}^2 \leq \begin{cases} C_u \,\Sigma_1\, \Sigma_2 & \textit{if } s = 2 \\ C\,\|u\|_{H^{1.1}(\Gamma)} \Sigma_1\, \Sigma_2 & \textit{if } s = 1.1, \end{cases} \quad (5.16)$$

$$\Sigma_1 := \sup_{P' \in \triangle_L^\Gamma} 2^{-xl(P')} \sum_{P \in \triangle_L^\Gamma} 2^{[x-s]l(P)} |a_{P',P}|, \quad (5.17)$$

$$\Sigma_2 := \sum_{l=-1}^{L-1} 2^{[2-s-x]l} \sup_{P \in \nabla_l^\Gamma} \sum_{P' \in \triangle_L^\Gamma} 2^{[x-2]l(P')} |a_{P',P}|. \quad (5.18)$$

Proof. In view of (5.11), we get, for $P_L u = \sum \xi_P \psi_P$,

$$\begin{aligned} \left\|[A_L^c - A_L^{c,q}]P_L u\right\|_{L^2(\Gamma)}^2 &= \left\| \sum_{P',P \in \triangle_L^\Gamma} a_{P',P} \xi_P \chi_{P'} \right\|_{L^2(\Gamma)}^2 \\ &\leq C\,2^{-2L} \sum_{\tilde{P} \in \triangle_L^\Gamma} \left| \sum_{P' \in \triangle_L^\Gamma} \sum_{P \in \triangle_L^\Gamma} a_{P',P} \xi_P \chi_{P'}(\tilde{P}) \right|^2 . \end{aligned}$$

Clearly, the function values $\chi_{P'}(\tilde{P})$ are non-negative and less than one. We apply the Cauchy-Schwarz inequality and some easy calculations to arrive at

$$\begin{aligned} \left\|[A_L^c - A_L^{c,q}]P_L u\right\|_{L^2(\Gamma)}^2 &\leq C\,2^{-2L} \sum_{\tilde{P} \in \triangle_L^\Gamma} \left| \sum_{P' \in \triangle_L^\Gamma} \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[x-s]l(P)} \right. \\ &\qquad \left. \times \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[s-x]l(P)} |\xi_P|^2 \, |\chi_{P'}|(\tilde{P})|^2 \right| \\ &\leq C\,2^{-2L} \Sigma_1 \sum_{\tilde{P} \in \triangle_L^\Gamma} \sum_{P' \in \triangle_L^\Gamma} 2^{xl(P')} \sum_{P \in \triangle_L^\Gamma} |a_{P',P}| 2^{[s-x]l(P)} |\xi_P|^2 |\chi_{P'}(\tilde{P})|^2. \end{aligned}$$

Now we observe that, for a fixed P', the number of $\tilde{P} \in \triangle_L^\Gamma$ such that $\chi_{P'}(\tilde{P}) > 0$ is less than $C2^{2[L-l(P')]}$. Using this as well as (5.6) and (5.7), we continue

$$\begin{aligned}
\left\| [A_L^c - A_L^{c,q}] P_L u \right\|^2_{L^2(\Gamma)} &\leq C\,\Sigma_1 \sum_{P\in\triangle_L^\Gamma} \sum_{P'\in\triangle_L^\Gamma} 2^{[x-2]l(P')} |a_{P',P}| 2^{[s-x]l(P)} |\xi_P|^2, \\
&\leq C\,\Sigma_1 \sum_{l=-1}^{L-1} 2^{[2-s-x]l} \sup_{P\in\nabla_l^\Gamma} \left[\sum_{P'\in\triangle_L^\Gamma} 2^{[x-2]l(P')} |a_{P',P}| \right] \\
&\qquad \times \sum_{P\in\nabla_l^\Gamma} 2^{2(s-1)l(P)} |\xi_P|^2 \;\leq\; C_u\,\Sigma_1\,\Sigma_2 .
\end{aligned}$$

■

6 The Estimation of the Errors due to the Approximate Parametrization and due to the Quadrature

6.1 The Far Field Estimate

In this subsection we suppose that the near field integrations are performed exactly and derive the convergence estimates for the far field case. The error estimate for the near field will be considered in Sects. 6.2 and 6.3, respectively. In view of Remark 5.1, it remains to prove

Lemma 6.1 *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ (cf. Sect. 3.5). If $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of the far field, i.e. of Sect. 4.1, then we get the estimates (5.15).*

Proof. **i)** It remains to estimate Σ_1 and Σ_2 (cf. Lemma 5.4). For the approximate parametrization and for the quadrature, we shall prove the error estimate

$$\begin{aligned}
|a_{P',P}| &= |a^{w,c}_{P',P} - a^{w,c,q}_{P',P}| \;\leq\; a^1_{P',P} + a^2_{P',P}\,, \\
a^1_{P',P} &:= C2^{-\mathbf{q}l_0} 2^{-2l(P)} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{q}l(P,P')} \mathrm{dist}\,(\Theta_{P'},\Psi_P)^{-\mathbf{r}-2-\mathbf{q}-\mathbf{m}_\vartheta}\,, \\
a^2_{P',P} &:= C2^{-2l(P)} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{m}L} \mathrm{dist}\,(\Theta_{P'},\Psi_P)^{-\mathbf{r}-2-\mathbf{m}_\vartheta}\,.
\end{aligned} \tag{6.1}$$

In accordance with the splitting into these two terms, we get two estimates of the form (5.16) which we denote by $C_u\,\Sigma_1^1\,\Sigma_2^1$ and $C_u\,\Sigma_1^2\,\Sigma_2^2$, respectively. Furthermore, we introduce the numbers

$$\begin{aligned}
\mathrm{dist} &:= \mathrm{dist}\,(\Theta_{P'},\Psi_P)\,, \\
\mathrm{M}_0 &:= \max\left\{2^{-l(P)}, 2^{-l(P')}\right\}, \\
\mathrm{M}_1 &:= \max\left\{2^{-l(P)}, 2^{-l(P')}, d2^{0.6\,L-0.7\,l(P)-0.9\,l(P')}\right\}, \\
\mathrm{M}_2 &:= \max\left\{2^{-l(P)}, 2^{-l(P')}, d2^{L-0.7\,l(P)-1.3\,l(P')}\right\}
\end{aligned} \tag{6.2}$$

(cf. the definition of the far field in Sect. 4.1 and the formulae (3.11) and (3.12)). Substituting the estimate $a^1_{P',P}$ and $2^{l(P,P')} \geq 2^{0.9\,L-l(P')}\mathrm{dist}^{-1.1}$ (cf. (4.1)) into the definition of Σ^1_1, we get

$$\begin{aligned}
\Sigma^1_1 &\leq C \sup_{P'\in\triangle^\Gamma_L} \sum_{l=-1}^{L-1} \sum_{\substack{P\in\nabla^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_2}} 2^{-sl}2^{-\mathbf{q}l_0}2^{-2l}2^{-\mathbf{m}_\vartheta l(P')}2^{-\mathbf{q}l(P,P')}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta-\mathbf{q}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L} \sup_{P'\in\triangle^\Gamma_L} 2^{(\mathbf{q}-\mathbf{m}_\vartheta)l(P')} \sum_{l=-1}^{L-1} 2^{-sl}\,2^{-2l} \sum_{\substack{P\in\nabla^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_2}} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta+0.1\,\mathbf{q}}.
\end{aligned}$$

Using the estimate

$$\begin{aligned}
2^{-2l} \sum_{P\in\nabla^\Gamma_l:\,\mathrm{dist}>\mathrm{M}_0} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta+0.1\,\mathbf{q}} &\leq C\int_{\{P\in\Gamma:\,|P'-P|>\mathrm{M}_0\}} \frac{\mathrm{d}_P\Gamma}{|P'-P|^{\mathbf{r}+2+\mathbf{m}_\vartheta-0.1\,\mathbf{q}}} \\
&\leq C\,\mathrm{M}_0{}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.1\,\mathbf{q}},
\end{aligned} \tag{6.3}$$

we continue

$$\begin{aligned}
\Sigma^1_1 \leq C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L} \sup_{-1\leq l(P')\leq L-1} 2^{(\mathbf{q}-\mathbf{m}_\vartheta)l(P')} \Bigg\{ &\sum_{l=-1}^{l(P')-1} 2^{l[-s+\mathbf{r}+\mathbf{m}_\vartheta-0.1\,\mathbf{q}]} \\
&+ 2^{l(P')[\mathbf{r}+\mathbf{m}_\vartheta-0.1\,\mathbf{q}]} \sum_{l=l(P')}^{L-1} 2^{-sl} \Bigg\} \leq C2^{-\mathbf{q}l_0}2^{-[s-\mathbf{r}]L}.
\end{aligned}$$

On the other hand, substituting the estimate $a^1_{P',P}$ and $2^{l(P,P')} \geq 2^{0.9\,L-l(P')}\mathrm{dist}^{-1.1}$ into the definition of Σ^1_2, we get

$$\begin{aligned}
\Sigma^1_2 &\leq C\sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla^\Gamma_l} \sum_{l'=-1}^{L-1} 2^{-2l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_2}} 2^{-\mathbf{q}l_0}2^{-2l}2^{-\mathbf{m}_\vartheta l'}2^{-\mathbf{q}l(P,P')}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta-\mathbf{q}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L} \sum_{l=-1}^{L-1} 2^{-sl} \sup_{P\in\triangle^\Gamma_l} \sum_{l'=-1}^{L-1} 2^{[\mathbf{q}-\mathbf{m}_\vartheta]l'}2^{-2l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_2}} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta+0.1\,\mathbf{q}}.
\end{aligned}$$

Using the estimate (6.3), we continue

$$\begin{aligned}
\Sigma^1_2 &\leq C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L} \sum_{l=-1}^{L-1} 2^{-sl} \left[\sum_{l'=-1}^{l} 2^{[\mathbf{r}+0.9\,\mathbf{q}]l'} + 2^{[\mathbf{r}+\mathbf{m}_\vartheta-0.1\,\mathbf{q}]l} \sum_{l'=l}^{L-1} 2^{[\mathbf{q}-\mathbf{m}_\vartheta]l'} \right] \\
&\leq C2^{-\mathbf{q}l_0}2^{-[s-\mathbf{r}]L}.
\end{aligned} \tag{6.4}$$

Next we turn to the estimates of Σ_1^2 and Σ_1^2. Analogously to the treatment of Σ_1^1 and Σ_1^1, we arrive at

$$\begin{aligned}
\Sigma_1^2 \;&\leq\; C \sup_{P'\in\triangle_L^\Gamma} \sum_{l=-1}^{L-1} \sum_{\substack{P\in\nabla_l^\Gamma:\\ \mathrm{M}_0\leq \mathrm{dist}\leq \mathrm{M}_2}} 2^{-sl}2^{-2l}2^{-\mathbf{m}_\vartheta l(P')}2^{-\mathbf{m}L}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta}\\
&\leq\; C2^{-\mathbf{m}L} \sup_{P'\in\triangle_L^\Gamma} 2^{-\mathbf{m}_\vartheta l(P')} \sum_{l=-1}^{L-1} 2^{-sl}\, 2^{-2l} \sum_{\substack{P\in\nabla_l^\Gamma:\\ \mathrm{M}_0\leq \mathrm{dist}\leq \mathrm{M}_2}} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta}.
\end{aligned}$$

Using an estimate like (6.3), we continue

$$\Sigma_1^2 \;\leq\; C2^{-\mathbf{m}L} \sup_{-1\leq l(P')\leq L-1} 2^{-\mathbf{m}_\vartheta l(P')} \left\{ \sum_{l=-1}^{l(P')-1} 2^{[4-s]l} + 2^{4l(P')} \sum_{l=l(P')}^{L-1} 2^{-sl} \right\} \;\leq\; C2^{-\mathbf{m}L}.$$

On the other hand, substituting the estimate $2^{-2l(P)}2^{-\mathbf{m}_\vartheta l(P')}2^{-\mathbf{m}L}\mathrm{dist}\,(\Theta_{P'},\Psi_P)^{-\mathbf{r}-2-\mathbf{m}_\vartheta}$ for $a^2_{P',P}$ into the definition of Σ_2^2, we get

$$\begin{aligned}
\Sigma_2^2 \;&\leq\; C \sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=-1}^{L-1} 2^{-2l'} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ \mathrm{M}_0\leq \mathrm{dist}\leq \mathrm{M}_2}} 2^{-2l}2^{-\mathbf{m}_\vartheta l'}2^{-\mathbf{m}L}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta}\\
&\leq\; C2^{-\mathbf{m}L} \sum_{l=-1}^{L-1} 2^{-sl} \sup_{P\in\triangle_l^\Gamma} \sum_{l'=-1}^{L-1} 2^{-\mathbf{m}_\vartheta l'}\, 2^{-2l'} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ \mathrm{M}_0\leq \mathrm{dist}\leq \mathrm{M}_2}} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta}\\
&\leq\; C2^{-\mathbf{m}L} \sum_{l=-1}^{L-1} 2^{-sl} \left[\sum_{l'=-1}^{l} 2^{\mathbf{r}l'} + 2^{[\mathbf{r}+\mathbf{m}_\vartheta]l} \sum_{l'=l}^{L-1} 2^{-\mathbf{m}_\vartheta l'} \right] \;\leq\; C2^{-\mathbf{m}L}.
\end{aligned}$$

ii) Let us prove (6.1). The first bound $a^1_{P,P'}$ is the bound for the error of the quadrature applied to the integral in $a^{w,c}_{P',P}$, where the parametrization is already replaced by the piecewise polynomial interpolation. Indeed, by standard estimates of $\mathbf{q}$-th order composite rules, the quadrature error is less than a constant times the measure $C2^{-2l(P)}$ of the domain of integration times the $\mathbf{q}$-th power of the step size of quadrature $2^{-l(Q)} \sim 2^{-l_0}2^{-l(P,P')}$ times the supremum of the $\mathbf{q}$-th order derivative of the integrand function. Due to the vanishing moments the test functional $\vartheta_{P'}$ acts like a difference formula of order $\mathbf{m}_\vartheta$ with improper scaling. Therefore, the $\mathbf{q}$-th order derivative of the integrand function can be estimated by the product of $C2^{-\mathbf{m}_\vartheta l(P')}$ and the $[\mathbf{m}_\vartheta+\mathbf{q}]$-th order derivative of the kernel function. Thanks to (2.5) the last factor is less than $C\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_\vartheta-\mathbf{q}}$. The replacement of the parametrization does not cause any problem since the supremum of the derivatives to the piecewise polynomial interpolations can be estimated by the supremum of the derivatives to the original parametrization mapping, i.e. it is bounded.

The second bound $a^2_{P',P}$ in (6.1) is the estimate for the error due to the replacement of the parametrization by the piecewise interpolation. To estimate the corresponding error it is sufficient to use the approximation order $\mathbf{m}+1$ instead of the actual order $\mathbf{m}_p+1$ of the piecewise polynomial interpolation of degree less than $\mathbf{m}_p$. As mentioned above, the test functional $\vartheta_{P'}$ can be considered to be a scaled version of a difference formula. Clearly, we get $|\kappa_m(\sigma)-\kappa'_m(\sigma)| \le C\,2^{-(\mathbf{m}+1)l(Q)}$ for $\sigma\in T_{\tau'}=\kappa_m^{-1}(\Gamma_{Q'})$ with $Q'\in\square^\Gamma_L$, i.e. $l(Q')=L$. Moreover, we obtain $|\nabla_\sigma\kappa_m(\sigma)-\nabla_\sigma\kappa'_m(\sigma)| \le C\,2^{-\mathbf{m}l(Q)}$ if ∇_σ is the gradient with respect to σ. From the smoothness assumptions on κ_m in Sect. 2.1 and on the integral kernel in Sect. 2.2, we conclude

$$|\mathcal{J}_m(\sigma)-\mathcal{J}'_m(\sigma)| \le C2^{-\mathbf{m}L}, \quad |\mathcal{J}_m(\sigma)|\le C, \quad |\mathcal{J}'_m(\sigma)|\le C,$$

$$\begin{aligned}
\left|k\left(\vartheta_{P'},\kappa_m(\sigma),n_{\kappa_m(\sigma)}\right)-k\left(\vartheta_{P'},\kappa'_m(\sigma),n'_{\kappa'_m(\sigma)}\right)\right| &\le C\frac{2^{-(\mathbf{m}+1)L}\;2^{-\mathbf{m}_\vartheta l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_\vartheta+1}}\;, \\
\left|k\left(\vartheta_{P'},\kappa_m(\sigma),n_{\kappa_m(\sigma)}\right)\right| &\le C\frac{2^{-\mathbf{m}L}\;2^{-\mathbf{m}_\vartheta l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_\vartheta}}\;, \\
\left|k\left(\vartheta_{P'},\kappa'_m(\sigma),n'_{\kappa'_m(\sigma)}\right)\right| &\le C\frac{2^{-\mathbf{m}L}\;2^{-\mathbf{m}_\vartheta l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_\vartheta}}\;,
\end{aligned} \tag{6.5}$$

where we have used the notation $\mathrm{dist}:=\mathrm{dist}(\Theta_{P'},\Psi_P)$ and the estimate $\mathrm{dist}>2^{-L}$ (cf. the definition of the far field in Sect. 4.1). Hence, we arrive at

$$\begin{aligned}
&\left|k\left(\vartheta_{P'},\kappa_m(\sigma),n_{\kappa_m(\sigma)}\right)\mathcal{J}_m(\sigma)\phi_{\tau,\iota}(\sigma)-k\left(\vartheta_{P'},\kappa'_m(\sigma),n'_{\kappa'_m(\sigma)}\right)\mathcal{J}'_m(\sigma)\phi_{\tau,\iota}(\sigma)\right| \\
&\qquad\qquad \le\; C2^{-\mathbf{m}L}\;2^{-\mathbf{m}_\vartheta l(P')}\mathrm{dist}^{-2-\mathbf{r}-\mathbf{m}_\vartheta}\;,
\end{aligned}$$

and the integral over T_τ of this difference is less than $a^2_{P',P}$ in (6.1). ∎

Lemma 6.2 *The number of necessary arithmetic operations for setting up the far field part of the stiffness matrix $A_L^{w,c,q}$, including the sparsity pattern $\mathcal{P}$, is less than $C2^{2l_0}L2^{2L}$.*

Proof. Clearly, the number of all arithmetic operations is bounded by a constant multiple of the number of all quadrature knots. Thus we count the number $\mathcal{N}$ of quadrature knots. For a fixed test functional $\vartheta_{P'}$ and for a fixed trial function ψ_P, the number of knots is less than $[2^{-l(P)}/2^{-l(Q)}]^2 \sim C2^{2l_0}2^{2[l(P,P')-l(P)]}$. In view of (4.1), the term $2^{l(P,P')}$ can be majorized by $2^{l(P)+1}+2^{0.9\,L-l(P')}\mathrm{dist}^{-1.1}$. By $\odot^\Gamma_l$ we denote the set of $P\in\nabla^\Gamma_l$ such that Ψ_P is not contained in the interior of a single patch Γ_m. Moreover, we set $\oslash^\Gamma_l:=\nabla^\Gamma_l\setminus\odot^\Gamma_l$. Summing up over all ψ_P and $\vartheta_{P'}$ and using the notation of the last proof, we arrive at

$$\mathcal{N} \;\le\; \sum_{P'\in\triangle^\Gamma_L}\sum_{l=-1}^{L-1}\left\{\sum_{\substack{P\in\oslash^\Gamma_l:\\ \mathrm{M}_0\le\mathrm{dist}\le\mathrm{M}_1}} C2^{2l_0}2^{2[l(P,P')-l(P)]} + \sum_{\substack{P\in\odot^\Gamma_l:\\ \mathrm{M}_0\le\mathrm{dist}\le\mathrm{M}_2}} C2^{2l_0}2^{2[l(P,P')-l(P)]}\right\}$$

$$
\begin{aligned}
&\leq \; C2^{2l_0}\sum_{l'=-1}^{L-1} 2^{2l'} \sup_{P'\in\nabla^\Gamma_{l'}} \sum_{l=-1}^{L-1} \left\{ \sum_{\substack{P\in\oslash^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_1}} 2^{-2l}\left[2^{2l}+\frac{2^{1.8L-2l'}}{\mathrm{dist}^{2.2}}\right]\right.\\
&\qquad\qquad \left. + \sum_{\substack{P\in\odot^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}\leq\mathrm{M}_2}} 2^{-2l}\left[2^{2l}+\frac{2^{1.8L-2l'}}{\mathrm{dist}^{2.2}}\right]\right\}\\
&\leq \; C2^{2l_0}\sum_{l'=-1}^{L-1} 2^{2l'} \sup_{P'\in\nabla^\Gamma_{l'}} \sum_{l=-1}^{L-1} \left\{ \sum_{\substack{P\in\oslash^\Gamma_l:\\ \mathrm{dist}\leq\mathrm{M}_1}} 1 + \sum_{\substack{P\in\odot^\Gamma_l:\\ \mathrm{dist}\leq\mathrm{M}_2}} 1\right\}\\
&\quad +C2^{2l_0}2^{1.8\,L}\sum_{l'=-1}^{L-1} \sup_{P'\in\nabla^\Gamma_{l'}} \sum_{l=-1}^{L-1} 2^{-2l} \sum_{\substack{P\in\oslash^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}}} \mathrm{dist}^{-2.2}\\
&\quad +C2^{2l_0}2^{1.8\,L}\sum_{l'=-1}^{L-1} \sup_{P'\in\nabla^\Gamma_{l'}} \sum_{l=-1}^{L-1} 2^{-l}\,2^{-l} \sum_{\substack{P\in\odot^\Gamma_l:\\ \mathrm{M}_0\leq\mathrm{dist}}} \mathrm{dist}^{-2.2}
\end{aligned}
$$

Using the definitions of M_0, M_1, and M_2 (cf. (6.2)) and applying the estimates (compare (6.3))

$$
\begin{aligned}
2^{-2l} \sum_{P\in\nabla^\Gamma_l:\,\mathrm{dist}>\mathrm{M}_0} \mathrm{dist}^{-2.2} &\leq C\,{\mathrm{M}_0}^{-0.2}, \qquad (6.6)\\
2^{-l} \sum_{P\in\odot^\Gamma_l:\,\mathrm{dist}>\mathrm{M}_0} \mathrm{dist}^{-2.2} &\leq C\,{\mathrm{M}_0}^{-1.2},
\end{aligned}
$$

we continue

$$
\begin{aligned}
\mathcal{N} &\leq C2^{2l_0}\sum_{l'=-1}^{L-1} 2^{2l'} \sum_{l=-1}^{L-1}\left\{\left[\frac{2^{-l}+2^{-l'}+d2^{0.6\,L-0.7\,l-0.9\,l'}}{2^{-l}}\right]^2+\right.\\
&\qquad \left.\left[\frac{2^{-l}+2^{-l'}+d2^{L-0.7\,l-1.3\,l'}}{2^{-l}}\right]\right\} \quad +\\
&\quad C2^{2l_0}2^{1.8\,L}\sum_{l'=-1}^{L-1}\left\{\sum_{l=-1}^{l'} 2^{0.2\,l}+\sum_{l=l'}^{L-1} 2^{0.2\,l'}\right\} + C2^{2l_0}2^{2L}\sum_{l'=-1}^{L-1}\left\{\sum_{l=-1}^{l'} 2^{0.2\,l}+\sum_{l=l'}^{L-1} 2^{-l}2^{1.2\,l'}\right\}\\
&\leq C2^{2l_0}L2^{2L}.
\end{aligned}
$$

∎

6.2 The Estimates for the First Part of the Near Field

Now we suppose that the far field integration and the integration of the second part of the near field are performed exactly and derive the convergence estimates for the first part of the near field. In view of Remark 5.1 it remains to prove

Lemma 6.3 *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ and that $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of Sect. 4.2, then we get the estimate (5.15).*

Proof. i) Like in Sect. 4.2 we distinguish the cases $l(P) \geq l(P')$ and $l(P) < l(P')$, and we start with $l(P) \geq l(P')$. Using Lemma 5.4, we have to estimate the sums Σ_1 and Σ_2. This time the estimate (6.1) holds with

$$a^1_{P',P} \;:=\; C2^{-\mathbf{q}l_0}2^{-2l(P)}2^{-\mathbf{q}l(P,P')}\mathrm{dist}\left(\mathrm{supp}\,\vartheta_{P'},\Psi_P\right)^{-\mathbf{r}-2-\mathbf{q}}, \tag{6.7}$$

$$a^2_{P',P} \;:=\; C2^{-2l(P)}2^{-\mathbf{m}L}\mathrm{dist}\left(\mathrm{supp}\,\vartheta_{P'},\Psi_P\right)^{-\mathbf{r}-2}. \tag{6.8}$$

Note that these estimates follow analogously to part ii) of the proof to Lemma 6.1. The only difference is that the vanishing moments of the test functional are not taken into account.

Again, in accordance with the splitting (6.1) into two terms, we get two estimates of the form (5.16) denoted by $C_u\,\Sigma_1^1\,\Sigma_2^1$ and $C_u\,\Sigma_1^2\,\Sigma_2^2$, respectively. We introduce dist := $\mathrm{dist}(\mathrm{supp}\,\vartheta_{P'},\Psi_P)$, and, similarly to part i) of the proof to Lemma 6.1, we conclude

$$\begin{aligned}
\Sigma_1^1 \;&\leq\; C \sup_{P'\in\triangle_L^\Gamma} \sum_{l=l(P')}^{L-1} \sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l(P')}}} 2^{-sl}2^{-\mathbf{q}l_0}2^{-2l}2^{-\mathbf{q}l(P,P')}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{q}}\\
&\leq\; C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L}\sup_{P'\in\triangle_L^\Gamma} 2^{\mathbf{q}l(P')}\sum_{l=l(P')}^{L-1}2^{-sl}\,2^{-2l}\sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l(P')}}}\mathrm{dist}^{-\mathbf{r}-2+0.1\,\mathbf{q}}.
\end{aligned}$$

Although the distance dist is less than $2^{-l(P')}$, we still have

$$\begin{aligned}
2^{-2l}\sum_{P\in\nabla_l^\Gamma:\,2^{-l(P')}>\mathrm{dist}>c_\Gamma 2^{-l}}\mathrm{dist}^{-\mathbf{r}-2+0.1\,\mathbf{q}} \;&\leq\; C\int_{\{P\in\Gamma:\,2^{-l(P')}>|P'-P|>c_\Gamma 2^{-l}\}}\frac{\mathrm{d}_P\Gamma}{|P'-P|^{\mathbf{r}+2+0.1\,\mathbf{q}}}\\
&\leq\; C[2^{-l(P')}]^{-\mathbf{r}+0.1\,\mathbf{q}}
\end{aligned} \tag{6.9}$$

due to the change from dist $=\mathrm{dist}(\Psi_P,\Theta_{P'})$ to dist $=\mathrm{dist}(\Psi_P,\mathrm{supp}\,\vartheta_{P'})$ and due to the fact that the support $\mathrm{supp}\,\vartheta_{P'}$ of the collocation test functional consist of a small number of points, only. Using (6.9), we continue

$$\Sigma_1^1 \;\leq\; C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L}\sup_{-1\leq l(P')\leq L-1}2^{l(P')[0.9\,\mathbf{q}+\mathbf{r}]}\sum_{l=l(P')}^{L}2^{-sl}\;\leq\; C2^{-\mathbf{q}l_0}2^{-[s-\mathbf{r}]L}.$$

On the other hand, substituting the estimate $a^1_{P',P}$ and $2^{l(P,P')} \geq 2^{0.9L-l(P')}\mathrm{dist}^{-1.1}$ into the definition of Σ^1_2, we get

$$\begin{aligned}\Sigma^1_2 \;&\leq\; C\sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla^\Gamma_l} \sum_{l'=-1}^{l} 2^{-2l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l'}}} 2^{-\mathbf{q}l_0}2^{-2l}2^{-\mathbf{q}l(P,P')}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{q}}\\ &\leq\; C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L}\sum_{l=-1}^{L-1} 2^{-sl} \sup_{P\in\triangle^\Gamma_l} \sum_{l'=-1}^{l} 2^{\mathbf{q}l'}\,2^{-2l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l'}}} \mathrm{dist}^{-\mathbf{r}-2+0.1\,\mathbf{q}}.\end{aligned}$$

Using the estimate (6.9), we continue

$$\Sigma^1_2 \;\leq\; C2^{-\mathbf{q}l_0}2^{-0.9\,\mathbf{q}L}\sum_{l=-1}^{L-1} 2^{-sl}\sum_{l'=-1}^{l} 2^{[\mathbf{r}+0.9\,\mathbf{q}]l'} \;\leq\; C2^{-\mathbf{q}l_0}2^{-[s-\mathbf{r}]L}. \tag{6.10}$$

Next we turn to the estimates of Σ^2_1 and Σ^2_1. Setting $x=1$ in the estimates of Lemma 5.4, proceeding analogously to the treatment of Σ^1_1 and Σ^1_1, and using the estimate (6.9), we arrive at

$$\begin{aligned}\Sigma^2_1 \;&\leq\; C\sup_{P'\in\triangle^\Gamma_L} 2^{-l(P')}\sum_{l=l(P')}^{L-1} \sum_{\substack{P\in\nabla^\Gamma_{l}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l(P')}}} 2^{[1-s]l}2^{-2l}2^{-\mathbf{m}L}\mathrm{dist}^{-\mathbf{r}-2}\\ &\leq\; C2^{-\mathbf{m}L}\sup_{P'\in\triangle^\Gamma_L} 2^{-l(P')}\sum_{l=l(P')}^{L-1} 2^{[1-s]l}\,2^{-2l} \sum_{\substack{P\in\nabla^\Gamma_{l}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l(P')}}} \mathrm{dist}^{-\mathbf{r}-2}\\ &\leq\; C2^{-\mathbf{m}L}\sup_{-1\leq l(P')\leq L-1} 2^{[\mathbf{r}-1]l(P')}\sum_{l=l(P')}^{L-1} 2^{[1-s]l}l^{\delta_{\mathbf{r},0}} \;\leq\; C2^{-\mathbf{m}L}.\end{aligned} \tag{6.11}$$

On the other hand, substituting the estimate $2^{-2l(P)}2^{-\mathbf{m}L}\mathrm{dist}\left(\mathrm{supp}\,\vartheta_{P'},\Psi_P\right)^{-\mathbf{r}-2}$ for $a^2_{P',P}$ into the definition of Σ^2_2, we get

$$\begin{aligned}\Sigma^2_2 \;&\leq\; C\sum_{l=-1}^{L-1} 2^{[1-s]l} \sup_{P\in\nabla^\Gamma_l} \sum_{l'=-1}^{l} 2^{-l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l'}}} 2^{-2l}2^{-\mathbf{m}L}\mathrm{dist}^{-\mathbf{r}-2}\\ &\leq\; C2^{-\mathbf{m}L}\sum_{l=-1}^{L-1} 2^{-[1+s]l} \sup_{P\in\triangle^\Gamma_l} \sum_{l'=-1}^{l} 2^{-l'} \sum_{\substack{P'\in\nabla^\Gamma_{l'}:\\ c_\Gamma 2^{-l}\leq \mathrm{dist}\leq 2^{-l'}}} \mathrm{dist}^{-\mathbf{r}-2}\\ &\leq\; C2^{-\mathbf{m}L}\sum_{l=-1}^{L-1} 2^{-[1+s]l}\sum_{l'=-1}^{l} 2^{-l'}2^{[\mathbf{r}+2]l} \;\leq\; C2^{-\mathbf{m}L}.\end{aligned} \tag{6.12}$$

ii) Now we consider the case $l(P) < l(P')$. Using Lemma 5.4, we have to estimate the sums Σ_1 and Σ_2. We set $\mathrm{Dist} := \mathrm{dist}(\Theta_{P'}, \Psi_P)$. This time the estimate (6.1) holds with

$$\begin{aligned} a^1_{P',P} &:= C 2^{-\mathbf{q} l_0} 2^{-\mathbf{q} l(P,P')} 2^{\mathbf{q} l(P)} 2^{-\mathbf{m}_\vartheta l(P')} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta} \qquad (6.13)\\ &= C 2^{-\mathbf{q} l_0} 2^{-0.95\,\mathbf{q} L} 2^{0.4\,\mathbf{q} l(P)} 2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta] l(P')} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}},\\ a^2_{P',P} &:= C 2^{-\mathbf{m} L} 2^{-\mathbf{m}_\vartheta l(P')} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta}. \qquad (6.14) \end{aligned}$$

Indeed, for the quadrature term $a^1_{P',P}$, we apply the error estimates from part ii) of the proof to Lemma 6.1 to each subtriangle $\Gamma_{Q'}$ of step ii) in Sect. 4.2.2. Note that, for any level l, there is only a bounded number of triangles $\Gamma_{Q'}$ of level l in the partition of step ii) with the bound independent of l. The distance of such a $\Gamma_{Q'}$ of level l to Θ_P can be estimated from below and above by constant times 2^{-l}. Using $2^{-l(P,P')} 2^{l(P)} \le C 2^{-0.95\,L+1.1\,l(P')+0.4\,l(P)} \mathrm{Dist}^{0.55}$ (cf. (4.10)) and adding up the standard quadrature estimates, we arrive at

$$\begin{aligned} a^1_{P',P} &\le C \sum_{l=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-2l} \left[2^{-l} 2^{-l_0} 2^{-l(P,P')} 2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta-\mathbf{q}}\\ &\le C 2^{-\mathbf{q} l_0} 2^{-0.95\,\mathbf{q} L} 2^{0.4\,\mathbf{q} l(P)} 2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta] l(P')} \mathrm{Dist}^{0.55\,\mathbf{q}} \sum_{l=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{l[\mathbf{r}+\mathbf{m}_\vartheta]}\\ &\le C 2^{-\mathbf{q} l_0} 2^{-0.95\,\mathbf{q} L} 2^{0.4\,\mathbf{q} l(P)} 2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta] l(P')} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}}. \end{aligned}$$

Proceeding similarly for the term due to the approximate parametrization, we conclude

$$\begin{aligned} a^2_{P',P} &\le C \sum_{l=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-2l} \left[2^{-L}\right]^{\mathbf{m}} \left[2^{-l}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta}\\ &\le C 2^{-\mathbf{m} L} 2^{-\mathbf{m}_\vartheta l(P')} \sum_{l=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{l[\mathbf{r}+\mathbf{m}_\vartheta]} \le C 2^{-\mathbf{m} L} 2^{-\mathbf{m}_\vartheta l(P')} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta}. \end{aligned}$$

Again, in accordance with the splitting (6.1) into two terms, we get two estimates of the form (5.16) denoted by $C_u\, \Sigma^1_1\, \Sigma^1_2$ and $C_u\, \Sigma^2_1\, \Sigma^2_2$, respectively. We choose the parameter $x = 0.5$ in the estimates of Lemma 5.4 and, similarly to part i) of the proof to Lemma 6.1, we conclude

$$\begin{aligned} \Sigma^1_1 &\le C \sup_{P' \in \triangle^\Gamma_L} 2^{-0.5\,l'} \sum_{l=-1}^{l(P')} \sum_{\substack{P \in \nabla^\Gamma_l:\\ c_\Gamma 2^{-l(P')} \le \mathrm{Dist} \le 2^{-l}}} \frac{2^{[0.5-s]l} 2^{-\mathbf{q} l_0} 2^{-0.95\,\mathbf{q} L} 2^{0.4\,\mathbf{q} l(P)} 2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta] l(P')}}{\mathrm{Dist}^{\mathbf{r}+\mathbf{m}_\vartheta-0.55\,\mathbf{q}}}\\ &\le C 2^{-\mathbf{q} l_0} 2^{-0.95\,\mathbf{q} L} \sup_{l(P'), P' \in \nabla^\Gamma_{l(P')}} 2^{[-0.5+1.1\,\mathbf{q}-\mathbf{m}_\vartheta] l(P')} \left\{ \sum_{l=-1}^{l(P')} 2^{[0.4\,\mathbf{q}+0.5-s] l} \right. \end{aligned}$$

$$\sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l(P')}\le \mathrm{Dist}\le 2^{-l}}} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}}\Bigg\}$$

$$\begin{aligned}
&\le C2^{-\mathbf{q}l_0}2^{-0.95\,\mathbf{q}L}\sup_{l(P')}2^{[-0.5+1.1\,\mathbf{q}-\mathbf{m}_\vartheta]l(P')}\sum_{l=-1}^{l(P')}2^{[0.4\,\mathbf{q}+0.5-s]l}\left[2^{-l(P')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}}\\
&\le C2^{-\mathbf{q}l_0}2^{-[s-\mathbf{r}]L}.
\end{aligned}$$

On the other hand, substituting $a^1_{P',P}$ and $2^{-l(P,P')}2^{l(P)}\le C2^{-0.95\,L+1.1\,l(P')+0.4\,l(P)}\mathrm{Dist}^{0.55}$ into the definition of Σ_2^1, we get

$$\begin{aligned}
\Sigma_2^1 &\le C\sum_{l=-1}^{L-1}2^{[2-s-0.5]l}\sup_{P\in\nabla_l^\Gamma}\sum_{l'=l}^{L-1}2^{[0.5-2]l'}\sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}\le \mathrm{Dist}\le 2^{-l}}}\frac{2^{-\mathbf{q}l_0}2^{-0.95\,\mathbf{q}L}2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta]l'}2^{0.4\,\mathbf{q}}}{\mathrm{Dist}^{\mathbf{r}+\mathbf{m}_\vartheta-0.55\,\mathbf{q}}}\\
&\le C2^{-\mathbf{q}l_0}2^{-0.95\,\mathbf{q}L}\sum_{l=-1}^{L-1}2^{[1.5-s+0.4\,\mathbf{q}]l}\sup_{P\in\nabla_l^\Gamma}\Bigg\{\sum_{l'=l}^{L-1}2^{[1.1\,\mathbf{q}-\mathbf{m}_\vartheta+0.5]l'}\\
&\qquad 2^{-2l'}\sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}\le \mathrm{Dist}\le 2^{-l}}}\mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}}\Bigg\}.
\end{aligned}$$

Using the estimate

$$\begin{aligned}
2^{-2l'}\sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}\ge \mathrm{dist}\ge 2^{-l}}}\mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}} &\le C\int_{\{P'\in\Gamma:\,c_\Gamma 2^{-l'}\ge\mathrm{dist}\ge 2^{-l}\}}\frac{\mathrm{d}_{P'}\Gamma}{\mathrm{dist}\,(P',\Psi_P)^{\mathbf{r}+\mathbf{m}_\vartheta-0.55\,\mathbf{q}}}\\
&\le C\int_0^{2^{-l}}\int_{2^{-l'}}^{x}y^{-\mathbf{r}-\mathbf{m}_\vartheta+0.55\,\mathbf{q}}\,\mathrm{d}y\,\mathrm{d}x\\
&\le C\,2^{-l}2^{[\mathbf{r}+\mathbf{m}_\vartheta-0.55\,\mathbf{q}-1]l'}, \qquad (6.15)
\end{aligned}$$

we continue

$$\Sigma_2^1 \le C2^{-\mathbf{q}l_0}2^{-0.95\mathbf{q}L}\sum_{l=-1}^{L-1}2^{[0.5-s+0.4\,\mathbf{q}]l}\sum_{l'=l}^{L-1}2^{[\mathbf{r}-0.5+0.55\,\mathbf{q}]l'} \le C2^{-\mathbf{q}l_0}2^{-[r-s]L}. \quad (6.16)$$

Next we turn to the parametrization estimates Σ_1^2 and Σ_2^2. Proceeding analogously to the treatment of Σ_1^1 and Σ_1^1 and using estimates like (6.15), we arrive at

$$\Sigma_1^2 \le C\sup_{P'\in\triangle_L^\Gamma}\sum_{l=-1}^{l(P')}\sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l(P')}\le \mathrm{Dist}\le 2^{-l}}}2^{-sl}2^{-\mathbf{m}L}2^{-\mathbf{m}_\vartheta l(P')}\mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta}$$

$$
\begin{aligned}
&\le\; C2^{-\mathbf{m}L} \sup_{P'\in\triangle_L^\Gamma} 2^{-\mathbf{m}_\vartheta l(P')} \sum_{l=-1}^{l(P')} 2^{-sl} \sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l(P')}\le \mathrm{Dist}\le 2^{-l}}} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\le\; C2^{-\mathbf{m}L} \sup_{l(P')=-1,\dots,L-1} 2^{-\mathbf{m}_\vartheta l(P')} \sum_{l=-1}^{l(P')} 2^{-sl} \left[2^{-l(P')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\le\; C2^{-\mathbf{m}L}
\end{aligned}
\tag{6.17}
$$

and at the estimate

$$
\begin{aligned}
\Sigma_2^2 \;&\le\; C \sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l}^{L-1} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}\le \mathrm{Dist}\le 2^{-l}}} 2^{-2l'} 2^{-\mathbf{m}L} 2^{-\mathbf{m}_\vartheta l'} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\le\; C2^{-\mathbf{m}L} \sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\triangle_l^\Gamma} \sum_{l'=l}^{L-1} 2^{-\mathbf{m}_\vartheta l'}\, 2^{-2l'} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}\le \mathrm{Dist}\le 2^{-l}}} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\le\; C2^{-\mathbf{m}L} \sum_{l=-1}^{L-1} 2^{[2-s]l} \sum_{l'=l}^{L-1} 2^{-\mathbf{m}_\vartheta l'} 2^{[\mathbf{r}+\mathbf{m}_\vartheta-1]l'} 2^{-l} \;\le\; C2^{-\mathbf{m}L}.
\end{aligned}
\tag{6.18}
$$

■

Lemma 6.4 *The number of necessary arithmetic operations for setting up that part of the near field of the stiffness matrix $A_L^{w,c,q}$ treated in Sect. 4.2 is less than $CL^2 2^{2L}$.*

Proof. Again we only have to count the number $\mathcal{N}$ of quadrature knots (cf. the proof to Lemma 6.2). First we count those used for the case $0 < \mathrm{dist}(\Psi_P, \Theta_{P'}) \le 2^{-l(P')}$ and $l(P) \ge l(P')$. For fixed ψ_P and $\vartheta_{P'}$, the number of knots is less than $C2^{2l_0}2^{2[l(P,P')-l(P)]}$. Using $2^{l(P,P')} \le 2^{l(P)+1} + 2^{0.9L-l(P')}\mathrm{dist}^{-1.1}$ (cf. (4.1)) and summing up over all ψ_P and $\vartheta_{P'}$, we get

$$
\begin{aligned}
\mathcal{N} \;&\le\; \sum_{P'\in\triangle_L^\Gamma} \sum_{l=l(P')}^{L-1} \sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l}\le \mathrm{dist}\le 2^{-l(P')}}} C2^{2l_0} 2^{2[l(P,P')-l]} \\
&\le\; C2^{2l_0} \sum_{l'=-1}^{L-1} 2^{2l'} \sup_{P'\in\nabla_{l'}^\Gamma} \sum_{l=l'}^{L-1} \sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l}\le \mathrm{dist}\le 2^{-l'}}} 2^{-2l} \left[2^{2l} + \frac{2^{1.8\,L-2l'}}{\mathrm{dist}^{2.2}}\right] \\
&\le\; C2^{2l_0} \sum_{l'=-1}^{L-1} 2^{2l'} \sum_{l=l'}^{L-1} \left[2^{-l'}/2^{-l}\right]^2 + C2^{2l_0}2^{1.8\,L} \sum_{l'=-1}^{L-1} \sum_{l=l'}^{L-1} 2^{-2l} \sum_{\substack{P\in\nabla_l^\Gamma:\\ c_\Gamma 2^{-l}\le \mathrm{dist}\le 2^{-l'}}} \mathrm{dist}^{-2.2} \\
&\le\; C2^{2l_0} L 2^{2L} + C2^{2l_0} 2^{1.8\,L} \sum_{l'=-1}^{L-1} \sum_{l=l'}^{L-1} 2^{0.2\,l} \;\le\; C2^{2l_0} L 2^{2L}.
\end{aligned}
$$

Here we have applied (6.6).

Next we consider the case $l(P) < l(P')$ of Sect. 4.2.2. We set Dist := $\mathrm{dist}(\partial\Psi_P, \Theta)$. For fixed ψ_P and $\vartheta_{P'}$, the number of knots is less than $CL2^{2l_0}2^{2[l(P,P')-l(P)]}$ since in the step ii) of Sect. 4.2.2 the number of triangles $\Gamma_{Q'}$ is less than CL and since each $\Gamma_{Q'}$ is split into $2^{2l_0}2^{2[l(P,P')-l(P)]}$ subtriangles in step iii). Using $2^{2l(P,P')} \leq 2^{2l(P)+2} + 2^{1.9\,L-2.2\,l(P')+1.2\,l(P)}\mathrm{Dist}^{-1.1}$ (cf. (4.10)) into and summing up over all ψ_P and $\vartheta_{P'}$, we get

$$\begin{aligned}
\mathcal{N} \;&\leq\; \sum_{P\in\triangle_L^\Gamma}^{L-1} \sum_{l'=l(P)}^{L-1} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}<\mathrm{Dist}\leq 2^{-l(P)}}} CL2^{2l_0}2^{2[l(P,P')-l(P)]} \\
&\leq\; C2^{2l_0}L\sum_{l=-1}^{L-1} 2^{2l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l}^{L-1} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}<\mathrm{Dist}\leq 2^{-l}}} 2^{-2l}\left[2^{2l} + 2^{1.9\,L-2.2\,l'+1.2\,l}/\mathrm{Dist}^{1.1}\right] \\
&\leq\; C2^{2l_0}L\sum_{l=-1}^{L-1} 2^{2l} \sum_{l'=l}^{L-1} \left[2^{-l}/2^{-l'}\right]^2 \\
&\quad + C2^{2l_0}L2^{1.9\,L}\sum_{l=-1}^{L-1} 2^{1.2\,l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l}^{L-1} 2^{-0.2\,l'}\, 2^{-2l'} \sum_{\substack{P'\in\nabla_{l'}^\Gamma:\\ c_\Gamma 2^{-l'}<\mathrm{Dist}\leq 2^{-l}}} \mathrm{Dist}^{-1.1} \\
&\leq\; C2^{2l_0}L^2 2^{2L} + C2^{2l_0}L2^{1.9L}\sum_{l=-1}^{L-1} 2^{1.2\,l}\sum_{l'=l}^{L-1} 2^{-0.2\,l'}2^{-l}2^{0.1\,l'} \;\leq\; C2^{2l_0}L^2 2^{2L}.
\end{aligned}$$

Here we have applied an estimate like (6.15). ∎

6.3 The Estimates for the Second Part of the Near Field

Now we suppose that the far field integration and the integration of the first part of the near field are performed exactly and derive the convergence estimates for the second part of the near field. In view of Remark 5.1 it remains to prove

Lemma 6.5 *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}$ and that $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of Sect. 4.3, then we get the estimate (5.15).*

Proof. **i)** First we look at the quadrature in Sect. 4.3.1. The case $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda) > 0$ can be treated completely analogously to the quadrature of Sect. 4.2.1 since even in Sect. 4.2.1 the quadrature is applied over the triangles Γ_{Q_μ} separately and since the vanishing moments of the test functionals are not used in the estimates for the quadrature of Sect. 4.2.1.

Moreover, the case $\mathrm{dist}(\Gamma_{Q_\mu}, P_\lambda) = 0$ can be included into the estimation in part i) of the proof to Lemma 6.3, too. We only have to check the estimates for $a^1_{P',P}$ and

$a^2_{P',P}$ in (6.7). The estimate $C2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}}[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}]^{\mathbf{q}}[2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{q}}$ for the quadrature error (cf. part ii) of the proof to Lemma 6.1 and the definition of the step size in (4.15)) applied to each of the subtriangles $\Gamma_{Q'}$ in the partition of step i) in Sect. 4.3.1 leads to

$$
\begin{aligned}
a^1_{P',P} &\leq C \sum_{l(Q')=l(P)+1}^{\mathbf{m}L} 2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}} \left[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{q}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-\mathbf{q}l(P,P')}2^{\mathbf{q}l(P)}L^{-\delta_{\mathbf{r},0}} \sum_{l(Q')=l(P)+1}^{\mathbf{m}L} \left[2^{-l(Q')}\right]^{-\mathbf{r}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-2l(P)}2^{-\mathbf{q}l(P,P')} \left[2^{-l(P)}\right]^{-\mathbf{r}-2-\mathbf{q}}.
\end{aligned}
$$

Hence, in comparison to the estimate $a^1_{P',P}$ we have a $2^{-l(P)}$ instead of $\operatorname{dist}(\operatorname{supp}\vartheta_{P'}, \Psi_P)$. In other words, the quadrature error terms corresponding to $\operatorname{dist}(\Gamma_{Q_\mu}, P_\lambda) = 0$ can be treated like the terms with $\operatorname{dist}(\Gamma_{Q_\mu}, P_\lambda) \sim 2^{-l(P)}$.

On the other hand, for the corresponding parametrization error, we get the upper estimate $C2^{-2l(Q')}[2^{-L}]^{\mathbf{m}+1}[2^{-l(Q')}]^{-2-\mathbf{r}}$ (cf. part ii) of the proof to Lemma 6.1 and use the improved convergence order $\mathbf{m}+2$ instead of $\mathbf{m}+1$ for the approximation by the piecewise polynomial approximation of degree $\mathbf{m}_p = \mathbf{q} \geq \mathbf{m}+1$). Applying this to each of the subtriangles $\Gamma_{Q'}$ in the partition of step i) in Sect. 4.3.1, we conclude

$$
\begin{aligned}
a^2_{P',P} &\leq C \sum_{l(Q')=l(P)+1}^{\mathbf{m}L} 2^{-2l(Q')} \left[2^{-L}\right]^{\mathbf{m}+1} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}} \\
&\leq C2^{-[\mathbf{m}+1]L} \sum_{l(Q')=l(P)+1}^{\mathbf{m}L} \left[2^{-l(Q')}\right]^{-\mathbf{r}} \leq C2^{-2l(P)}2^{-\mathbf{m}L} \left[2^{-l(P)}\right]^{-\mathbf{r}-2}.
\end{aligned}
$$

Here we have estimated the logarithmic term CL appearing in the case $\mathbf{r}=0$ by using the additional factor 2^{-L} of the interpolation. Hence, the parametrization error terms with $\operatorname{dist}(\Gamma_{Q_\mu}, P_\lambda) = 0$ can be treated like the terms with $\operatorname{dist}(\Gamma_{Q_\mu}, P_\lambda) \sim 2^{-l(P)}$.

ii) Next we consider the quadrature in Sect. 4.3.2. We suppose that the approximate values $a^{w,c,q}_{P',\Gamma^e_m}$ and $a^{w,c,q}_{P',\lambda,\Gamma^e_m}$ are known exactly. We defer the analysis of their approximation to part iii) of the present proof. Now the case $\operatorname{Dist} := \operatorname{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) > 0$ can be treated analogously to the quadrature of Sect. 4.2.2. We only have to check the estimates for $a^1_{P',P}$ and $a^2_{P',P}$ in (6.13) and (6.14). As usual, the estimate for the quadrature error over the triangle $\Gamma_{Q'}$ of step i) in Sect. 4.3.2.1 is

$$
C2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}\right]^{\mathbf{q}} [2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{m}_\vartheta-\mathbf{q}} \sup_{R\in\Gamma_{Q'}} |\psi_P(R)|
$$

(cf. part ii) of the proof to Lemma 6.1). The supremum of ψ_P is bounded if the distance of $\Gamma_{Q'}$ to Ψ_P is less than $C2^{-l(P)}$, i.e. if $l(Q') \geq l(P)$. For $l(Q') < l(P)$, we get the bound $2^{l(P)-l(Q')}$. Applying this to each of the subtriangles $\Gamma_{Q'}$ in the partition of step i) in Sect.

4.3.2.1, we get

$$\begin{aligned}
a^1_{P',P} &\leq C \sum_{l(Q')=-3}^{l(P)} 2^{-2l(Q')} 2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(Q')} 2^{-l_0} 2^{-l(P,P')} 2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta-\mathbf{q}} \frac{2^{l(P)}}{2^{l(Q')}} \\
&\quad + C \sum_{l(Q')=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{-2l(Q')} 2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(Q')} 2^{-l_0} 2^{-l(P,P')} 2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta-\mathbf{q}} \\
&\leq C 2^{-\mathbf{q} l_0} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{q} l(P,P')} 2^{[\mathbf{q}+1] l(P)} \sum_{l(Q')=-3}^{l(P)} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta+1} \\
&\quad + C 2^{-\mathbf{q} l_0} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{q} l(P,P')} 2^{\mathbf{q} l(P)} \sum_{l(Q')=l(P)+1}^{-{}^2\log \mathrm{Dist}} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\leq C 2^{-\mathbf{q} l_0} 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{q} l(P,P')} 2^{\mathbf{q} l(P)} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta}.
\end{aligned}$$

In other words, the quadrature error terms corresponding to Dist > 0 can be treated like the terms in Sect. 4.2.2 with the distance $\mathrm{dist}(\Psi_P, \Theta_{P'})$ of the same size.

On the other hand, for the parametrization error, we arrive at the usual upper bound

$$C \sup_{R \in \Gamma_{Q'}} |\psi_P(R)|\, 2^{-2l(Q')} 2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-L}\right]^{\mathbf{m}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}+\mathbf{m}_\vartheta}$$

(cf. part ii) of the proof to Lemma 6.1). Applying this to each of the subtriangles $\Gamma_{Q'}$ in the partition of step i) in Sect. 4.3.2.1, we conclude

$$\begin{aligned}
a^2_{P',P} &\leq C \sum_{l(Q')=-3}^{l(P)} 2^{l(P)-l(Q')} 2^{-2l(Q')} 2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-L}\right]^{\mathbf{m}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}+\mathbf{m}_\vartheta} + \\
&\quad C \sum_{l(Q')=l(P)+1}^{-{}^2\log \mathrm{Dist}} 2^{-2l(Q')} 2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-L}\right]^{\mathbf{m}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}+\mathbf{m}_\vartheta} \\
&\leq C 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{m} L} 2^{l(P)} \sum_{l(Q')=-3}^{l(P)} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta+1} + \\
&\quad C 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{m} L} \sum_{l(Q')=l(P)+1}^{-{}^2\log \mathrm{Dist}} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\leq C 2^{-\mathbf{m}_\vartheta l(P')} 2^{-\mathbf{m} L} \mathrm{Dist}^{-\mathbf{r}-\mathbf{m}_\vartheta}.
\end{aligned}$$

Hence, the parametrization error terms with $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) > 0$ can be treated like the terms with the distance $\mathrm{dist}(\Psi_P, \Theta_{P'})$ of the same size as $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'})$ corresponding to the quadrature of Sect. 4.2.2 (cf. part ii) of the proof to Lemma 6.3).

If $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) = 0$, we split $\vartheta_{P'}$ into the linear combination of point functionals at the points P_λ, and, setting Dist $:= 2^{-l(P')}$, we derive the estimates (6.13) and (6.14) for $a^1_{P',P}$ and $a^2_{P',P}$. The usual bounds $C 2^{-2l(Q')} L^{-\delta_{\mathbf{r},0}} [2^{-l(Q')} 2^{-l_0} 2^{-l(P,P')} 2^{l(P)}]^{\mathbf{q}} [2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{q}}$

resp. $C2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}}2^{-\mathbf{m}_\vartheta l(P')}[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}]^{\mathbf{q}}[2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{q}+\mathbf{m}_\vartheta}$ for the quadrature error (cf. part ii) of the proof to Lemma 6.1) applied to each of the subtriangles $\Gamma_{Q'}$ in the partition of step ii') in Sect. 4.3.2 lead to

$$\begin{aligned}
a^1_{P',P} &\leq C \sum_{l(Q')=l(P)+1}^{l(P')} 2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}L^{-\delta_{\mathbf{r},0}} \left[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{q}-\mathbf{m}_\vartheta} \\
&\quad + C \sum_{l(Q')=l(P')+1}^{\mathbf{m}L} 2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}} \left[2^{-l(Q')}2^{-l_0}2^{-l(P,P')}2^{l(P)}\right]^{\mathbf{q}} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{q}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-\mathbf{m}_\vartheta l(P')}2^{-\mathbf{q}l(P,P')}2^{\mathbf{q}l(P)} \sum_{l(Q')=l(P)+1}^{l(P')} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\quad + C2^{-\mathbf{q}l_0}2^{-\mathbf{q}l(P,P')}2^{\mathbf{q}l(P)}L^{-\delta_{\mathbf{r},0}} \sum_{l(Q')=l(P')+1}^{\mathbf{m}L} \left[2^{-l(Q')}\right]^{-\mathbf{r}} \\
&\leq C2^{-\mathbf{q}l_0}2^{-\mathbf{q}l(P,P')}2^{\mathbf{q}l(P)}2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(P')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta},
\end{aligned}$$

which is (6.13) with Dist replaced by $2^{-l(P')}$. On the other hand, for the parametrization error, we arrive at the usual upper bound $C2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}2^{-[\mathbf{m}+1]L}[2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{m}_\vartheta}$ resp. $C2^{-2l(Q')}2^{-[\mathbf{m}+1]L}[2^{-l(Q')}]^{-2-\mathbf{r}}$ (cf. part ii) of the proof to Lemma 6.1 and notice that $\mathbf{m}_p \geq \mathbf{m}+1$). Applying this to each of the subtriangles $\Gamma_{Q'}$ in the partition of step ii') in Sect. 4.3.2.2, we conclude

$$\begin{aligned}
a^2_{P',P} &\leq C \sum_{l(Q')=l(P)+1}^{l(P')} 2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}2^{-[\mathbf{m}+1]L} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta} \\
&\quad + C \sum_{l(Q')=l(P')+1}^{\mathbf{m}L} 2^{-2l(Q')}2^{-[\mathbf{m}+1]L} \left[2^{-l(Q')}\right]^{-2-\mathbf{r}} \\
&\leq C2^{-[\mathbf{m}+1]L}2^{-\mathbf{m}_\vartheta l(P')} \sum_{l(Q')=l(P)+1}^{l(P')} \left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} + C2^{-[\mathbf{m}+1]L} \sum_{l(Q')=l(P')+1}^{\mathbf{m}L} \left[2^{-l(Q')}\right]^{-\mathbf{r}} \\
&\leq CL^{\delta_{\mathbf{r},0}}2^{-[\mathbf{m}+1]L}2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(P')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta} \leq C2^{-\mathbf{m}L}2^{-\mathbf{m}_\vartheta l(P')} \left[2^{-l(P')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta}.
\end{aligned}$$

Hence, the quadrature and parametrization error terms with $\mathrm{dist}(\partial\Gamma_{Q_\mu}, \Theta_{P'}) = 0$ can be treated like the terms with $\mathrm{dist}(\Gamma_{Q_\mu}, \Theta_{P'}) \sim 2^{-l(P)}$ corresponding to the quadrature of Sect. 4.2.2 (cf. part ii) of the proof to Lemma 6.3).

iii) The "quadrature" error $a^1_{P',\Gamma^e_m,p}$ of neglecting the approximation $a^{w,c,q}_{P',\Gamma^e_m,p}$ for the corresponding integral in the case $l(P') \geq \frac{\mathbf{m}}{\mathbf{m}_\vartheta}L$ is less than $C2^{-\mathbf{m}_\vartheta l(P')}$ since the integrand is $\mathbf{m}_\vartheta$ times continuously differentiable by assumption (cf. Sect. 2.2) and since $\vartheta_{P'}$ has $\mathbf{m}_\vartheta$ vanishing moments. For the other case $l(P') < \frac{\mathbf{m}}{\mathbf{m}+2}L$, we really have to estimate the quadrature error $a^1_{P',\Gamma^e_m,p}$ and the parametrization error $a^2_{P',\Gamma^e_m,p}$, respectively. Now applying the usual upper bounds $C2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}}[2^{-l(Q')}2^{-\zeta L}2^{\zeta' l(P')}]^{\mathbf{q}}[2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{q}}$ resp.

$C2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}L^{-\delta_{\mathbf{r},0}}[2^{-l(Q')}2^{-\zeta L}2^{\zeta' l(P')}]^{\mathbf{q}}[2^{-l(Q')}]^{-2-\mathbf{r}-\mathbf{q}-\mathbf{m}_\vartheta}$ for the quadrature error to each of the subtriangles $\Gamma_{Q'}$ in the partition of step ii) in Sect. 4.3.3.2, we get the error estimate

$$\begin{aligned}
a^1_{P',\Gamma^e_m,p} &\leq C\sum_{l(Q')=-3}^{l(P')} 2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}L^{-\delta_{\mathbf{r},0}}\left[2^{-l(Q')}2^{-\zeta L}2^{\zeta' l(P')}\right]^{\mathbf{q}}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{q}-\mathbf{m}_\vartheta}\\
&\quad +C\sum_{l(Q')=l(P')+1}^{\mathbf{m}L} 2^{-2l(Q')}L^{-\delta_{\mathbf{r},0}}\left[2^{-l(Q')}2^{-\zeta L}2^{\zeta' l(P')}\right]^{\mathbf{q}}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{q}}\\
&\leq C2^{-\mathbf{m}_\vartheta l(P')}2^{-\mathbf{q}\zeta L}2^{\mathbf{q}\zeta' l(P')}\sum_{l(Q')=-3}^{l(P')}\left[2^{-l(Q')}\right]^{-\mathbf{r}-\mathbf{m}_\vartheta}+\\
&\quad C2^{-\mathbf{q}\zeta L}2^{\mathbf{q}\zeta' l(P')}L^{-\delta_{\mathbf{r},0}}\sum_{l(Q')=l(P')+1}^{\mathbf{m}L}\left[2^{-l(Q')}\right]^{-\mathbf{r}}\\
&\leq C2^{[\mathbf{r}+\mathbf{q}\zeta']l(P')}2^{-\mathbf{q}\zeta L}.
\end{aligned}$$

On the other hand, for the parametrization error over the level $l(Q')$ triangles, we arrive at the usual upper bound

$$C2^{-2l(Q')}\begin{cases} 2^{-\mathbf{m}_\vartheta l(P')}\left[2^{-L}\right]^{\mathbf{m}}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta} & \text{if } l(Q')\leq l(P')\\ \left[2^{-L}\right]^{\mathbf{m}+1}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}} & \text{if } l(P')<l(Q'). \end{cases}$$

Applying this to each of the subtriangles $\Gamma_{Q'}$ in the partition of step ii) in Sect. 4.3.3.2, we conclude

$$\begin{aligned}
a^2_{P',\Gamma^e_m,p} &\leq \sum_{l(Q')=-3}^{l(P')} C2^{-2l(Q')}2^{-\mathbf{m}_\vartheta l(P')}\left[2^{-L}\right]^{\mathbf{m}+1}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}-\mathbf{m}_\vartheta}\\
&\quad +\sum_{l(Q')=l(P')}^{\mathbf{m}L} C2^{-2l(Q')}\left[2^{-L}\right]^{\mathbf{m}+1}\left[2^{-l(Q')}\right]^{-2-\mathbf{r}} \leq C2^{\mathbf{r}l(P')}2^{-[\mathbf{m}+0.5]L}.
\end{aligned}$$

Now we take into account that, for the computation of an $a^{w,c,q}_{P',P}$, the trial function ψ_P is presented as a linear combination of the three basic linear polynomials p, where the constant p has a bounded coefficient and where the coefficients of the linear functions $p(\kappa_m(\tau))\mapsto(\tau-[\kappa_m]^{-1}(P_\lambda))_i$, $i=1,2$ are bounded by $2^{l(P)}$. Consequently, the resulting error $a^i_{P',P}$, $i=1,2$ for $a^{w,c,q}_{P',P}$ is bounded by $2^{l(P)}a^i_{P',\Gamma^e_m,p}$.

In view of Lemma 5.4, we have to estimate the sums Σ^i_1 and Σ^i_2 with $a_{P',P}$ replaced by $a^i_{P',P}$, $i=1,2$. Choosing $i=1$, we obtain

$$\begin{aligned}
\Sigma^1_1 &\leq \sup_{P'\in\triangle^\Gamma_L,\, l(P')\geq\frac{\mathbf{m}}{\mathbf{m}+2}L}\ \sum_{P\in\triangle^\Gamma_L:\, l(P)<l(P'),\,\dots} 2^{-sl(P)}C2^{l(P)}2^{-\mathbf{m}_\vartheta l(P')}\\
&\quad +\sup_{P'\in\triangle^\Gamma_L,\, l(P')<\frac{\mathbf{m}}{\mathbf{m}+2}L}\ \sum_{P\in\triangle^\Gamma_L:\, l(P)<l(P'),\,\dots} 2^{-sl(P)}C2^{l(P)}2^{[\mathbf{r}+\mathbf{q}\zeta']l(P')}2^{-\mathbf{q}\zeta L}
\end{aligned}$$

$$
\begin{aligned}
&\le C \sup_{l(P')\ge \frac{\mathbf{m}}{\mathbf{m}+2}L} 2^{-\mathbf{m}_\vartheta l(P')} \sum_{l(P)=-1}^{l(P')-1} 2^{[1-s]l(P)} \\
&\quad + C2^{-\mathbf{q}\zeta L} \sup_{l(P')<\frac{\mathbf{m}}{\mathbf{m}+2}L} 2^{[\mathbf{r}+\mathbf{q}\zeta']l(P')} \sum_{l(P)=-1}^{l(P')} 2^{[1-s]l(P)} \\
&\le C2^{-\mathbf{m}_\vartheta \frac{\mathbf{m}}{\mathbf{m}+2}L} + C2^{-\mathbf{q}\zeta L}2^{[\mathbf{r}+\mathbf{q}\zeta'][\frac{\mathbf{m}}{\mathbf{m}+2}L]} \le C2^{-\mathbf{m}L}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\Sigma_2^1 &\le \sum_{l=-1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{[2-s]l} \sup_{P\in\nabla_l^\Gamma} \Bigg\{ \sum_{l'=l+1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{-2l'} \sum_{P'\in\nabla_{l'}^\Gamma:\,\dots} C2^l 2^{[\mathbf{r}+\mathbf{q}\zeta']l'} 2^{-\mathbf{q}\zeta L} \\
&\qquad + \sum_{l'=\frac{\mathbf{m}}{\mathbf{m}+2}L}^{L-1} 2^{-2l'} \sum_{P'\in\nabla_{l'}^\Gamma:\,\dots} C2^l 2^{-\mathbf{m}_\vartheta l'} \Bigg\} \\
&\quad + \sum_{l=\frac{\mathbf{m}}{\mathbf{m}+2}L}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l+1}^{L-1} 2^{-2l'} \sum_{P'\in\nabla_{l'}^\Gamma:\,\dots} C2^l 2^{-\mathbf{m}_\vartheta l'} \\
&\le C \sum_{l=-1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{[3-s]l} \Bigg\{ \sum_{l'=l+1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{[-2+\mathbf{r}+\mathbf{q}\zeta']l'} 2^{-\mathbf{q}\zeta L} + \sum_{l'=\frac{\mathbf{m}}{\mathbf{m}+2}L}^{L-1} 2^{[-2-\mathbf{m}_\vartheta]l'} \Bigg\} \left[\frac{2^{-l}}{2^{-l'}}\right]^2 \\
&\quad + C \sum_{l=\frac{\mathbf{m}}{\mathbf{m}+2}L}^{L-1} 2^{[3-s]l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l+1}^{L-1} 2^{[-2-\mathbf{m}_\vartheta]l'} \left[\frac{2^{-l}}{2^{-l'}}\right]^2 \\
&\le C2^{-\mathbf{m}L}.
\end{aligned}
$$

The dots in the last formulae stand for the restriction to pairs of P and P' for which the quadrature approximation to $a^{w,c}_{P',P}$ is treated in Sect. 4.3.2.1. Similarly, we conclude

$$
\begin{aligned}
\Sigma_1^2 &\le \sup_{P'\in\triangle_L^\Gamma,\, l(P')<\frac{\mathbf{m}}{\mathbf{m}+2}L} \sum_{P\in\triangle_L^\Gamma:\,\dots} 2^{-sl(P)} C2^{l(P)} 2^{\mathbf{r}l(P')} 2^{-[\mathbf{m}+0.5]L} \\
&\le C2^{-\mathbf{m}L}, \\
\Sigma_2^2 &\le \sum_{l=-1}^{L-1} 2^{[2-s]l} \sup_{P\in\nabla_l^\Gamma} \sum_{l'=l+1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{-2l'} \sum_{P'\in\nabla_{l'}^\Gamma:\,\dots} C2^l 2^{\mathbf{r}l'} 2^{-[\mathbf{m}+0.5]L} \\
&\le C2^{-[\mathbf{m}+0.5]L} \sum_{l=-1}^{L-1} 2^{[3-s]l} \sum_{l'=l+1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L-1} 2^{[-2+\mathbf{r}]l'} \left[\frac{2^{-l}}{2^{-l'}}\right]^2 \\
&\le C2^{-\mathbf{m}L}.
\end{aligned}
$$

iv) The last error to be estimated is the error due to the neglect of the singular integrals over triangles of the level $\mathbf{m}L$. Such a neglect occurs in all the three subsections of Sect. 4.3. However, if $\mathbf{r} = -1$ or if $\mathbf{r} = 0$ and the kernel $k(P,Q)$ satisfies the Mikhlin-Gireaud condition (cf. Sect. 2.2 and cf. e.g. [26]), then the value of such an integral is less than $2^{-\mathbf{m}L}$. Since we commit such an error at most once in every entry $a^{w,q,c}_{P',P}$ of the stiffness matrix, Lemma 5.4 with the choice $x = 1$ implies the global error estimate $\|[A^c_L - A^{c,w}_L]P_L u\|_{L^2(\Gamma)} \le C2^{-\mathbf{m}L}$ for $s = 1.1$ and $s = 2$. ■

Lemma 6.6 *The number of necessary arithmetic operations for setting up the part of the near field of the stiffness matrix $A^{w,c,q}_L$ treated in Sect. 4.3 is less than $CL^2 2^{2L}$ if $\mathbf{r} = -1$ and less than $CL^3 2^{2L}$ if $\mathbf{r} = 0$.*

Proof. We have seen that each quadrature term of Sect. 4.3 computed over a Ψ_P or a Γ_{Q_μ} can be included into the estimates of Sect. 4.2. In particular, each quadrature for an entry of the second part of the near field requires the same number of quadrature knots as a corresponding entry of the first part of the near field. The only exception is that, due to the logarithmic term in the levels of the uniform refinements according to (4.13) and (4.15), there arises an additional factor L in the complexity if $\mathbf{r} = 0$. Consequently, we get the same complexity estimate as in Lemma 6.4 for $\mathbf{r} = -1$ and the same complexity multiplied by L for $\mathbf{r} = 0$.

It remains to count the quadrature knots for Sect. 4.3.3.2. For a fixed $\vartheta_{P'}$, the number of knots is less than $CL[2^{\zeta L - \zeta' l(P')}]^2$. Hence, the number of all arithmetic operations for the computation according to Sect. 4.3.3.2 is less than

$$C \sum_{l(P')=-1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L} 2^{2l(P')} L \left[2^{\zeta L - \zeta' l(P')}\right]^2 \;\le\; C2^{2\zeta L} L \sum_{l(P')=-1}^{\frac{\mathbf{m}}{\mathbf{m}+2}L} 2^{2l(P')[1-\zeta']} \;\le\; CL2^{2L}.$$

■

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