Numerische lineare Algebra WS 2018/2019

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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

3. Numerical Range

1. The „Numerical Range“ of the matrix $A \in \mathbb{C}^{n\times n}$ is the set $W(A) \subset \mathbb{C}$, defined as

$$W(A) := \{ < Ax, x > \quad | \quad x \in \mathbb{C}^n, ||x|| = 1 \} .$$

Proof the following statements (a) and (b)

(a) $\sigma(A) \subset W(A)$

(b) If A is a normal matrix, then $W(A) = \mathrm{conv}(\sigma(A))$.

(c) For arbitrary A the set $W(A)$ is convex. (Hausdorff-Toeplitz-Theorem)

2. We disassemble $A \in \mathbb{C}^{n\times n}$ in the following parts

$$A^+ := \frac{A + A^*}{2}$$
$$A^- := \frac{A - A^*}{2}$$

define therby

- λ^+_{min} := smallest eigenvalue of A^+
- λ^+_{max} := largest eigenvalue of A^+
- λ^-_{min} := smallest imaginary part of the eigenvalues of A^-
- λ^-_{max} := largest imaginary part of the eigenvalues of A^-

and consider the rectangle

$$B(A) := [\lambda^+_{min}, \lambda^+_{max}] + i[\lambda^-_{min}, \lambda^-_{max}]$$

in the complex plane.

(a) Which symmetries of the rectangle $B(A)$ respective to the real and imaginary axe for $A \in \mathbb{R}^{n\times n}$ we can see ?

(b) Proof: $W(A) \subset B(A)$.

(c) Draft $B(A)$ and $W(A)$ for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi)] \end{bmatrix}$$

für $\phi = \frac{\pi}{2}$. (Hint: Can you use (b) from the last problem ?)

(d) We restrict for $A \in \mathbb{R}^{n\times n}$ our $W(A)$ with

$$W^{\mathbb{R}}(A) := \{ < Ax, x > \quad | \quad x \in \mathbb{R}^n, ||x|| = 1 \} .$$

to $W^{\mathbb{R}}(A)$. Describe this set $W^{\mathbb{R}}(A)$ and add it to the sketch in (c).

3. For $A \in \mathbb{C}^{n\times n}$ and $W(A)$ like above we define with $\rho(A)$ the spectral radius and with

$$r(A) := \max_{z\in W(A)} |z|$$

the „numerical radius“ of A.

(a) Proof:

$$\rho(A) \leq r(A) \leq ||A||_2 \leq 2r(A)$$

(b) Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times n}$ und $r(A)$, $r(B)$ their numerical radii. It is well known:

$$||AB|| \leq ||A||\ ||B||.$$

Proof or rebut the inequality:

$$r(AB) \leq r(A)\ r(B)$$

4. Show for $A \in \mathbb{C}^{n\times n}$ with eigenvalues λ_i

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq ||A||_F^2 = \operatorname{tr}(A^*A)$$

and for the case that A is normal

$$\sum_{i=1}^{n} |\lambda_i|^2 = ||A||_F^2.$$