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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

4. Gaussian Elimination

Please hand in questions 1,3,4,5. For the Matlab question a printout of the output of your Matlab routines is fine.

1. Let A be an $n \times n$ square matrix. Show using Gaussian elimination that $Ax = b$ has a unique solution iff (if and only if) the diagonal entries u_{jj} of U from the LU factorization of $A = LU$ are nonzero. (Hint: Consider determinants).

2. Suppose that the $n \times n$ matrix A is written in block-form, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{m,m}$ and $A_{22} \in \mathbb{R}^{n-m,n-m}$ as well as A_{11} and A being invertible.

   (a) Verify that

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

   Here $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur-complement of A.

   (b) What are the conditions on the Schur-complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ (provided A_{11}^{-1} exists) for A to be invertible?

3. Show that if Gaussian elimination with partial pivoting is applied to a matrix $A \in \mathbb{R}^{4,4}$ that we can write

$$L_3P_3L_2P_2L_1P_1 = L_3^{'}L_2^{'}L_1^{'}P_3P_2P_1$$

and define $L_j^{'}$.

4. Perform (by hand) Gaussian elimination with partial pivoting on the following matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}.$$

Based on Question 3 write down the factorization $PA = LU$. Verify your result by using Matlab's `lu` command. Implement Gaussian elimination without pivoting and compare the results when solving for the right hand side `b=sum(A,2)`. Now again compare LU with and without (your function) pivoting for the matrix

$$A = \begin{bmatrix} 1e-6 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

with the right hand side given by `b=sum(A,2)`. You want to look at the residual in both cases `norm(b-Ax)`.

5. Show that for a matrix A that is strictly column diagonally dominant, i.e.,

$$|a_{jj}| > \sum_{i=1, i\neq j}^{n} |a_{ij}|,$$

Gaussian elimination with partial pivoting does not need to perform any row interchanges.