Numerische lineare Algebra WS 2018/2019

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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

5. Sherman-Morrison und Moore-Penrose-Inverse

1. Proof the **Sherman-Morrison-Formula** for $A \in \mathbb{R}^{n\times n}$, $U, V \in \mathbb{R}^{n\times k}$ with $k \ll n$.

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I_k + V^T A^{-1} U)^{-1} V^T A^{-1}$$

Wherefore this formula could be useful ?

2. Let $A \in \mathbb{R}^{n\times n}$ with large n occupied like follows

$$A = \begin{bmatrix} * & * & & * & * & * & * \\ * & * & & & & & * \\ \vdots & & & \ddots & & & \vdots \\ * & & & & & * & * \\ * & * & & * & * & * & * \end{bmatrix}$$

Find an effective way to solve the linear system $Ax = b$.

3. Recapitulate the relationships between normal equations, least square problems and overdetermined linear systems.

4. Let $A \in \mathbb{R}^{m\times n}$ and $r := rank(A) = min\{m, n\}$. Show that

$$A^\dagger = \begin{cases} (A^T A)^{-1} A^T & \text{if} \quad r = n \le m \\ A^T (AA^T)^{-1} & \text{if} \quad r = m \le n \\ A^{-1} & \text{if} \quad r = m = n \end{cases}$$

5. Decide: The Moore-Penrose-Inverse is

   a) the generalized inverse.

   b) one generalized inverse.

6. Proof the following relations:

   a)

$$\begin{aligned} (A^\dagger)^\dagger &= A \\ (A^\dagger)^T &= (A^T)^\dagger \\ 0^\dagger &= 0 \\ (\lambda A)^\dagger &= \frac{1}{\lambda} A^\dagger \quad \forall \lambda \neq 0 \end{aligned}$$

   b) Let $A = FG$ with $A \in \mathbb{R}^{m\times n}$, $rank(A) = r$, $F \in \mathbb{R}^{m\times r}$, $G \in \mathbb{R}^{r\times n}$, then holds $A^\dagger = G^\dagger F^\dagger (= G^T(GG^T)^{-1}(F^T F)^{-1} F^T)$.

   c) For $A = UBV^T$ with orthogonal $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{m\times n}$, it holds $A^\dagger = VB^\dagger U^T$.

   d) If $A = \begin{bmatrix} B \\ 0 \end{bmatrix}$, then $A^\dagger = \begin{bmatrix} B^\dagger & 0 \end{bmatrix}$.