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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

10. Invariant subspaces

1. Recapitulate the algorithm of the vector iteration for the computation of the eigenvalue with maximum absolute value. Which conditions must be fulfilled for convergence ?

2. Adapt the algorithm in 1. to get the eigenvalue with the smallest norm.

3. Let $A \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times p}, p \leq n, rank(X) = p$ and

   $$F := (X^*X)^{-1}X^*AX$$

   the so called „generalized Rayleigh-Quotient“ .

   Proof that (v, λ) is an eigenpair of F if and only if (Xv, λ) is an eigenpair of A.

   Which consequences has this for A and the subspace $\mathbb{X}$, defined by X?

   What is the shape of F in the special case $X = [x_1, \cdots x_p]$ with $x_i, i = 1 \cdots p$ as eigenvectors of A?

4. Extend the algorithm of challenge 1 for the computation of more than one eigenvalues and eigenvectors. Which condition of challenge 1 can bypassed ?

5. Let $a, b \in \mathbb{R}$ mit $|a| \leq 1$ und $|b| \leq 1$. Proof in preparation of the following challenges, that:

   $$ab + \sqrt{1-a^2}\sqrt{1-b^2} \leq 1$$

6. For convergence considerations et cetera it is usual to compare the iterated solution with a known exact solution. For one dimensional subspaces, given exactly by $v_e \in \mathbb{R}^n$ and iterated with $v \in \mathbb{R}^n$ this can be done in an easy manner by normalization, direction adaption and consideration of $||v - v_e||$.

   Now we want to develope such an comparison for p-dimensional subspaces. Which possibilities can you see to do this ?

7. Illustrate the idea at the example of two (zero containing) planes in $\mathbb{R}^3$.

8. Now develop an algorithm to compute all needed values in an effective manner.