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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

9. Projections and Krylov spaces

1. Proof the following invariances of krylov spaces $\mathcal{K}_m(A, \mathbf{v})$:
   1.1. $\mathcal{K}_m(\alpha A, \beta\mathbf{v}) = \mathcal{K}_m(A, \mathbf{v}) \quad \forall \alpha, \beta \in \mathbb{C} \setminus 0$
   1.2. $\mathcal{K}_m(A - \sigma I, \mathbf{v}) = \mathcal{K}_m(A, \mathbf{v}) \quad \forall \sigma \in \mathbb{C}$
   1.3. $\mathcal{K}_m(TAT^{-1}, T\mathbf{v}) = T\mathcal{K}_m(A, \mathbf{v}) \quad \forall$ regular $T \in \mathbb{C}^{n\times n}$

2. 2.1. Let $\mathbf{x}_1, \ldots, \mathbf{x}_M$ eigenvectors of a matrix $A \in \mathbb{C}^{n\times n}$ to pairwise different eigenvalues $\lambda_1, \ldots, \lambda_M \in \mathbb{C} \setminus \{0\}$. Moreover let $\mathbf{v} = \beta_1\mathbf{x}_1 + \ldots + \beta_M\mathbf{x}_M$ with $\beta_j \neq 0$ for all $j = 1, \ldots, M$.
   Proof, that for the grade $d(A, \mathbf{v})$ (in german: Abbruchindex) of the krylov-sequence $A^m\mathbf{v}$ holds: $d(A, \mathbf{v}) = M$.

   2.2. Let $A \in \mathbb{C}^{n\times n}$ invertable and $\mathbf{v} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$. Show that

   $$d = d(A, \mathbf{v}) = \min\{m : A^{-1}\mathbf{v} \in \mathcal{K}_m(A, \mathbf{v})\}.$$

3. Let $P_1 = P_{\mathcal{R}_1,\mathcal{S}_1}$ and $P_2 = P_{\mathcal{R}_2,\mathcal{S}_2}$ projections in $\mathbb{C}^n$.

   3.1. Proof that:

   $$P_1P_2 = 0 \quad \Longrightarrow \quad \mathcal{R}_1 \cap \mathcal{R}_2 = \{\mathbf{0}\},$$

   and, if P_1 and P_2 are orthogonal it holds

   $$P_1P_2 = 0 \quad \Longleftrightarrow \quad \mathcal{R}_1 \cap \mathcal{R}_2 = \{\mathbf{0}\}.$$

   Find a counterexample which shows that the equivalence in the non orthogonal case is not true.

   3.2. Proof that: $Q = P_1 + P_2$ is a projection if and only if $P_1P_2 = P_2P_1 = 0$.
   Determine for that case image and kernal of Q and an und examine with this if Q is orthogonal if P_1 and P_2 are orthogonal.

   3.3. Show: If $P_1P_2 = P_2P_1$, then $Q = P_1P_2$ is a projection.
   Determine again under this assumption image and kernal of Q and show that Q is orthogonal if this is true for P_1 and P_2.

   3.4. Proof that

   $$P_{\mathcal{W}_m} = P_{\mathcal{V}_m} \cdots P_{\mathcal{V}_1},$$

   where are $\mathcal{V}_j = span\{\mathbf{v}_j\}^\perp$, $\mathcal{W}_m = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}^\perp$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ a orthonormal base(of $\mathcal{K}_m(A, \mathbf{b})$).