TU-Chemnitz, Fakultät für Mathematik
Vorlesung: Prof. Dr. Martin Stoll **Übung: Dr. Roman Unger**
Homepage zur Übung: `https://www.tu-chemnitz.de/mathematik/wire/WS1819/nla.php`

1. Basics

1. For the following matrices compute the eigenvalues, their algebraic and geometric multiplicities, their eigenvectors and, if needed, their generalized eigenvectors.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & -4 \\ -2 & 1 & -2 \end{bmatrix},$$

$$E = \begin{bmatrix} 3 & -10 & -10 \\ 0 & 3 & 0 \\ 0 & -5 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

2. **„The big Six Matrix Decompositions“**
Repeat the properties of the following matrix factorizations:

   2.1. Cholesky-Decomposition
   2.2. (Pivotised) LU-Decomposition
   2.3. QR-Decomposition
   2.4. Spectral Decomposition
   2.5. Schur-Decomposition
   2.6. Singular Value Decomposition (SVD)

   Think about existence and uniquess. Which algorithms you know for the decompositions ?

3. Show the uniquess of the sparse QR-Decomposition $A = Q_1 R_1$, if $A \in \mathbb{R}^{m\times n} (m > n)$ has full column rank. $Q_1 \in \mathbb{R}^{m\times n}$ has orthonormal columns and $R_1 \in \mathbb{R}^{n\times n}$ is a upper triangular matrix. Moreover, R_1 comes from the lower triangular factor G from the Cholesky-Decomposition of $A^T A$ by the identity $R_1 = G^T$.

4. For $A \in \mathbb{R}^{m\times n}$ with $s = rank(A)$ we have the Singular Value Decomposition (SVD)

$$A = U\Sigma V^T$$

   with orthonormal $U = [u_1, \cdots u_m] \in \mathbb{R}^{m\times m}$ and orthonormal $V = [v_1, \cdots v_n] \in \mathbb{R}^{n\times n}$.
   Proof that:

   a) If $k < s$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, then:

$$\min_{rank(B)=k} ||A - B||_2 = ||A - A_k||_2 = \sigma_{k+1}$$

   b)

$$\begin{aligned} ker(A) &= span\{v_{s+1}, \cdots v_n\} \\ im(A) &= span\{u_1, \cdots u_s\} \end{aligned}$$

   c)

$$||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_s^2}$$