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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

6. Iterative Methods I

Please hand in questions 1,2,4.

1. Implement Richardson iteration, Jacobi iteration and Gauss-Seidel iteration! Test your code on the Poisson matrix of dimension n for some values of n! (`A=gallery('poisson',n)`) and right-hand side `b=sum(A,2)`.

2. (a) Show that for a symmetric matrix A that is strictly diagonally dominant the Jacobi iteration always converges.

   (b) We consider the Jacobi iteration for the following matrix

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & 0 & 0 \\ 0 & 1 & 15 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 495 & 1 \\ 0 & 0 & 0 & 0 & 1 & 500 \end{bmatrix}.$$

   Give an estimate of the spectral radius of the iteration matrix based on the observations from the first homework.

3. Show that the Gauss-Seidel iteration always converges for a matrix with the assumptions from the previous question part (a).

4. We consider the Peaceman-Rachford ADI iteration described by

$$(H + \rho I)u_{k+1/2} = (\rho I - V)u_k + b \tag{1}$$
$$(V + \rho I)u_{k+1} = (\rho I - H)u_{k+1/2} + b, \tag{2}$$

   which solves the linear system $(H + V)u = b$.

   (a) Show that this can be written as $u_{k+1} = Gu_k + f$. What is G and f?

   (b) Verify the splitting $M = \frac{1}{2\rho}(H + \rho I)(V + \rho I)$ and $N = \frac{1}{2\rho}(H - \rho I)(V - \rho I)$. Show also that f from part (a) can be written as $M^{-1}b$.

5. Take the $n \times n$ matrix of the following form

$$T = \begin{bmatrix} \alpha & -1 & & \\ -1 & \alpha & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & \alpha \end{bmatrix}$$

with α a real parameter. Verify that the eigenvalues are given by $\lambda_j = \alpha - 2\cos(j\theta)$ with $\theta = \frac{\pi}{n+1}$ and the corresponding eigenvector is given by

$$q_j = [\sin(j\theta), \sin(2j\theta), \ldots, \sin(nj\theta)]^T .$$

When is the matrix T positive definite?

6. Let $A \in \mathbb{R}^{n\times n}$ symmetric. Proof (i) and (ii) of the following (nice) conditions of A:
   (i) All eigenvalues λ_i of A are real.
   (ii) Eigenvectors to different eigenvalues are orthogonal.
   (iii) Algebraic and geometric multiplicities of the eigenvalues are equal.
7. A is a real symmetric positive definite matrix and $A = D - L - U$ the decomposition for the Jacobi-Iteration with the matrix $B = D^{-1}(L + U)$. Proof that B has only real eigenvalues.