Numerische lineare Algebra WS 2018/2019

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Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

2. Elementary Transformations

There will be no work to hand in but you will be asked to perform one of the questions on the board.

1. Familiarize yourself with the first chapter in the book "Iterative methods for sparse linear systems" by Yousef Saad. `http://www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf`

2. Let A be a 4×4 matrix to which the following operations are applied

   2.1. double column 2,
   2.2. halve row 1,
   2.3. add row 2 to row 4,
   2.4. interchange columns 1 and 2,
   2.5. substract row 1 from each of the other rows,
   2.6. replace column 4 by column 3,
   2.7. delete column 2 (so that the column dimension is reduced by 1).

   (a) Write this as the product of 8 matrices.
   (b) Write this as the product of three matrices UAV with A the original matrix.

3. An important role in numerical linear algebra is played by matrices used for similarity transformations of the original matrix A to obtain a matrix SAS^{-1} with more desirable properties. Especially the Housholder reflection and Givens rotation are such transformations, which we will consider in the next tasks.

   Let $v \in \mathbb{R}^n$ with $||v|| = 1$. Define the term „Reflection on a hyperplane" and deduce a reflection matrix $H \in \mathbb{R}^{n\times n}$ which maps an arbitrary vektor $x \in \mathbb{R}^n$ to his mirrored image $y \in \mathbb{R}^n$ respective to the hyperplane $v^\perp$. (We can assume the hyperplane contains 0, such that it is an $n-1$ dimensional subspace.)

4. Determine some characteristics (determinant, eigenvalues, invertibility, $\cdots$) of the matrix H.

5. Now compute a vector $v \in \mathbb{R}^n$ and with v a Housholder matrix H, which maps an arbitrary vector $x \in \mathbb{R}^n$ in such a way, that the image Hx is a multiple of a given vector $a \in \mathbb{R}^n$:

   $$Hx = \alpha a \qquad \alpha \in \mathbb{R}.$$

   Use the formulas for $x = [1, 1, 1, 1]^T$ and $a = e_1 = [1, 0, 0, 0]$, compute v, H and the scaling factor α.

6. Develope an (effective!) algorithm (Matlab-notation) for the computation of such problems for given arbitrary $a \in \mathbb{R}^n$.

7. Givens roations defined as matrices

$$G(i,k,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

where $c = cos(\theta)$ and $s = sin(\theta)$. These matrices are used to eliminate entries from vectors and matrices. Derive expressions for c and s based on the fact that this matrix is used to eliminate the entry in position k of the vector x using the vector element in position j, i.e., $y := G(i, k, \theta)x$ with $y_k = 0$. Using this show that the matrix G is orthogonal.

8. Show that for any positive definite matrix $A \in \mathbb{R}^{n,n}$ (see 1.11 in the book from Question 1) all diagonal entries must be positive.

9. (a) Let A be an $n \times n$ matrix, with entries a_{ij}. For $i \in \{1, \ldots, n\}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the ith row. Let $D(a_{ii}, R_i)$ be the closed disc centered at a_{ii} with radius R_i. Prove that every eigenvalue of A lies within at least one of the discs $D(a_{ii}, R_i)$. [Hint: Start using the eigenvalue relation $Ax = \lambda x$.]

   (b) Show using the previous result that the matrix

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & 0 & 0 \\ 0 & -1 & 15 & 1 & 0 & 0 \\ 0 & 0 & -1 & 20 & 1 & 0 \\ 0 & 0 & 0 & -1 & 25 & 1 \\ 0 & 0 & 0 & 0 & -1 & 30 \end{bmatrix}$$

   is invertible.

   (c) Assuming that *Theorem: If the union of k discs is disjoint from the union of the other $n - k$ discs then the former union contains exactly k and the latter $n - k$ eigenvalues of A* holds, show that the matrix A from part (b) has only real eigenvalues.