SIAM J. Sci. Comput.
Vol. 34, No. 2, pp. A659–A682

EFFICIENT ITERATIVE SOLVERS FOR STOCHASTIC GALERKIN DISCRETIZATIONS OF LOG-TRANSFORMED RANDOM DIFFUSION PROBLEMS*

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Abstract. We consider the numerical solution of a steady-state diffusion problem where the diffusion coefficient is the exponent of a random field. The standard stochastic Galerkin formulation of this problem is computationally demanding because of the nonlinear structure of the uncertain component of it. We consider a reformulated version of this problem as a stochastic convection-diffusion problem with random convective velocity that depends linearly on a fixed number of independent truncated Gaussian random variables. The associated Galerkin matrix is nonsymmetric but sparse and allows for fast matrix-vector multiplications with optimal complexity. We construct and analyze two block-diagonal preconditioners for this Galerkin matrix for use with Krylov subspace methods such as the generalized minimal residual method. We test the efficiency of the proposed preconditioning approaches and compare the iterative solver performance for a model problem posed in both diffusion and convection-diffusion formulations.

Key words. stochastic Galerkin method, finite elements, Karhunen–Loève expansion, lognormal random field, convection-diffusion problem, preconditioning, algebraic multigrid

AMS subject classifications. 35R60, 60H15, 60H35, 65N30, 65F10, 65F08

DOI. 10.1137/110836675

1. Introduction.

We are interested in constructing efficient numerical methods for the steady-state diffusion equation where the diffusion coefficient is a positive random field of specific structure. The problem is

$$-\nabla \cdot (\exp(a)\nabla u) = f, \tag{1.1}$$

posed on a bounded domain $D \subset \mathbb{R}^2$ together with appropriate boundary conditions. The log-transformed diffusion coefficient $a = a(\boldsymbol{x}, \omega)$ is a random field; that is, for each elementary event ω in a given probability space $(\Omega, \mathfrak{A}, P)$ we obtain a scalar function $a(\cdot, \omega)$ varying in the physical domain D. Such problems arise, for example, from groundwater flow simulations, where the permeability is often modeled as a *lognormal* random field; see [19, 51].

The nonlinearity of the diffusion coefficient complicates the numerical solution of (1.1). For example, as we show in section 2, if stochastic Galerkin methods are combined with a polynomial chaos expansion of $\exp(a)$, then computations must be carried out with matrices of dense structure that are expensive to use. However, the particular form of this coefficient allows for a “log-transformed” reformulation

*Submitted to the journal’s Methods and Algorithms for Scientific Computing section June 7, 2011; accepted for publication (in revised form) December 22, 2011; published electronically March 13, 2012.
http://www.siam.org/journals/sisc/34-2/83667.html
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of the problem as a *convection-diffusion problem*. This approach is mentioned, for example, in [51, section 1.4] and [40]. Multiplying both sides of (1.1) by $\exp(-a)$ and rearranging, we obtain the equation

$$-\Delta u + \boldsymbol{w} \cdot \nabla u = f \exp(-a) \tag{1.2}$$

with (advective) velocity $\boldsymbol{w} = -\nabla a$. Indeed, since

$$\exp(-a)\nabla \cdot (\exp(a)\nabla u) = \exp(-a)\exp(a)\Delta u + \exp(-a)\nabla \exp(a) \cdot \nabla u = \Delta u + \nabla a \cdot \nabla u$$

we have arrived at (1.2). Notably, the convection-diffusion problem (1.2) contains the gradient of the log-transformed diffusion coefficient a on the left-hand side of the equation; the fact that this is linear in a will lead to discrete problems with simpler structure and *sparse* system matrices. The inverse of the term $\exp(a)$ appearing on the right-hand side of the equation raises no special difficulties.

In this study, we will explore the stochastic Galerkin discretization [22] of the convection-diffusion formulation of the diffusion problem and show that the associated discrete systems can be solved efficiently by iterative methods. In particular, we show that preconditioning operators derived from a matrix associated with the mean of a can be combined with Krylov subspace methods to give convergence rates that are independent of the characteristic mesh size of the spatial discretization and only slightly sensitive to parameters of the stochastic discretization, for example, the standard deviation of a. We consider two ways to construct preconditioners, one based on a discrete diffusion operator, and the other derived from a discrete convection-diffusion operator. For both methods, the required computations needed to implement the preconditioning operation can be done efficiently using multigrid methods, leading to textbook multigrid behavior.

An outline of the paper is as follows. In section 2, we review the treatment of the original stochastic steady-state diffusion equation (1.1) by Galerkin methods, describe the details of the random field model for this formulation, and discuss the iterative solution of the Galerkin equations. In section 3, we present the convection-diffusion formulation (1.2) in detail, introduce its weak formulation, and describe the construction and properties of the associated Galerkin matrix. In addition, we touch upon some aspects of stability and implementation for the convection-diffusion formulation. Section 4 contains the major contribution of this work. Here, we present the preconditioning methodology for the discrete convection-diffusion equations and derive bounds on eigenvalues and convergence rates associated with it. In section 5, we demonstrate the effectiveness of the new preconditioning strategies for the discrete convection-diffusion formulation of the problem and compare their performance with solvers for the discrete diffusion formulation. Finally, in section 6, we make some concluding observations.

2. Stochastic steady-state diffusion problem.

Our point of departure is the diffusion equation (1.1) with stochastic diffusion coefficient of the form $\exp(a)$. A formal statement of the problem is to find a random field $u(\boldsymbol{x}, \omega)$ satisfying

$$\begin{aligned} -\nabla \cdot (\exp(a(\boldsymbol{x}, \omega))\nabla u(\boldsymbol{x}, \omega)) &= f(\boldsymbol{x}) && \text{in } D \times \Omega, \\ u(\boldsymbol{x}, \omega) &= g(\boldsymbol{x}) && \text{on } \partial D_D \times \Omega, \\ \boldsymbol{n} \cdot \nabla u(\boldsymbol{x}, \omega) &= 0 && \text{on } \partial D_N \times \Omega \text{ a.e.} \end{aligned} \tag{2.1}$$

For simplicity, we consider deterministic boundary conditions on $\partial D = \partial D_D \cup \partial D_N$ and a deterministic source term $f = f(\boldsymbol{x})$.

2.1. Log-transformed diffusion coefficient. Assume that the function a is a *Gaussian* random field; then it can be specified in terms of its mean value $a_0(\boldsymbol{x}) = \langle a(\boldsymbol{x}, \cdot)\rangle$ and covariance function

$$\mathrm{Cov}(\boldsymbol{x}, \boldsymbol{y}) = \langle (a(\boldsymbol{x}, \cdot) - a_0(\boldsymbol{x}))(a(\boldsymbol{y}, \cdot) - a_0(\boldsymbol{y}))\rangle\,, \qquad \boldsymbol{x}, \boldsymbol{y} \in D.$$

Here, $\langle\cdot\rangle$ denotes the expectation with respect to the probability measure P. A widely used tool for the representation of Gaussian random fields is the Karhunen–Loève (KL) expansion [32]

$$a(\boldsymbol{x}, \omega) = a_0(\boldsymbol{x}) + \sigma \sum_{m=1}^{\infty} \sqrt{\lambda_m} a_m(\boldsymbol{x}) \xi_m(\omega), \tag{2.2}$$

where $\{\xi_m\}_{m=1}^{\infty}$ are *independent* standard Gaussian random variables, $\sigma^2 = \mathrm{Cov}(\boldsymbol{x}, \boldsymbol{x})$ denotes the (for simplicity, spatially constant) variance of a, $(\lambda_m, a_m)_{m=1}^{\infty}$ are the eigenpairs of the integral operator $C : L^2(D) \to L^2(D)$,

$$(Cu)(\boldsymbol{x}) = \int_D c(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y})\, d\boldsymbol{y}, \qquad (Ca_m)(\boldsymbol{x}) = \lambda_m a_m(\boldsymbol{x}), \tag{2.3}$$

and the kernel function is $c(\boldsymbol{x}, \boldsymbol{y}) := \mathrm{Cov}(\boldsymbol{x}, \boldsymbol{y})/\sigma^2$. We assume that the eigenvalues λ_m in (2.2) are arranged in decreasing order. In actual computations, only the first $M + 1$ terms in the KL expansion are retained, yielding an approximation of a of the form

$$a^{(M)}(\boldsymbol{x}, \omega) = a_0(\boldsymbol{x}) + \sigma \sum_{m=1}^{M} \sqrt{\lambda_m} a_m(\boldsymbol{x}) \xi_m(\omega). \tag{2.4}$$

In the current treatment of stochastic diffusion problems by Galerkin methods it is standard to assume that the random diffusion coefficient can be bounded a.e. in $D \times \Omega$ by *deterministic constants* in order to obtain a well-posed weak formulation [1, 3, 4, 5, 13, 18]. A major complication arising from lognormal diffusion coefficients is, however, that realizations of a (and thus $\exp(a)$) cannot be uniformly bounded with respect to $\omega \in \Omega$. Instead, we have

$$0 < e_{\min}(\omega) \le \exp(a(\boldsymbol{x}, \omega)) \le e_{\max}(\omega) < \infty \qquad \text{a.e. in } D \times \Omega,$$

where $e_{\min}(\omega)$ and $e_{\max}(\omega)$ are *random variables*; see [20] and [24]. Note that each individual realization of $\exp(a)$ is bounded away from zero and infinity. A well-posed weak formulation of (2.1) can be obtained under additional assumptions on the regularity of the source term f and $e_{\min}^{-1}$ (see [2]). Other approaches involve weak formulations in weighted versions of standard Sobolev spaces; see the works [20, 24, 34].

Since the focus of our study is on iterative solvers for the discrete Galerkin equations we do not adopt these advanced techniques here and instead follow the standard approach. To this end, we modify our model for the log-transformed diffusion coefficient slightly and make the simplifying assumption that the random variables $\{\xi_m\}_{m=1}^{M}$ in the KL expansion (2.4) of a are independent and have a *truncated* Gaussian density of the form

$$\rho_m(\xi_m) = (2\Phi(c/s) - 1)^{-1} \times (\sqrt{2\pi} s)^{-1} \times \exp(-\xi_m^2/(2s^2)) \times \mathbb{1}_{[-c,c]}(\xi_m) \tag{2.5}$$

for $m = 1, \dots, M$. Above, $\Phi(\cdot)$ denotes the standard Gaussian cumulative distribution function, $c > 0$ denotes a cut-off parameter, and the constant $s > 0$ is chosen such that ξ_m has variance one. Now, $|\xi_m| \le c$ and for sufficiently regular covariance functions it can be shown that the KL eigenfunctions in (2.2) are bounded (see [42, Theorem 2.24]). In summary, then, $|a^{(M)}(\boldsymbol{x}, \omega)|$ is bounded a.e. in $D \times \Omega$, and thus it follows that

$$0 < e_{\min} \le \exp(a^{(M)}(\boldsymbol{x}, \omega)) \le e_{\max} < \infty \qquad \text{a.e. in } D \times \Omega, \tag{2.6}$$

where $e_{\min}$ and $e_{\max}$ are deterministic constants.

We mention that stochastically linear models of the form (2.4) with independent random variables often approximate Gaussian random fields irrespective of the distribution of their random parameters (see [26, section 2]). In addition, the idea of reformulating the original lognormal diffusion problem (1.1) as a convection-diffusion problem is applicable to *all* log-transformed diffusion coefficients of the form (2.2) regardless of the statistical properties of the random variables $\{\xi_m\}_{m=1}^{\infty}$ in (2.2).

2.2. Stochastic Galerkin formulation. With $a^{(M)}(\boldsymbol{x}, \omega)$ of (2.4) used for $a(\boldsymbol{x}, \omega)$, the weak formulation of (2.1) is to find $u \in H_0^1(D) \otimes L_\rho^2(\Gamma)$ such that

$$\int_\Gamma \int_D \exp(a^{(M)}) \nabla u \cdot \nabla v \, \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} = \int_\Gamma \int_D f v \, \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} \qquad \forall v \in H_0^1(D) \otimes L_\rho^2(\Gamma). \tag{2.7}$$

Here, $H_0^1(D) = \{v \in H^1(D) \colon v|_{\partial D_D} = 0\} \subset H^1(D)$ is a subset of the standard Sobolev space $H^1(D)$, and $L_\rho^2(\Gamma)$ denotes the Sobolev space of square-integrable functions on the joint image $\Gamma = \Gamma_1 \times \cdots \times \Gamma_M$ of the truncated Gaussian random variables $\xi_m(\Omega) = \Gamma_m = [-c, c]$ weighted by the M-variate probability density function $\rho(\boldsymbol{\xi}) = \rho_1(\xi_1)\rho_2(\xi_2) \cdots \rho_M(\xi_M)$. Since the log-transformed diffusion coefficient in (2.7) satisfies (2.6), the well-posedness of this problem follows (see, for example, [3]).

We will consider piecewise (bi)linear finite elements for the physical discretization and complete M-variate (generalized) chaos polynomials of total degree $\le d$ for the stochastic discretization. In other words, the finite-dimensional physical subspace is

$$X_h = \operatorname{span}\{\phi \in H_0^1(D) \colon \phi|_K \in P_1(K) \quad \forall K \in \mathscr{T}_h\}, \tag{2.8}$$

where $K \in \mathscr{T}_h$ denotes an element (triangle, rectangle) in a triangulation $\mathscr{T}_h$ of the physical domain D, and $P_1(K)$ is the space of (bi)linear functions on K. Defining $n_{\boldsymbol{x}} = \dim(X_h)$, we write $X_h = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{n_{\boldsymbol{x}}}\}$.

For the stochastic discretization, we use as shape functions the (generalized) chaos polynomials $\psi_{\boldsymbol{\alpha}}$ consisting of products of orthonormal (univariate) polynomials $\psi_0, \psi_1, \dots, \psi_n, \dots$, $\deg(\psi_n) = n$, generated by the truncated Gaussian probability density function in (2.5). (These polynomials are known as *Rys* polynomials; see [21] and the references therein.) That is,

$$\psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{m=1}^{M} \psi_{\boldsymbol{\alpha}_m}(\xi_m) \, . \tag{2.9}$$

Here, $\boldsymbol{\alpha} \in \mathscr{I} = \mathbb{N}_0^M$ is a multi-index with M components. We define

$$\mathscr{I}_d = \{\boldsymbol{\alpha} \in \mathscr{I} \colon |\boldsymbol{\alpha}| \le d\},$$

where $|\boldsymbol{\alpha}| := \alpha_1 + \alpha_2 + \cdots + \alpha_M$, and write

$$S_d = \operatorname{span}\{\psi_{\boldsymbol{\alpha}} \colon \boldsymbol{\alpha} \in \mathscr{I}_d\}. \tag{2.10}$$

We will enumerate the stochastic shape functions $\{\psi_{\boldsymbol{\alpha}}\}$ with multi-indices $\boldsymbol{\iota}(j) \in \mathscr{I}_d$, where $j \in \{1, \ldots, n_{\boldsymbol{\xi}}\}$, $n_{\boldsymbol{\xi}} = \dim(S_d) = |\mathscr{I}_d| = \binom{M+d}{M}$. Discretization using the spaces (2.8) and (2.10) produces the discrete Galerkin equations

$$\sum_{i=1}^{n_x} \sum_{j=1}^{n_{\boldsymbol{\xi}}} \int_\Gamma \int_D \exp(a^{(M)}) \nabla\phi_i \cdot \nabla\phi_k \, \psi_{\boldsymbol{\iota}(j)} \psi_{\boldsymbol{\iota}(\ell)} \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} \, u_{i,j}^{(\mathrm{diff})} = \int_\Gamma \int_D f \phi_k \, \psi_{\boldsymbol{\iota}(\ell)} \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} \tag{2.11}$$

for $k = 1, \ldots, n_{\boldsymbol{x}}$ and $\ell = 1, \ldots, n_{\boldsymbol{\xi}}$.

2.3. Generalized polynomial chaos expansion of $\exp(a)$. As detailed in section 2.1, the diffusion coefficient $\exp(a^{(M)})$ in (2.11) is a nonlinear function of the components of the KL expansion of a of (2.2). It can be represented using a generalized polynomial chaos expansion [50],

$$\begin{aligned} \exp(a^{(M)}(\boldsymbol{x}, \omega)) &= \sum_{\boldsymbol{\alpha} \in \mathscr{I}} a_{\boldsymbol{\alpha}}(\boldsymbol{x}) \psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}(\omega)), \\ a_{\boldsymbol{\alpha}}(\boldsymbol{x}) &= \left\langle \exp(a^{(M)}) \psi_{\boldsymbol{\alpha}} \right\rangle = \exp(a_0(\boldsymbol{x})) \prod_{m=1}^{M} \left\langle \exp(\sigma \sqrt{\lambda_m} \xi_m) \psi_{\alpha_m}(\xi_m) \right\rangle. \end{aligned} \tag{2.12}$$

Again, $\boldsymbol{\alpha} \in \mathscr{I} = \mathbb{N}_0^M$ is a multi-index. For results on the convergence of such an expansion, see [16]. Since analytic expressions for the chaos coefficients $a_{\boldsymbol{\alpha}}$ are in general unavailable, we compute these with the help of Gauss quadrature rules generated by the truncated Gaussian density (2.5). Inserting the expansion (2.12) into the Galerkin equations (2.11), we arrive at

$$\sum_{\boldsymbol{\alpha} \in \mathscr{I}_{2d}} \sum_{i=1}^{n_x} \sum_{j=1}^{n_{\boldsymbol{\xi}}} \int_D a_{\boldsymbol{\alpha}} \nabla\phi_i \cdot \nabla\phi_k \, d\boldsymbol{x} \left\langle \psi_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\iota}(j)} \psi_{\boldsymbol{\iota}(\ell)} \right\rangle u_{i,j}^{(\mathrm{diff})} = \int_D f \phi_k \, d\boldsymbol{x} \left\langle \psi_{\boldsymbol{\iota}(\ell)} \right\rangle. \tag{2.13}$$

Above, we have used the notation $\langle g(\boldsymbol{\xi}) \rangle = \int_\Gamma g(\boldsymbol{\xi}) \rho(\boldsymbol{\xi}) \, d\boldsymbol{\xi}$. Note in particular that $\left\langle \psi_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\iota}(j)} \psi_{\boldsymbol{\iota}(\ell)} \right\rangle = 0$ for $\boldsymbol{\alpha} \in \mathscr{I} \backslash \mathscr{I}_{2d}$, so that the chaos expansion of $\exp(a^{(M)})$ in (2.12) is implicitly truncated in (2.13) and involves chaos polynomials of total degree $\leq 2d$ in the M random variables. Introducing the finite element matrices

$$[A_{\boldsymbol{\alpha}}]_{i,k} = \int_D a_{\boldsymbol{\alpha}} \nabla\phi_k \cdot \nabla\phi_i \, d\boldsymbol{x}, \qquad i, k = 1, \ldots, n_{\boldsymbol{x}}, \quad \boldsymbol{\alpha} \in \mathscr{I}_{2d}, \tag{2.14}$$

and the stochastic Galerkin matrices [17]

$$[G_{\boldsymbol{\alpha}}]_{j,\ell} = \left\langle \psi_{\boldsymbol{\alpha}} \psi_{\boldsymbol{\iota}(\ell)} \psi_{\boldsymbol{\iota}(j)} \right\rangle, \qquad j, \ell = 1, \ldots, n_{\boldsymbol{\xi}}, \quad \boldsymbol{\alpha} \in \mathscr{I}_{2d}, \tag{2.15}$$

we have arrived at the Kronecker product representation of the matrix $\widehat{A}$ associated with the Galerkin equations (2.13) of the original stochastic diffusion problem,

$$\widehat{A} = \sum_{\boldsymbol{\alpha} \in \mathscr{I}_{2d}} G_{\boldsymbol{\alpha}} \otimes A_{\boldsymbol{\alpha}}. \tag{2.16}$$

This matrix is symmetric positive-definite, so the conjugate gradient (CG) method [28] can be used for iterative solution of the Galerkin system. It is known that $\widehat{A}$ is ill-conditioned with respect to the mesh size h, the standard deviation σ of the log-transformed diffusion coefficient $a^{(M)}$, and the total degree d of the chaos polynomials;

see [17, 37, 48]. Ill-conditioning with respect to the spatial mesh size can be handled by a mean-based block-diagonal preconditioner derived by approximating the random diffusion coefficient $\exp(a^{(M)})$ by its mean value $\langle \exp(a^{(M)}) \rangle$; see [23, 31, 35, 36]. Analysis in [36] shows that the spectrum of the mean-based preconditioned Galerkin matrix is independent of h. (The analysis is performed for stochastically linear diffusion coefficients, but it carries over to stochastically nonlinear diffusion coefficients.) The robustness with respect to σ and d can be improved using a "Kronecker product" preconditioner developed in [48].

Unfortunately, the matrix-vector products required by CG are expensive. Although each finite element matrix $A_{\boldsymbol{\alpha}}$ in (2.16) is a sparse $n_{\boldsymbol{x}} \times n_{\boldsymbol{x}}$ matrix, the global Galerkin matrix $\widehat{A}$ is block dense [33]. As a result, matrix-vector multiplications with the fully assembled $\widehat{A}$ require $O(n_{\boldsymbol{\xi}}^2 n_{\boldsymbol{x}})$ work. Storing only the Kronecker factors in (2.16) does not cure this problem. In fact, matrix-vector multiplications with $\widehat{A}$ in the form (2.16) cost at least $O(n_{\boldsymbol{x}} n_{\boldsymbol{\xi}}^2 n_d)$ operations, where $n_d = |\mathscr{I}_{2d}|/|\mathscr{I}_d| \ll n_{\boldsymbol{\xi}}$; see [48]. These costs do not scale linearly in the total number of unknowns and thus preclude from the outset the design of iterative solvers with optimal, i.e., $O(n_{\boldsymbol{x}} n_{\boldsymbol{\xi}})$, complexity.

2.4. Karhunen–Loève expansion of $\exp(a)$.

A sparse Galerkin matrix can be obtained by representing the diffusion coefficient $\exp(a)$ in (1.1) using a KL expansion; see [49] for details of this approach. Precisely,

$$\exp(a(\boldsymbol{x}, \omega)) = e_0(\boldsymbol{x}) + \sum_{m=1}^{\infty} \sqrt{\nu_m} e_m(\boldsymbol{x}) \eta_m(\omega). \tag{2.17}$$

Here, $(\nu_m, e_m)_{m=1}^{\infty}$ are the eigenpairs of the covariance integral operator defined as in (2.3) with the covariance function of $\exp(a)$ as the kernel function. If the sum in (2.17) is approximated using $\widetilde{M}$ terms, the resulting approximate diffusion coefficient is a *linear* function of $\widetilde{M}$ basic random variables $\{\eta_m\}_{m=1}^{\widetilde{M}}$. In this case, the Galerkin matrix analogous to (2.16) will be sparse (see, for example, [36]) and allows for inexpensive matrix-vector products.

We will not pursue this approach here. Note that the random variables $\{\eta_m\}$ are uncorrelated but dependent, and their probability density functions are not known in general, nor is their joint density function. This issue is addressed in [49], where the marginal densities of each random variable are estimated, and the joint density is estimated using the assumption that the variables $\{\eta_m\}$ are statistically independent. The stochastic discretization is then carried out in terms of generalized polynomial chaos functions of the form (2.9) generated by the (estimated) density functions of the random variables $\{\eta_m\}$ in (2.17). A postprocessing step is required to approximate the actual law of probability of the random solution. Moreover, it is not clear how the decay of the eigenvalues in the KL expansion of $\exp(a)$ relates to the decay of the respective eigenvalues in the KL expansion of the log-transformed diffusion coefficient a. The eigenvalue decay determines the number terms to be retained in order to parameterize and approximate the random input. Depending on the correlation length and standard deviation of a, it can happen that the number of variables required in the expansion of $\exp(a)$ is (significantly) larger than the number of variables to be retained in the expansion of a.

3. Reformulation as a convection-diffusion problem.

In this section, we use the reformulated convection-diffusion variant of the problem, (1.2), in combination with a stochastically linear expansion of the gradient ∇a of the log-transformed diffusion coefficient, to generate an alternative discrete Galerkin system built from sparse matrices. A formal statement of the transformed problem is to find the random field $u(\boldsymbol{x},\omega)$ satisfying a.e.

$$\begin{aligned} -\Delta u(\boldsymbol{x},\omega) + \boldsymbol{w}(\boldsymbol{x},\omega)\cdot\nabla u(\boldsymbol{x},\omega) &= f(\boldsymbol{x})\exp(-a(\boldsymbol{x},\omega)) && \text{in } D\times\Omega,\\ u(\boldsymbol{x},\omega) &= g(\boldsymbol{x}) && \text{on } \partial D_D\times\Omega,\\ \boldsymbol{n}\cdot\nabla u(\boldsymbol{x},\omega) &= 0 && \text{on } \partial D_N\times\Omega, \end{aligned} \tag{3.1}$$

where the velocity $\boldsymbol{w}(\boldsymbol{x},\omega) = -\nabla a(\boldsymbol{x},\omega)$ is a vector-valued random field. As above, we use the truncated KL expansion $a^{(M)}$ of (2.4) as the log-transformed diffusion coefficient a. In addition, we make the assumption that the gradient of $a^{(M)}$ is well-defined. We will discuss this point in section 3.3 below.

Following the procedure in section 2.2, the weak formulation of (3.1) is to find $u \in H_0^1(D)\otimes L_\rho^2(\Gamma)$ such that for all $v \in H_0^1(D)\otimes L_\rho^2(\Gamma)$

$$\int_\Gamma\int_D \nabla u\cdot\nabla v\rho\,d\boldsymbol{x}\,d\boldsymbol{\xi} + \int_\Gamma\int_D v\,\boldsymbol{w}_M\cdot\nabla u\rho\,d\boldsymbol{x}\,d\boldsymbol{\xi} = \int_\Gamma\int_D f\exp(-a^{(M)})v\,\rho\,d\boldsymbol{x}\,d\boldsymbol{\xi}\,, \tag{3.2}$$

where $\boldsymbol{w}_M = -\nabla a^{(M)}$. Since the log-transformed diffusion coefficient $a^{(M)}$ is uniformly bounded in our model (see (2.6)), we can choose the modified test function $v\cdot\exp(-a^{(M)}) \in H_0^1(D)\otimes L_\rho^2(\Gamma)$ in the weak formulation (2.7). Then it is easy to see that the weak formulation (3.2) of the convection-diffusion version is equivalent to the corresponding original weak formulation (2.7). Hence, the well-posedness of (3.2) follows from the well-posedness of (2.7).

Remark 3.1. The connection between diffusion and convection-diffusion problems through logarithmic transformation is documented in the literature; see, for example, [38] and [46, 47]. In the latter works, the existence of a velocity potential is assumed, which enables a recasting of the convection-diffusion formulation (1.2) as a diffusion problem of the form (1.1). Here, we are following the opposite strategy in order to construct efficient iterative solvers for our model problem.

3.1. Galerkin equations. Use of the finite-dimensional subspace $X_h \subset H_0^1(D)$ in (2.8) and $S_d \subset L_\rho^2(\Gamma)$ in (2.10) yields a conforming Galerkin finite element discretization of the stochastic convection-diffusion problem (3.1). The discretized Galerkin equations for the $n_{\boldsymbol{x}}\cdot n_{\boldsymbol{\xi}}$ degrees of freedom (d.o.f.) that define $u \in X_h\otimes S_d$ can be derived from (3.2) as

$$\begin{aligned} &\sum_{i=1}^{n_{\boldsymbol{x}}}\sum_{j=1}^{n_{\boldsymbol{\xi}}}\left[\int_D \nabla\phi_i\cdot\nabla\phi_k\,d\boldsymbol{x} - \int_D \phi_k\,\nabla a_0\cdot\nabla\phi_i\,d\boldsymbol{x}\right]\left\langle\psi_{\boldsymbol{\iota}(j)}\psi_{\boldsymbol{\iota}(\ell)}\right\rangle u_{i,j}^{(\mathrm{cd})}\\ &-\sum_{m=1}^{M}\sum_{i=1}^{n_{\boldsymbol{x}}}\sum_{j=1}^{n_{\boldsymbol{\xi}}}\int_D \sigma\sqrt{\lambda_m}\phi_k\nabla a_m\cdot\nabla\phi_i\,d\boldsymbol{x}\left\langle\xi_m\psi_{\boldsymbol{\iota}(j)}\psi_{\boldsymbol{\iota}(\ell)}\right\rangle u_{i,j}^{(\mathrm{cd})}\\ &= \sum_{\boldsymbol{\alpha}\in\mathscr{I}_d}\int_D t_{\boldsymbol{\alpha}} f\phi_k\,d\boldsymbol{x}\left\langle\psi_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\iota}(\ell)}\right\rangle \end{aligned} \tag{3.3}$$

for $k = 1,\dots,n_{\boldsymbol{x}}$ and $\ell = 1,\dots,n_{\boldsymbol{\xi}}$. In (3.3), $t_{\boldsymbol{\alpha}} = \left\langle\exp(-a^{(M)})\psi_{\boldsymbol{\alpha}}\right\rangle$, $\boldsymbol{\alpha}\in\mathscr{I}$, denotes the generalized polynomial chaos coefficients of $\exp(-a^{(M)})$. Since $\left\langle\psi_{\boldsymbol{\alpha}}\psi_{\boldsymbol{\iota}(\ell)}\right\rangle = 0$ for $\boldsymbol{\alpha}\in\mathscr{I}\backslash\mathscr{I}_d$, the chaos expansion of $\exp(-a^{(M)})$ above is implicitly truncated. We define the finite element matrices

$$[L]_{i,k} = \int_D \nabla\phi_k \cdot \nabla\phi_i \, d\boldsymbol{x}, \tag{3.4a}$$

$$[N_0]_{i,k} = -\int_D \phi_i \, \nabla a_0 \cdot \nabla\phi_k \, d\boldsymbol{x}, \tag{3.4b}$$

$$[N_m]_{i,k} = -\int_D \sigma\sqrt{\lambda_m}\phi_i \, \nabla a_m \cdot \nabla\phi_k \, d\boldsymbol{x}, \quad i,k = 1,\dots,n_{\boldsymbol{x}}, \; m = 0,\dots,M. \tag{3.4c}$$

The matrix L corresponds to a discretized Laplace operator in the spatial domain, and N_0 and N_m correspond to the convection operators $\nabla a_0\cdot$ and $\sigma\sqrt{\lambda_m}\nabla a_m\cdot$, respectively. In addition, we define the stochastic Galerkin matrices

$$[G_0]_{j,\ell} = \left\langle \psi_{\boldsymbol{\iota}(\ell)}\psi_{\boldsymbol{\iota}(j)} \right\rangle, \tag{3.5}$$

$$[G_m]_{j,\ell} = \left\langle \xi_m\psi_{\boldsymbol{\iota}(\ell)}\psi_{\boldsymbol{\iota}(j)} \right\rangle, \qquad j,\ell = 1,\dots,n_{\boldsymbol{\xi}}, \; m = 1,\dots,M. \tag{3.6}$$

Thus, noting that $G_0 = I$ is the $n_{\boldsymbol{\xi}} \times n_{\boldsymbol{\xi}}$ identity matrix in our setting, the Galerkin matrix associated with the Galerkin equations (3.3) of the stochastic convection-diffusion problem is given by

$$\widehat{C} = I \otimes (L + N_0) + \sum_{m=1}^{M} G_m \otimes N_m. \tag{3.7}$$

We mention that the nonsymmetric matrices N_m, $m = 0,\dots,M$, in (3.4b) and (3.4c) are in general not skew-symmetric, since the terms ∇a_0 and $\sigma\sqrt{\lambda_m}\nabla a_m$ are in general not divergence-free.

3.2. Stability considerations. The convection-diffusion problem in (3.1) can be written as

$$-\sigma^{-1}\Delta u + \widetilde{\boldsymbol{w}} \cdot \nabla u = \sigma^{-1} f e^{-a},$$

where the velocity is

$$\widetilde{\boldsymbol{w}} = -\sigma^{-1}\nabla a = -\sigma^{-1}\nabla a_0 - \sum_{m=1}^{M} \sqrt{\lambda_m}\nabla a_m \xi_m.$$

Thus, for large values of σ, the stochastic convective term dominates the diffusive term $-\sigma^{-1}\Delta u$ as well as the mean convective term $-\sigma^{-1}\nabla a_0 \cdot \nabla u$. In this situation, the matrix $\widehat{C}$ in (3.7) could have qualities like those associated with convection-dominated flow problems, where for large mesh Péclet numbers, stabilization techniques might be required (see [15]).

We examine this question using the benchmark problems from section 5 below, where complete specification of the PDEs and discretizations are given. In particular, we compute two quantities:

$$\mathrm{Pr}_c = P(\omega \in \Omega \colon \|\boldsymbol{w}_c(\omega)\| \le 2/h), \tag{3.8}$$

the probability of the event that the *stochastic element Péclet number* $\|\boldsymbol{w}_c\|h/2$ is less than or equal to one, where $\boldsymbol{w}_c(\omega) = -\nabla a^{(M)}(\boldsymbol{x}_c, \omega)$ denotes the random velocity evaluated at a finite element centroid $\boldsymbol{x}_c \in D$; and the *mean element Péclet number*

$$\bar{P}_c = \sqrt{\langle \|\boldsymbol{w}_c\|^2 h^2/4 \rangle} = (h/2)\left(\|\nabla a_0(\boldsymbol{x}_c)\|^2 + \sigma^2 \sum_{m=1}^{M} \lambda_m \|\nabla a_m(\boldsymbol{x}_c)\|^2 \right)^{1/2}. \tag{3.9}$$

Using bilinear elements and a variety of discretization mesh sizes (see Tables 3.1–3.2), we estimated Pr_c in (3.8) in each element via Monte Carlo integration with 2.5×10^5 samples. We found that $\max\{\mathrm{Pr}_c\} = 1.00$ and $\min\{\mathrm{Pr}_c\} = 1.00$ for all but one combination of n and σ, where $\min\{\mathrm{Pr}_c\} = 0.99$. Table 3.1 shows the maximum values of $\bar{P}_c$ over all elements, for the benchmark Examples 5.1 and 5.2. These two examples differ in the number M of terms in $a^{(M)}$ and correlation length of the covariance function. Table 3.2 shows the analogous quantities for Examples 5.3–5.4. It can be seen that in all of these examples the mean element Péclet number is smaller than one.

Table 3.1
Examples 5.1–5.2: *Maximum of* $\bar{P}_c$ *of* (3.9) *over all elements in an* $n \times n$ *mesh.*

	$M = 5$, $\ell = 1$			$M = 10$, $\ell = 0.5$		
$n = h^{-1}$	$\sigma = 0.1$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 0.1$	$\sigma = 1.0$	$\sigma = 2.0$
16	0.006	0.062	0.124	0.012	0.124	0.247
32	0.003	0.031	0.062	0.006	0.062	0.124
64	0.002	0.015	0.031	0.003	0.031	0.062
128	0.000	0.008	0.015	0.002	0.015	0.031

Table 3.2
Examples 5.3–5.4: *Maximum of* $\bar{P}_c$ *of* (3.9) *over all elements in an* $n \times n$ *mesh.*

	$M = 5$, $\ell = 1$			$M = 10$, $\ell = 0.5$		
$n = h^{-1}$	$\sigma = 0.1$	$\sigma = 1.0$	$\sigma = 2.0$	$\sigma = 0.1$	$\sigma = 1.0$	$\sigma = 2.0$
16	0.607	0.609	0.615	0.607	0.616	0.645
32	0.308	0.309	0.312	0.308	0.312	0.327
64	0.155	0.156	0.157	0.155	0.157	0.164
128	0.078	0.078	0.079	0.078	0.079	0.082

These data suggest that stabilization is not needed for these benchmark problems. In contrast to what happens for flow problems, where flows can be highly convection-dominated and give rise to Péclet numbers of many orders of magnitude in size, here we do not expect the standard deviation σ to be larger than $O(1)$. Since the components of $\{\nabla a_m\}$ are bounded for sufficiently regular covariance functions of a (see [42, Theorem 2.24]), this trend is likely to be representative, and we do not expect stabilization to be required.

We also mention that if stabilization were actually needed, it could be implemented cheaply when linear finite elements are used for the spatial discretization defining X_h in (2.8) and the streamline diffusion methodology (see [15] and the references therein) is applied to the convection-diffusion problem. In this case, the term

$$\left\langle \int_D \delta(\boldsymbol{w} \cdot \nabla\phi_i)(\boldsymbol{w} \cdot \nabla\phi_k)\, \psi_{\boldsymbol{\iota}(j)}\psi_{\boldsymbol{\iota}(\ell)} \right\rangle$$

is added to the left-hand side of the Galerkin equations, where $\boldsymbol{w} = -\nabla a^{(M)}$ denotes the stochastic velocity. If the stabilization parameter $\delta > 0$ is a deterministic constant, then the streamline diffusion method yields a *stochastically quadratic* problem formulation, since $\boldsymbol{w}$ depends linearly on a fixed number of independent truncated Gaussian random variables. It is easy to see that the associated Galerkin matrix has at most $2M^2 + 2M + 1$ nonzero blocks per row, and thus it is block sparse (hence sparse) for $M^2 \ll n_{\boldsymbol{\xi}}$.

3.3. Expansion of $\nabla \boldsymbol{a}$. The convection-diffusion formulation requires ∇a. To this end, we differentiate the truncated KL expansion (2.4) of a and arrive at

$$\nabla a^{(M)}(\boldsymbol{x},\omega) = \nabla a_0 + \sigma \sum_{m=1}^{M} \sqrt{\lambda_m} \nabla a_m(\boldsymbol{x}) \xi_m(\omega).$$

This allows us to formulate the weak version of the stochastic convection-diffusion problem in terms of the exact same random variables that appear in the original KL expansion of a.

The gradient of a_m can be computed using the integral eigenproblem equation

$$\int_D c(\boldsymbol{x},\boldsymbol{y}) a_m(\boldsymbol{y})\, d\boldsymbol{y} = \lambda_m a_m(\boldsymbol{x}). \tag{3.10}$$

Assume that a is continuously differentiable in the mean-square sense; necessary and sufficient for this is the existence and continuity of ∇a_0 and the partial derivatives $(\partial^2/\partial x_j \partial y_k) c(\boldsymbol{x},\boldsymbol{y})$, $\boldsymbol{x},\boldsymbol{y} \in D$, $j,k = 1,2$ (see, e.g., [12, Chapter 2]). Then, by Lebesgue's dominated convergence theorem (see, e.g., [43, Chapter 2]), we can interchange the order of differentiation and integration in (3.10) and obtain

$$\frac{\partial a_m}{\partial x_i}(\boldsymbol{x}) = \lambda_m^{-1} \int_D \frac{\partial c}{\partial x_i}(\boldsymbol{x},\boldsymbol{y}) a_m(\boldsymbol{y})\, d\boldsymbol{y}, \qquad i = 1,2. \tag{3.11}$$

In the experiments in section 5, we consider random fields with Gaussian covariance function

$$\mathrm{Cov}(\boldsymbol{x},\boldsymbol{y}) = \sigma^2 \exp(-(r/\ell)^2), \quad r = \|\boldsymbol{x}-\boldsymbol{y}\|_2,$$

which satisfies the assumptions above. Other admissible covariance functions can be found in the Matérn family (see [45, section 2.7]) with smoothness parameter $\nu > 1$.

The assembly of the finite element matrices $N_0, N_1, \ldots, N_M$ in (3.4b) and (3.4c) requires the evaluation of (3.11) at certain quadrature nodes $\{\boldsymbol{x}_\ell\} \in D$. We discretize the action of the integral operator on the right-hand side of (3.11) by a Galerkin projection onto the subspace Z_h of piecewise constant functions on a given triangulation $\mathscr{T}_h$ of D. Let $Z_h = \mathrm{span}\{\pi_1(\boldsymbol{x}), \ldots, \pi_n(\boldsymbol{x})\} \subset L^2(D)$. With $\partial a_m(\boldsymbol{x})/\partial x_i \approx \sum_j a_{m,j}^{(i)} \pi_j(\boldsymbol{x})$, $i = 1,2$, and $a_m(\boldsymbol{x}) \approx \sum_j a_{m,j}^{(0)} \pi_j(\boldsymbol{x})$, it follows that the n coefficients that determine the partial derivative of an (approximate) eigenfunction $a_m \in Z_h$ satisfy the Galerkin equations

$$\sum_{j=1}^{n} a_{m,j}^{(i)} \int_D \pi_j(\boldsymbol{x}) \pi_k(\boldsymbol{x})\, d\boldsymbol{x} = \lambda_m^{-1} \sum_{j=1}^{n} a_{m,j}^{(0)} \int_D \int_D \frac{\partial c}{\partial x_i}(\boldsymbol{x},\boldsymbol{y}) \pi_j(\boldsymbol{y}) \pi_k(\boldsymbol{x})\, d\boldsymbol{y}\, d\boldsymbol{x}$$

for $k = 1, \ldots, n$ and $i = 1,2$. The matrix formulation of these equations reads

$$Q\, \boldsymbol{a}_m^{(i)} = \lambda_m^{-1} K\, \boldsymbol{a}_m^{(0)}, \tag{3.12}$$

where $Q \in \mathbb{R}^{n \times n}$ denotes the Gramian matrix of $\{\pi_1, \ldots, \pi_n\}$ with respect to the $L^2(D)$ inner product, $K \in \mathbb{R}^{n \times n}$ is a discrete integral operator with kernel function $\partial c/\partial x_i$, the vector $\boldsymbol{a}_m^{(0)} \in \mathbb{R}^n$ contains the coefficients of the KL eigenfunction a_m, and the vectors $\boldsymbol{a}_m^{(i)} \in \mathbb{R}^n$ contain the coefficients of $\partial a_m/\partial x_i$, $i = 1,2$, respectively. The action of Q^{-1} can be computed in $O(n)$ operations since Q is a diagonal matrix in this setting. However, the matrix K in (3.12) is in general a dense $n \times n$ matrix. Therefore we approximate K by a hierarchical $\mathscr{H}^2$-matrix (see [6, 8, 27] and the references therein), which allows us to perform matrix-vector products with K in $O(n)$ operations. The assembly of an $\mathscr{H}^2$-matrix costs $O(n \log n)$ operations. The vector $\boldsymbol{a}_m^{(0)}$ solves a discrete version of the KL integral eigenproblem (3.10). An efficient way to compute approximate KL eigenpairs is outlined in [14], where hierarchical matrix techniques are combined with a thick-restart Lanczos method. This approach requires $O(n \log n)$ operations. In summary, then, approximations to the gradients ∇a_m of KL eigenfunctions can be obtained with $O(n \log n)$ complexity.

An alternative way to obtain a stochastically linear expansion of ∇a is by direct computation of the KL expansion of the vector-valued random field ∇a (see [39, Chapter 10]). An algorithm for computing the vector KL eigenpairs with $O(n \log n)$ complexity is presented in [11]. However, the (marginal and joint) densities of the random variables in such a KL expansion are in general not known a priori and must be inferred from the probability law of ∇a.

The fundamental difference between the two expansion strategies lies in the order of truncation and differentiation. In the first approach, we truncate the KL expansion of a and then compute the gradient $\nabla a^{(M)}$. In the second approach, the KL expansion of ∇a is computed and then truncated. We used the first approach in the computations described below. We mention that for mean-square continuously differentiable a its KL expansion can be differentiated termwise and $\nabla a^{(M)}$ converges to ∇a in the mean-square sense uniformly on D for $M \to \infty$ (see [30][1]).

3.4. Piecewise smooth coefficient $\exp(a)$. The convection-diffusion formulation (3.1) requires a well-defined gradient ∇a. However, it can happen that the diffusion coefficient $\exp(a)$ is well-defined but the gradient ∇a does not exist. Such situations occur, for example, for piecewise constant mean values $\langle a \rangle$. This issue can be addressed as follows in the case where the log-transformed diffusion coefficient a has the form $a(\boldsymbol{x}, \omega) = \widehat{a}(\omega) + \widetilde{a}(\boldsymbol{x}, \omega)$, where $\widehat{a}$ is a random variable and $\exp(-\widetilde{a})$ and $\nabla \widetilde{a}$ exist for $(\boldsymbol{x}, \omega) \in D \times \Omega$ a.e. Then we can write $\exp(a) = \exp(\widehat{a}) \exp(\widetilde{a})$, and, following the ideas in section 1, we obtain

$$\begin{aligned} -\exp(-\widetilde{a}) \nabla \cdot (\exp(a) \nabla u) &= -\exp(-\widetilde{a}) \exp(\widehat{a}) \nabla \cdot (\exp(\widetilde{a}) \nabla u) \\ &= -\exp(\widehat{a}) \Delta u - \exp(-\widetilde{a}) \exp(\widehat{a}) \nabla \exp(\widetilde{a}) \cdot \nabla u \\ &= -\exp(\widehat{a}) \Delta u - \exp(\widehat{a}) \nabla \widetilde{a} \cdot \nabla u. \end{aligned}$$

The associated weak formulation is as follows: Find $\boldsymbol{u} \in H_0^1(D) \otimes L_\rho^2(\Gamma)$ such that

$$\int_\Gamma \int_D \exp(\widehat{a}) \nabla u \cdot \nabla v \, \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} + \int_\Gamma \int_D \exp(\widehat{a}) \, v \, \boldsymbol{w} \cdot \nabla u \, \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi} = \int_\Gamma \int_D f e^{-\widetilde{a}} v \, \rho \, d\boldsymbol{x} \, d\boldsymbol{\xi}$$

for all test functions $v \in H_0^1(D) \otimes L_\rho^2(\Gamma)$. The velocity is $\boldsymbol{w} = -\nabla \widetilde{a}$. In this problem $\widehat{a}$ is assumed random and independent of $\boldsymbol{x} \in D$; the formulation above is also meaningful for deterministic, piecewise constant $\widehat{a}$.

4. Iterative solvers for the convection-diffusion formulation. We now consider the iterative solution of the discretized stochastic Galerkin convection-diffusion equations of (3.3). Crucially, the associated Galerkin matrix (3.7) discretizes a

[1]This result is proved in [30, Theorem A.2] for random functions $a : D \times \Omega \to \mathbb{R}$ defined on $D \subset \mathbb{R}$, but it carries over to random fields defined on $D \subset \mathbb{R}^d$ with $d > 1$.

stochastically linear problem. Thus, it is block-sparse with at most $2M+1$ nonzero blocks per row [36]. Moreover, each block

$$\left\langle \psi_{\boldsymbol{\iota}(\ell)}\psi_{\boldsymbol{\iota}(j)} \right\rangle (L+N_0) + \sum_{m=1}^{M} \left\langle \xi_m \psi_{\boldsymbol{\iota}(\ell)}\psi_{\boldsymbol{\iota}(j)} \right\rangle N_m$$

is a sparse $n_{\boldsymbol{x}} \times n_{\boldsymbol{x}}$ matrix. (The matrices L and $N_0, N_1, \ldots, N_M$ all have the same sparsity pattern.) Therefore, matrix-vector products with $\widehat{C}$ can be performed in $O(n_{\boldsymbol{x}} n_{\boldsymbol{\xi}})$ operations for $M \ll n_{\boldsymbol{\xi}}$, which is optimal in terms of the total number of unknowns. To solve (3.3), Krylov subspace methods such as GMRES [41] designed for nonsymmetric systems must be used, and it is critical that preconditioners be used to achieve fast convergence.

4.1. Symmetric part of the Galerkin matrix. Before turning to preconditioning approaches we identify the symmetric part of the Galerkin matrix $\widehat{C}$ in (3.7),

$$\widehat{H} = \frac{1}{2}(\widehat{C} + \widehat{C}^\top) = I \otimes L + I \otimes \frac{1}{2}(N_0 + N_0^\top) + \sum_{m=1}^{M} G_m \otimes \frac{1}{2}(N_m + N_m^\top).$$

To explore the symmetric part of the matrices $N_0, \ldots, N_M$, we utilize the relation

$$\nabla \cdot (\boldsymbol{r} s\, t) = \boldsymbol{r} \cdot \nabla(s\, t) + s\, t \nabla \cdot \boldsymbol{r} = t \boldsymbol{r} \cdot \nabla s + s \boldsymbol{r} \cdot \nabla t + s\, t \nabla \cdot \boldsymbol{r}.$$

Then, letting $s = \phi_i$, $t = \phi_k$ for $\phi_i, \phi_k \in X_h \subset H_0^1(D)$, and integrating over the domain D, we obtain

$$\int_{\partial D_N} \boldsymbol{n} \cdot \boldsymbol{r} \phi_i \phi_k \, ds = \int_D \nabla \cdot \boldsymbol{r} \phi_i \phi_k \, d\boldsymbol{x} + \int_D \phi_k \boldsymbol{r} \cdot \nabla \phi_i \, d\boldsymbol{x} + \int_D \phi_i \boldsymbol{r} \cdot \nabla \phi_k \, d\boldsymbol{x}.$$

By choosing $\boldsymbol{r} = -\nabla a_0$ and $\boldsymbol{r} = -\sigma\sqrt{\lambda_m}\nabla a_m$, respectively, we arrive at $\frac{1}{2}(N_m + N_m^\top) = H_m$, where the symmetric matrices H_m are defined as

$$[H_0]_{i,k} = \frac{1}{2}\left(\int_D \phi_i\, \phi_k\, \Delta a_0 \, d\boldsymbol{x} - \int_{\partial D_N} \phi_i \phi_k \boldsymbol{n} \cdot \nabla a_0 \, ds \right), \tag{4.1a}$$

$$[H_m]_{i,k} = \frac{1}{2}\sigma\sqrt{\lambda_m}\left(\int_D \phi_i\, \phi_k\, \Delta a_m \, d\boldsymbol{x} - \int_{\partial D_N} \phi_i \phi_k \boldsymbol{n} \cdot \nabla a_m \, ds \right), \tag{4.1b}$$

$$i, k = 1, \ldots, n_{\boldsymbol{x}}, \; m = 1, \ldots, M.$$

In summary, then, the symmetric part $\widehat{H}$ of the Galerkin matrix $\widehat{C}$ reads

$$\widehat{H} = I \otimes (L + H_0) + \sum_{m=1}^{M} G_m \otimes H_m. \tag{4.2}$$

4.2. Diffusion preconditioner. First, we consider the matrix $\widehat{P}_L = I \otimes L$ as a preconditioner for the stochastic convection-diffusion Galerkin matrix $\widehat{C}$ in (3.7). We note that $\widehat{P}_L$ is a symmetric positive-definite block-diagonal matrix representing only the deterministic diffusive part of the stochastic convection-diffusion operator. We are interested in bounds on the generalized field of values

$$\mathrm{W}(\widehat{C}, \widehat{P}_L) = \left\{ \frac{\boldsymbol{v}^H \widehat{C} \boldsymbol{v}}{\boldsymbol{v}^H \widehat{P}_L \boldsymbol{v}} : \boldsymbol{v} \in \mathbb{C}^{n_{\boldsymbol{x}} n_{\boldsymbol{\xi}}}, \; \boldsymbol{v} \neq \boldsymbol{0} \right\}.$$

THEOREM 4.1. *The generalized field of values of the Galerkin matrix* $\widehat{C}$ *in* (3.7) *with respect to the preconditioner* $\widehat{P}_L = I \otimes L$ *is contained in the circle*

$$\{z \in \mathbb{C}\colon |z-1| \le 2\, c_D\, \delta_L\}, \qquad \delta_L = \|\nabla a_0\|_\infty + \sigma \nu_{d+1} \sum_{m=1}^{M} \sqrt{\lambda_m} \|\nabla a_m\|_\infty, \tag{4.3}$$

where $\|\boldsymbol{w}\|_\infty = \sup_{\boldsymbol{x}\in D} |\boldsymbol{w}(\boldsymbol{x})|$, ν_{d+1} *denotes the largest root of the univariate, orthonormal Rys polynomial of exact degree* $d+1$, *and* $c_D > 0$ *is a constant independent of* h, σ, *and* d.

Proof. Observe that $\widehat{C} = I \otimes L + \sum_{m=0}^{M} G_m \otimes N_m = \widehat{P}_L + \widehat{N}$, and hence

$$\mathrm{W}(\widehat{C}, \widehat{P}_L) = 1 + \mathrm{W}(\widehat{N}, \widehat{P}_L).$$

We enclose $\mathrm{W}(\widehat{N}, \widehat{P}_L)$ in a circle, which must be shifted by $+1$ in order to enclose $\mathrm{W}(\widehat{C}, \widehat{P}_L)$. To this end, we recall that the field of values is subadditive, i.e., $\mathrm{W}(A+B) \subset \mathrm{W}(A) + \mathrm{W}(B)$ for matrices A, $B \in \mathbb{C}^n$, where the set sum contains sums of all possible pairs [29, Property 1.2.7]. Furthermore, if the matrix A is normal, then $\mathrm{W}(A \otimes B) = \mathrm{conv}(\mathrm{W}(A) \cdot \mathrm{W}(B))$ the convex hull of the product of all possible pairs from $\mathrm{W}(A)$ and $\mathrm{W}(B)$; see [29, Theorem 4.2.16]. Again, if A is normal, $\mathrm{W}(A) = \mathrm{conv}(\lambda(A))$ is the convex hull of the spectrum of A. Thus, if A is symmetric, then $\mathrm{W}(A)$ is a closed real line segment whose endpoints are the smallest and largest eigenvalue of A, respectively. Therefore, the generalized field of values $\mathrm{W}(\widehat{N}, \widehat{P}_L)$ can be bounded as follows:

$$\mathrm{W}(\widehat{N}, \widehat{P}_L) \subset \sum_{m=0}^{M} \mathrm{W}(G_m \otimes N_m, I \otimes L) = \sum_{m=0}^{M} \mathrm{conv}(\mathrm{conv}(\lambda(G_m)), \mathrm{W}(N_m, L)). \tag{4.4}$$

Next we wish to bound quantities $|z|$, where $z \in \mathrm{W}(N_m, L)$, $m = 0, 1, \ldots, M$. To this end, we establish a bound for finite element shape functions u, $v \in X_h \subset H_0^1(D)$. We utilize the Cauchy–Schwarz inequality and Friedrich's inequality (see, e.g., [9]) and arrive at

$$\begin{aligned}\left|\int_D v\boldsymbol{w} \cdot \nabla u \, d\boldsymbol{x}\right| &\le \int_D |v| (w_x^2 + w_y^2)^{1/2} \left((\partial u/\partial x)^2 + (\partial u/\partial y)^2\right)^{1/2} \, d\boldsymbol{x} \\ &\le \|\boldsymbol{w}\|_\infty \left(\int_D v^2 \, d\boldsymbol{x}\right)^{1/2} \left(\int_D (\partial u/\partial x)^2 + (\partial u/\partial y)^2 \, d\boldsymbol{x}\right)^{1/2} \\ &= \|\boldsymbol{w}\|_\infty \|v\| \, \|\nabla u\| \le \|\boldsymbol{w}\|_\infty c_D \|\nabla v\| \|\nabla u\|,\end{aligned}$$

where we have introduced $\|\boldsymbol{w}\|_\infty = \sup_{\boldsymbol{x}\in D} |\boldsymbol{w}(\boldsymbol{x})|$. The constant $c_D > 0$ appears in Friedrich's inequality and is independent of the characteristic mesh size h. Now, we wish to translate this bound into matrix notation. For a vector $\boldsymbol{v} \in \mathbb{R}^{n_x}$ we define the corresponding finite element shape function $v \in X_h$ by $v(\boldsymbol{x}) = \sum_i v_i \phi_i(\boldsymbol{x})$. Then the above estimate reads

$$|\boldsymbol{v}^\top N_m \boldsymbol{u}| \le \|\boldsymbol{w}_m\|_\infty c_D (\boldsymbol{v}^\top L \boldsymbol{v})^{1/2} (\boldsymbol{u}^\top L \boldsymbol{u})^{1/2}, \qquad m = 0, 1, \ldots, M, \tag{4.5}$$

where we have defined $\boldsymbol{w}_0 = -\nabla a_0$, and $\boldsymbol{w}_m = -\sigma\sqrt{\lambda_m} \nabla a_m$, $m = 1, \ldots, M$. Now, for $z \in \mathrm{W}(N_m, L)$ there is a vector $\boldsymbol{w} \in \mathbb{C}^{n_x} \backslash \{\boldsymbol{0}\}$, such that $z = \boldsymbol{w}^H N_m \boldsymbol{w} / \boldsymbol{w}^H L \boldsymbol{w}$.

We decompose $\boldsymbol{w} = \boldsymbol{u} + i\boldsymbol{v}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^{n_x}$, and, utilizing the estimate (4.5), we obtain

$$\begin{aligned} |\boldsymbol{w}^H N_m \boldsymbol{w}| &\le |\boldsymbol{u}^\top N_m \boldsymbol{u}| + |\boldsymbol{v}^\top N_m \boldsymbol{v}| + |\boldsymbol{u}^\top N_m \boldsymbol{v}| + |\boldsymbol{v}^\top N_m \boldsymbol{v}| \\ &\le \|\boldsymbol{w}_m\|_\infty c_D \left(\boldsymbol{u}^\top L \boldsymbol{u} + \boldsymbol{v}^\top L \boldsymbol{v} + 2(\boldsymbol{u}^\top L \boldsymbol{u})^{1/2} (\boldsymbol{v}^\top L \boldsymbol{v})^{1/2} \right) \\ &\le \|\boldsymbol{w}_m\|_\infty c_D \left(\boldsymbol{u}^\top L \boldsymbol{u} + \boldsymbol{v}^\top L \boldsymbol{v} + \boldsymbol{u}^\top L \boldsymbol{u} + \boldsymbol{v}^\top L \boldsymbol{v} \right) \\ &= 2\|\boldsymbol{w}_m\|_\infty c_D (\boldsymbol{u}^\top L \boldsymbol{u} + \boldsymbol{v}^\top L \boldsymbol{v}) = 2\|\boldsymbol{w}_m\|_\infty c_D\, \boldsymbol{w}^H L \boldsymbol{w}. \end{aligned}$$

Hence, for $z \in \mathrm{W}(N_m, L)$, $m = 0, 1, \dots, M$, we arrive at the estimate

$$|z| \le 2\|\boldsymbol{w}_m\|_\infty c_D. \tag{4.6}$$

Finally, it is well known that $\lambda(G_m) \subseteq [-\nu_{d+1}, \nu_{d+1}]$, $m = 1, \dots, M$, where ν_{d+1} denotes the largest root of the univariate orthonormal Rys polynomial of degree $d+1$ (see, e.g., [17]). Thus, combining (4.4) with (4.6) and the estimate on the spectral interval of G_m, we have arrived at

$$|z| \le 2c_D \left(\|\boldsymbol{w}_0\|_\infty + \sigma \nu_{d+1} \sum_{m=1}^{M} \|\boldsymbol{w}_m\|_\infty \right)$$

for $z \in \mathrm{W}(\widehat{N}, \widehat{P}_L)$, which completes the proof. ▯

Remark 4.2. The components of ∇a_m, $m = 0, \dots, M$, are bounded for sufficiently regular covariance functions of a; see again [42, Theorem 2.24].

From Theorem 4.1 we conclude that the field of values of $\widehat{C}$ preconditioned by $\widehat{P}_L$ is insensitive to the characteristic mesh size h. For sufficiently small values of σ, the generalized field of values $\mathrm{W}(\widehat{C}, \widehat{P}_L)$ is tightly clustered around $z = 1$ and does not contain the origin; thus we can expect fast convergence of GMRES (see, e.g., [25]). However, the radius of the circle around the generalized field of values increases when σ increases. In addition, since the zeros of ψ_{d+1} alternate with those of ψ_{d+2} [21, Theorem 1.20] we have $\nu_{d+1} < \nu_{d+2}$ for two consecutive total degrees d. Fortunately, all zeros of the chaos polynomials are located in the interior of the support interval of the corresponding density [21, Theorem 1.19], meaning that $\nu_{d+1} \le c$ in our setting, where c is the cut-off parameter in (2.5). Thus, the impact of d on the GMRES convergence behavior is limited. In summary, the GMRES convergence behavior is determined by σ and d and may deteriorate for large values of σ.

4.3. Mean-based preconditioner. Next, we choose the matrix $\widehat{P}_0 = I \otimes (L + N_0)$ as preconditioner for the stochastic convection-diffusion Galerkin matrix $\widehat{C}$ in (3.7). In particular, $\widehat{P}_0$ contains a convective term when $\nabla \langle a \rangle \ne 0$, and it is identical to the diffusion preconditioner $\widehat{P}_L$ when $\nabla \langle a \rangle \equiv 0$. In general, $\widehat{P}_0$ is not symmetric and may not be positive-definite.

THEOREM 4.3. *Assume N_0 is a positive-semidefinite matrix. Then the eigenvalues of the Galerkin matrix $\widehat{C}$ in* (3.7) *preconditioned by $\widehat{P}_0 = I \otimes (L + N_0)$ (from the left- or right-hand side) are contained in the circle*

$$\{z \in \mathbb{C} \colon |z - 1| \le 2\, c_D\, \delta_0\}, \qquad \delta_0 = \sigma \nu_{d+1} \sum_{m=1}^{M} \sqrt{\lambda_m} \|\nabla a_m\|_\infty, \tag{4.7}$$

where $\|\boldsymbol{w}\|_\infty = \sup_{\boldsymbol{x} \in D} |\boldsymbol{w}(\boldsymbol{x})|$, ν_{d+1} denotes the largest root of the univariate, orthonormal Rys polynomial of exact degree $d+1$, and $c_D > 0$ is a constant independent of h, σ, and d.

Proof. Given the assumptions stated in the theorem, observe that the matrix $L+N_0$ is positive-definite, and hence so is $\widehat{P}_0 = I \otimes (L+N_0)$. Now, let $\boldsymbol{w} \in \mathbb{C}^{n_x n_{\boldsymbol{\xi}}} \setminus \{\boldsymbol{0}\}$ and $\lambda \in \mathbb{C}$ denote an eigenvector and the corresponding eigenvalue of the matrix $\widehat{P}_0^{-1}\widehat{C}$, respectively. Then the Rayleigh quotient reads

$$\lambda = \frac{\boldsymbol{w}^H \widehat{C} \boldsymbol{w}}{\boldsymbol{w}^H \widehat{P}_0 \boldsymbol{w}} = \frac{\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w} + \sum_{m=1}^M \boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w}}{\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w}}$$
$$= 1 + \sum_{m=1}^M \frac{\boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w}}{\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w}}.$$

Hence we obtain the estimate

$$|\lambda - 1| \le \sum_{m=1}^M \left| \frac{\boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w}}{\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w}} \right|. \tag{4.8}$$

Next, we wish to bound the quantities $|\boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w}|$, $m = 1, \dots, M$, in terms of $|\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w}|$. By linearity and the properties of the Kronecker product it suffices to establish such a bound for vectors $\boldsymbol{w} = \boldsymbol{w}_\ell \otimes \boldsymbol{w}_r$, where $\boldsymbol{w}_\ell \in \mathbb{C}^{n_{\boldsymbol{\xi}}}$ and $\boldsymbol{w}_r \in \mathbb{C}^{n_x}$. Then $\boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w} = (\boldsymbol{w}_\ell^H G_m \boldsymbol{w}_\ell)(\boldsymbol{w}_r^H N_m \boldsymbol{w}_r)$. We proceed by bounding the factors in this expression separately.

It was observed in the proof of Theorem 4.1 that the spectrum of G_m is contained in the interval $[-\nu_{d+1}, \nu_{d+1}]$. Thus, we arrive at $\boldsymbol{w}_\ell^H G_m \boldsymbol{w}_\ell \le \nu_{d+1} \boldsymbol{w}_\ell^H \boldsymbol{w}_\ell$. Furthermore, in the proof of Theorem 4.1 we have established the bound

$$|\boldsymbol{w}_r^H N_m \boldsymbol{w}_r| \le 2\sigma\sqrt{\lambda_m}\|\nabla a_m\|_\infty c_D \boldsymbol{w}_r^H L \boldsymbol{w}_r, \quad m = 1, \dots, M, \tag{4.9}$$

where $c_D > 0$ is a constant independent of the characteristic mesh size h. Now, let $\boldsymbol{w}_r = \boldsymbol{u} + i\boldsymbol{v}$, $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^{n_x}$. We write $N_0 = H_0 + S_0$, where $H_0 = (N_0 + N_0^\top)/2$ and $S_0 = (N_0 - N_0^\top)/2$. That is, we split the matrix N_0 into its symmetric part H_0 and skew-symmetric part S_0, respectively. Then

$$\boldsymbol{w}_r^H (L+N_0)\boldsymbol{w}_r = \boldsymbol{u}^\top (L+H_0)\boldsymbol{u} + \boldsymbol{v}^\top (L+H_0)\boldsymbol{v} + 2i\boldsymbol{u}^\top S_0 \boldsymbol{v}.$$

Recall that the matrix H_0 is positive-semidefinite by assumption. (L is positive-definite.) Moreover, for $z = a + ib \in \mathbb{C}$, there holds $|z| = \sqrt{a^2+b^2} \ge |a|$. Thus, the bound

$$|\boldsymbol{w}_r^H (L+N_0)\boldsymbol{w}_r| \ge \boldsymbol{u}^\top (L+H_0)\boldsymbol{u} + \boldsymbol{v}^\top (L+H_0)\boldsymbol{v} \ge \boldsymbol{u}^\top L \boldsymbol{u} + \boldsymbol{v}^\top L \boldsymbol{v} = \boldsymbol{w}_r^H L \boldsymbol{w}_r$$

follows. Combining this estimate and (4.9), we obtain the bound

$$|\boldsymbol{w}_r^H N_m \boldsymbol{w}_r| \le 2\sigma\sqrt{\lambda_m}\|\nabla a_m\|_\infty c_D |\boldsymbol{w}_r^H (L+N_0)\boldsymbol{w}_r|, \quad m = 1, \dots, M.$$

Finally, we arrive at

$$\begin{aligned} |\boldsymbol{w}^H G_m \otimes N_m \boldsymbol{w}| &= |\boldsymbol{w}_\ell^H G_m \boldsymbol{w}_\ell||\boldsymbol{w}_r^H N_m \boldsymbol{w}_r| \\ &\le \nu_{d+1} \boldsymbol{w}_\ell^H \boldsymbol{w}_\ell \, 2\sigma\sqrt{\lambda_m}\|\nabla a_m\|_\infty c_D |\boldsymbol{w}_r^H (L+N_0)\boldsymbol{w}_r| \\ &= 2c_D\sigma\nu_{d+1}\sqrt{\lambda_m}\|\nabla a_m\|_\infty |\boldsymbol{w}^H I \otimes (L+N_0)\boldsymbol{w}|, \quad m = 1, \dots, M. \end{aligned}$$

Combining these bounds with (4.8) completes the proof. ∎

Remark 4.4. The convection matrix N_0 defined in (3.4b) is positive-semidefinite iff the matrix H_0 in (4.1a) is positive-semidefinite. The latter condition is satisfied if $\Delta a_0 \geq 0$ on D and $\boldsymbol{n} \cdot a_0 = 0$ along ∂D_N, for example.

Remark 4.5. The statement of Theorem 4.3 can be generalized by assuming that there exists a constant ϵ, $0 \leq \epsilon < 1$, such that $\boldsymbol{w}^\top H_0 \boldsymbol{w} \geq -\epsilon \boldsymbol{w}^\top L \boldsymbol{w}$ for all vectors $\boldsymbol{w} \in \mathbb{R}^{n_x}$. Then the eigenvalues of $\widehat{C}$ preconditioned by $\widehat{P}_0$ are contained in the circle $\{z \in \mathbb{C}\colon |z-1| \leq 2\, c_D\, \delta_0 (1-\epsilon)^{-1}\}$, where δ_0 is defined in (4.7). Clearly, if the matrix N_0 (that is, H_0) is positive-semidefinite, we can choose $\epsilon = 0$ and the assertion of Theorem 4.3 follows.

We note that the spectral inclusion regions in (4.3) and (4.7) coincide for $\nabla a_0 \equiv 0$, since $\widehat{P}_L = \widehat{P}_0$ in this case. In general, however, if the mean value a_0 of the log-transformed diffusion coefficient is such that $\nabla a_0 \not\equiv 0$, it follows that $\delta_0 < \delta_L$. That is, the spectral inclusion bounds for the matrix $\widehat{P}_0^{-1}\widehat{C}$ are tighter than the bounds for $\widehat{P}_L^{-1}\widehat{C}$, and we expect the mean-based preconditioner to outperform the diffusion preconditioner. This will be explored in section 5.

Theorem 4.3 tells us that the spectral inclusion bounds for the matrix $\widehat{C}$ preconditioned by $\widehat{P}_0$ are insensitive to the characteristic mesh size h. For sufficiently small values of σ, the eigenvalues of $\widehat{P}_0^{-1}\widehat{C}$ are tightly clustered around $z = 1$. The radius of the circle containing the eigenvalues increases when σ and/or d increase. However, the impact of d is limited since $\nu_{d+1} \leq c$ in our setting (see the discussion at the end of section 4.2).

Remark 4.6. The results in Theorems 4.1 and 4.3 can be generalized as follows. First, the inclusion regions in (4.3) and (4.7) hold for any stochastic Galerkin convection-diffusion discretization as long as the velocity $\boldsymbol{w}$ depends linearly on a finite number M of independent truncated Gaussian random variables. Second, if $\boldsymbol{w}$ can be represented in terms of M independent, identically distributed, not necessarily truncated Gaussian, basic random variables $\{\xi_m\}_{m=1}^M$ with probability density function ρ_m, then the spectrum of the matrices G_m, $m = 1, \ldots, M$, is contained in the interval $[\theta_{d+1}, \Theta_{d+1}]$, where θ_{d+1} and Θ_{d+1} denote the smallest and largest root of the univariate orthonormal polynomial of degree $d+1$ generated by ρ_m (see, e.g., [17]). Hence the quantity ν_{d+1} in (4.3) and (4.7) must be replaced by $\max\{|\theta_{d+1}|, |\Theta_{d+1}|\}$ for other basic random variables.

4.4. Practical preconditioners. The application of the proposed preconditioners $\widehat{P}_L$ and $\widehat{P}_0$ in conjunction with Krylov subspace methods requires the solution of $n_{\boldsymbol{\xi}}$ linear systems with the sparse coefficient matrices L and $L + N_0$, respectively, in each iteration.

These operations can be done efficiently using geometric or algebraic multigrid methods; see, e.g., [10]. Moreover, these operations can be replaced with "approximate solves" in which the action of L^{-1} or $(L + N_0)^{-1}$ is replaced by application of a small number of multigrid steps. In experiments described below, the actions of L^{-1} and $(L + N_0)^{-1}$ are replaced by application of one V-cycle of an algebraic multigrid method (AMG) (see [44]). We denote these preconditioners by $\widehat{P}_{L,amg}$ and $\widehat{P}_{0,amg}$, respectively. One V-cycle costs $O(n_{\boldsymbol{x}})$ operations, and thus one application of $\widehat{P}_{L,amg}$ and $\widehat{P}_{0,amg}$ incurs a computational cost of $O(n_{\boldsymbol{x}} n_{\boldsymbol{\xi}})$, which is essentially as cheap as a matrix-vector product with the Galerkin matrix $\widehat{C}$.

5. Numerical experiments. The experimental setting is as follows.

EXAMPLE 5.1. *We consider the diffusion problem* (2.1) *and the associated convection-diffusion formulation* (3.1) *on the unit square domain* $D = (0,1) \times (0,1)$ *with* *zero source term $f \equiv 0$. The northern and eastern boundaries are no-flow boundaries, that is, $\boldsymbol{n} \cdot \nabla u = 0$. We impose $u = 0$ along the southern boundary, and $u = 1$ along the western boundary, respectively. The log-transformed diffusion coefficient has constant mean value $a_0 = 1$, and covariance function*

$$\mathrm{Cov}(\boldsymbol{x}, \boldsymbol{y}) = \sigma^2 \exp(-(r/\ell)^2), \quad r = \|\boldsymbol{x} - \boldsymbol{y}\|_2,$$

where $\ell > 0$ denotes the correlation length. With this choice of mean value and covariance function, a is mean-square continuously differentiable. In this example, we set $\ell = 1$. The physical discretization uses $n \times n$ square bilinear finite elements on D. For the stochastic discretization we employ complete chaos polynomials in $M = 5$ truncated Gaussian random variables with cut-off parameter $c = 2.575$ (see (2.5)*) and capture more than 98% of the total variance of a, which is $\int_D \mathrm{Cov}(\boldsymbol{x}, \boldsymbol{x})\, d\boldsymbol{x} = \sigma^2 |D|$ in this setting.*

EXAMPLE 5.2. *The problem is as in Example* 5.1*, but with a smaller correlation length $\ell = 0.5$ and $M = 10$ truncated Gaussian random variables in the expansion of a. Again, we capture more than 98% of the total variance of a.*

EXAMPLE 5.3. *The problem is as in Example* 5.1*, the exceptions being the boundary conditions and mean value of a. We use $a_0 = 1 + 10x^2$; thus $\Delta a_0 = 20 \geq 0$. This time, the northern and southern boundaries are no-flow boundaries. Note that $\boldsymbol{n} \cdot \nabla a_0 = 0$ along these boundaries. In addition, we impose $u = 1$ along the western boundary, and $u = 0$ along the eastern boundary. With this choice of a_0 and boundary conditions, the assumptions of Theorem* 4.3 *hold; see Remark* 4.4*.*

EXAMPLE 5.4. *The problem is as in Example* 5.3 *with a smaller correlation length $\ell = 0.5$ and $M = 10$ random variables.*

In our implementation, we utilize software based on the IFISS package [44] for the physical finite element discretization and the AMG V-cycle as part of the practical preconditioners in section 4.4, respectively. The gradient $\nabla a^{(M)}$ is computed as described in section 3.3 with software based upon the HLib package [7].

5.1. Preconditioned GMRES. We solve the discretized convection-diffusion Galerkin equations (3.3) for Examples 5.1–5.4. We explore the performance of (right preconditioned) GMRES in conjunction with the diffusion preconditioner $\widehat{P}_L$ introduced in section 4.2, the mean-based preconditioner $\widehat{P}_0$ (section 4.3), and their practical versions (section 4.4). The stopping criterion is $\|r_k\|_2 < 10^{-8}\|b\|_2$ in all experiments, where for a linear system $Ax = b$ and kth approximate solution x_k, r_k is the residual $b - Ax_k$.

5.1.1. Diffusion preconditioner. We start with Examples 5.1 and 5.2 and employ the preconditioner $\widehat{P}_L = I \otimes L$. This is also the mean-based preconditioner $\widehat{P}_0$, since $\nabla a_0 \equiv 0$ in these examples. Tables 5.1–5.2 show preconditioned GMRES iteration counts for the AMG ($\widehat{P}_{L,amg}$) and exact ($\widehat{P}_L$) versions of the diffusion preconditioner. Replacing $\widehat{P}_L$ by $\widehat{P}_{L,amg}$ does not yield a significant increase of the iteration count, so the computational cost will be much lower with $\widehat{P}_{L,amg}$. For standard deviations σ less than or equal to one, iteration counts are completely insensitive to the mesh width $h = n^{-1}$ and largely insensitive to the total degree of the chaos polynomials d, Theorem 4.1 suggests. For fixed degree d, the number of GMRES iterations required to satisfy the stopping criterion decreases slightly as the mesh is refined. For fixed mesh width, the observed iteration count is almost independent of d and σ, except for the case of (relatively large) standard deviation $\sigma = 2$ in Example 5.2, where the preconditioner breaks down. Iterations for $n = 128, d = 5$ terminated before the stopping criterion was satisfied due to memory limitations. This is not inconsistent with the analysis of section 4, which establishes convergence bounds only for small enough σ. We found no difficulties for $\sigma = 1.5$ (see Table 5.2).

Table 5.1
Example 5.1: GMRES iteration counts with AMG and exact (in parentheses) versions of the diffusion preconditioner $\widehat{P}_L = I \otimes L$. In this example, $M = 5$ and $\ell = 1$.

n	σ	$d=1$		$d=2$		$d=3$		$d=4$		$d=5$		$d=6$	
16	0.1	5	(4)	6	(5)	6	(5)	6	(5)	6	(5)	6	(5)
32	-	5	(4)	6	(5)	6	(5)	6	(5)	6	(5)	6	(5)
64	-	5	(4)	6	(4)	6	(4)	6	(4)	6	(4)	6	(4)
128	-	5	(4)	6	(4)	6	(4)	6	(4)	6	(4)	6	(4)
16	1.0	9	(9)	11	(11)	13	(12)	14	(13)	15	(13)	16	(14)
32	-	9	(8)	11	(10)	12	(12)	13	(12)	14	(12)	15	(13)
64	-	8	(8)	10	(10)	12	(11)	12	(11)	13	(12)	14	(12)
128	-	8	(7)	10	(9)	11	(10)	12	(11)	12	(11)	13	(11)
16	2.0	13	(12)	18	(18)	23	(22)	27	(25)	31	(27)	35	(30)
32	-	12	(12)	17	(17)	21	(21)	25	(23)	29	(26)	32	(28)
64	-	11	(11)	16	(16)	20	(19)	24	(22)	27	(24)	29	(26)
128	-	11	(10)	14	(15)	19	(18)	22	(20)	25	(23)	27	(24)

Table 5.2
Example 5.2: GMRES iteration counts with AMG and exact (in parentheses) versions of the diffusion preconditioner $\widehat{P}_L = I \otimes L$. In this example, $M = 10$ and $\ell = 0.5$.

n	σ	$d=1$		$d=2$		$d=3$		$d=4$		$d=5$	
16	0.1	6	(5)	6	(5)	6	(5)	6	(6)	6	(6)
32	-	6	(5)	6	(5)	6	(5)	6	(5)	6	(5)
64	-	6	(4)	6	(5)	6	(5)	6	(5)	6	(5)
128	-	6	(4)	6	(4)	6	(5)	6	(5)	6	(5)
16	1.0	11	(11)	16	(15)	18	(17)	20	(19)	22	(21)
32	-	11	(10)	14	(14)	17	(16)	18	(18)	20	(19)
64	-	10	(10)	13	(13)	16	(15)	17	(17)	19	(18)
128	-	10	(9)	13	(12)	15	(14)	16	(15)	17	(17)
16	1.5	14	(14)	22	(22)	29	(28)	39	(37)	50	(49)
32	-	14	(13)	21	(21)	28	(26)	36	(34)	44	(44)
64	-	13	(13)	20	(19)	26	(25)	33	(31)	41	(40)
128	-	12	(12)	19	(17)	24	(23)	30	(28)	37	(36)
16	2.0	17	(17)	33	(32)	65	(63)	170	(168)	563	(550)
32	-	17	(16)	32	(31)	59	(58)	176	(174)	513	(501)
64	-	16	(15)	30	(29)	56	(55)	158	(154)	533	(521)
128	-	15	(14)	28	(26)	54	(53)	148	(145)	-	-

5.1.2. Mean-based preconditioner. Tables 5.3–5.4 show GMRES iteration counts for Examples 5.3 and 5.4 using both the mean-based preconditioner $\widehat{P}_0$ and the diffusion preconditioner $\widehat{P}_L$ (note that they are different for these examples), as well as the variants using AMG. These results reproduce the trends (insensitivity to h and d) seen in section 5.1.1 for the diffusion preconditioner, and similar trends can be seen for the mean-based preconditioner. As expected, performance of $\widehat{P}_0$, which takes account of the fact that $\nabla a_0 \neq 0$, is superior to that of $\widehat{P}_L$. Since the computational costs of the two preconditioners are identical, the mean-based preconditioner is more efficient. Use of practical AMG versions leads to insignificant changes in iteration counts.

TABLE 5.3

Example 5.3: GMRES iteration counts with AMG and exact (in parentheses) versions of the mean-based preconditioner $\widehat{P}_0 = I \otimes (L + N_0)$ and diffusion preconditioner $\widehat{P}_L = I \otimes L$. In this example, $M = 5$ and $\ell = 1$.

		$\widehat{P}_0 = I \otimes (L+N_0)$				$\widehat{P}_L = I \otimes L$			
n	σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=1$	$d=2$	$d=3$	$d=4$
16	0.1	5 (4)	5 (4)	5 (5)	5 (5)	24 (22)	25 (24)	25 (26)	26 (26)
32	-	6 (4)	6 (4)	6 (4)	6 (4)	23 (22)	25 (25)	25 (25)	25 (26)
64	-	6 (4)	6 (4)	6 (4)	6 (4)	23 (21)	24 (24)	24 (24)	24 (24)
128	-	5 (4)	5 (4)	5 (4)	5 (4)	21 (21)	22 (22)	22 (22)	22 (23)
16	1.0	8 (7)	9 (9)	10 (10)	10 (10)	27 (26)	30 (31)	32 (33)	33 (34)
32	-	8 (7)	9 (8)	9 (9)	10 (10)	26 (25)	29 (30)	31 (32)	32 (33)
64	-	7 (7)	8 (8)	9 (9)	9 (9)	25 (24)	28 (28)	30 (30)	30 (31)
128	-	7 (6)	8 (8)	8 (8)	9 (9)	23 (23)	26 (27)	28 (28)	29 (29)
16	2.0	10 (10)	12 (12)	14 (14)	16 (16)	28 (28)	34 (34)	37 (38)	39 (40)
32	-	10 (9)	12 (12)	14 (14)	15 (15)	28 (27)	33 (33)	36 (37)	38 (39)
64	-	9 (9)	11 (11)	13 (13)	14 (14)	27 (26)	31 (31)	34 (35)	36 (37)
128	-	9 (9)	11 (11)	12 (12)	13 (14)	25 (24)	29 (30)	32 (33)	34 (34)

TABLE 5.4

Example 5.4: GMRES iteration counts with AMG and exact (in parentheses) versions of the mean-based preconditioner $\widehat{P}_0 = I \otimes (L + N_0)$ and diffusion preconditioner $\widehat{P}_L = I \otimes L$. In this example, $M = 10$ and $\ell = 0.5$.

		$\widehat{P}_0 = I \otimes (L+N_0)$				$\widehat{P}_L = I \otimes L$			
n	σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=1$	$d=2$	$d=3$	$d=4$
16	0.1	6 (5)	6 (5)	6 (5)	6 (5)	24 (24)	26 (27)	27 (27)	27 (27)
32	-	6 (5)	6 (5)	6 (5)	6 (5)	24 (23)	25 (26)	26 (27)	27 (27)
64	-	6 (4)	6 (5)	6 (5)	6 (5)	23 (22)	24 (24)	25 (25)	25 (26)
128	-	6 (4)	6 (4)	6 (4)	6 (5)	21 (21)	23 (23)	23 (24)	24 (24)
16	1.0	10 (10)	12 (12)	14 (14)	15 (15)	28 (29)	32 (33)	35 (35)	37 (38)
32	-	10 (9)	12 (12)	13 (13)	14 (15)	28 (28)	32 (32)	34 (34)	36 (36)
64	-	9 (9)	11 (11)	13 (12)	14 (14)	26 (26)	30 (31)	32 (33)	34 (35)
128	-	9 (9)	11 (10)	12 (12)	13 (13)	25 (25)	29 (29)	30 (31)	32 (33)
16	2.0	14 (14)	20 (20)	24 (24)	30 (30)	31 (31)	37 (38)	42 (43)	47 (48)
32	-	13 (13)	19 (20)	24 (24)	28 (29)	30 (31)	37 (37)	41 (42)	46 (47)
64	-	13 (13)	18 (19)	23 (23)	27 (28)	29 (29)	35 (35)	39 (40)	44 (45)
128	-	12 (12)	17 (18)	21 (22)	26 (26)	28 (27)	33 (34)	37 (38)	41 (42)

5.2. Comparison of formulations. Finally, we compare the performance of iterative solvers for the discretized Galerkin equations (2.13) associated with the diffusion formulation and those arising from the convection-diffusion formulation (3.3) for the model problem in Example 5.1. For the physical discretization we use a 64×64 grid of square bilinear elements, yielding $n_{\boldsymbol{x}} = 4{,}096$. The parameters of the stochastic discretization are summarized in Table 5.5. Recall that the Kronecker product representation of the convection-diffusion Galerkin matrix $\widehat{C}$ in (3.7) requires $M + 1 = 6$ terms in Example 5.1. We employ right-preconditioned GMRES with the diffusion preconditioner $\widehat{P}_L = I \otimes L$ (see section 4.2) to solve the discretized convection-diffusion Galerkin equations. For the linear system of equations associated with the diffusion formulation we employ the CG method in conjunction with the Kronecker product preconditioner $\widehat{P}_1 = G \otimes A_0$ from [48] (where it is shown that this choice is superior to a mean-based preconditioner $I \otimes A_0$). Here, $A_0 \in \mathbb{R}^{n_x \times n_x}$ denotes the finite element stiffness matrix in (2.14) with coefficient function $\langle \exp(a) \rangle$, and $G \in \mathbb{R}^{n_{\boldsymbol{\xi}} \times n_{\boldsymbol{\xi}}}$

TABLE 5.5

Example 5.1: *Number of stochastic d.o.f.* $n_{\boldsymbol{\xi}}$*, total number of d.o.f.* $n_x \cdot n_{\boldsymbol{\xi}}$*, and number of terms* $|\mathscr{I}_{2d}|$ *in Kronecker product representation* (2.16) *of the Galerkin matrix associated with the diffusion formulation.*

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$n_{\boldsymbol{\xi}}$	6	21	56	126	252	462
$\lvert\mathscr{I}_{2d}\rvert$	21	126	462	1,287	3,003	6,188
$n_x \cdot n_{\boldsymbol{\xi}}$	24,576	86,016	229,376	516,096	1,032,192	1,982,352

TABLE 5.6

Iterative solver types and preconditioners, and costs of matrix-vector products, for the diffusion and convection-diffusion formulations.

Formulation	Solver	Preconditioner	Cost mat-vecs
Diffusion	CG	$\widehat{P}_1 = G \otimes A_0$	$O(n_x n_{\boldsymbol{\xi}}^2)$
Convection-diffusion	GMRES	$\widehat{P}_L = I \otimes L$	$O(n_x n_{\boldsymbol{\xi}})$

TABLE 5.7

Example 5.1: *Iteration counts for the discretized diffusion formulation and for the discretized convection-diffusion formulation.*

	Diffusion: CG					Convection-diffusion: GMRES				
σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
0.1	7	8	8	9	9	5	6	6	6	6
0.2	8	9	11	12	12	6	6	6	6	6
0.4	10	13	16	19	21	6	7	7	8	8
0.6	12	18	23	28	32	7	8	9	9	9
0.8	15	23	32	41	49	8	9	10	11	11
1.0	17	29	43	57	72	8	10	12	12	13
2.0	32	78	152	257	393	11	16	20	24	27

is a specific linear combination of the stochastic Galerkin matrices $\{G_{\boldsymbol{\alpha}} \colon \boldsymbol{\alpha} \in \mathscr{I}_{2d}\}$. The solver methodologies are summarized in Table 5.6. The stopping criterion is $\|r_k\|_2 < 10^{-8}\|b\|_2$ for both approaches. Once again, we replaced the action of L^{-1} and A_0^{-1} by one AMG V-cycle.

We present the preconditioned CG and GMRES iteration count along with iteration time and setup time for the preconditioners in Tables 5.7, 5.8, and 5.9, respectively. Timings are elapsed times in seconds. The numerical experiments were performed on a single processor of a four-processor quad-core Linux machine with 128 GB RAM using MATLAB 7.10.

The trends for the CG and GMRES iteration counts presented in Table 5.7 agree well with the observations made in [48] and in section 5.1.1. As previously shown in [48], the CG iteration counts deteriorate for large values of the standard deviation σ of the log-transformed diffusion coefficient and the degree d of the chaos polynomial. In contrast, as has been shown above, performance of GMRES is only slightly sensitive to variations in σ and d, and at most 27 GMRES iterations are required for the range of parameters considered here. Moreover, the amount of time needed for the diffusion formulation is dramatically larger. This is because the associated Galerkin matrix is block-dense so that matrix-vector products are very expensive (see Table 5.6). Because it depends linearly on the stochastic random variables, the convection-diffusion formulation avoids this difficulty, and the costs of matrix-vector products are significantly lower. Finally, the setup costs for the Kronecker product preconditioner $\widehat{P}_1$ increase with the degree d of the chaos polynomials (see [48] for details), whereas the

TABLE 5.8
Example 5.1: Total iteration time (in seconds) for the discretized diffusion formulation and for the discretized convection-diffusion formulation.

	Diffusion: CG					Convection-diffusion: GMRES				
σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
0.1	0.35	5.64	53	517	2,712	0.28	0.88	2.71	6.23	13.1
0.2	0.33	6.11	89	682	2,972	0.24	0.83	2.34	6.26	12.9
0.4	0.38	9.88	108	1,122	5,916	0.24	0.98	2.75	8.37	17.3
0.6	0.46	11.4	155	1,358	8,521	0.29	1.17	3.60	9.53	19.8
0.8	0.58	17.2	218	2,104	11,486	0.33	1.26	4.03	11.6	23.8
1.0	0.65	17.5	288	3,214	17,905	0.33	1.41	4.83	12.7	29.4
2.0	1.18	50.0	1,003	12,031	100,503	0.46	2.32	8.68	28.3	71.9

TABLE 5.9
Example 5.1: Setup time (in seconds) for the AMG versions of the Kronecker product preconditioner $\widehat{P}_1 = G \otimes A_0$ and for the diffusion preconditioner $\widehat{P}_L = I \otimes L$, respectively.

	Diffusion: $\widehat{P}_1 = G \otimes A_0$					Convection-diffusion: $\widehat{P}_L = I \otimes L$				
σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$
0.1	1.11	1.10	1.29	1.74	3.14	1.12	1.05	1.05	1.05	1.07
0.2	1.06	1.10	1.29	1.76	3.17	1.06	1.05	1.05	1.05	1.05
0.4	1.05	1.10	1.28	1.77	3.05	1.05	1.05	1.05	1.05	1.06
0.6	1.05	1.11	1.28	1.76	3.10	1.05	1.06	1.05	1.06	1.05
0.8	1.06	1.10	1.28	1.76	3.06	1.05	1.05	1.05	1.06	1.06
1.0	1.06	1.11	1.30	1.80	3.30	1.06	1.06	1.05	1.05	1.05
2.0	1.06	1.11	1.28	1.74	3.02	1.05	1.05	1.06	1.05	1.05

TABLE 5.10
Example 5.1: Maximal variation $\max_{i,j}|u_{i,j}^{(\mathrm{cd})} - u_{i,j}^{(\mathrm{diff})}|$ of the $n_x \cdot n_{\boldsymbol{\xi}}$ solution coefficients associated with the convection-diffusion formulation (3.3) and the diffusion formulation (2.13).

σ	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	
0.1	6.410	6.410	6.410	6.410	6.410	$\times 10^{-5}$
0.2	1.319	1.282	1.282	1.282	1.282	$\times 10^{-4}$
0.4	9.840	2.564	2.564	2.564	2.564	$\times 10^{-4}$
0.6	31.744	3.846	3.846	3.846	3.846	$\times 10^{-4}$
0.8	71.424	5.129	5.129	5.129	5.129	$\times 10^{-4}$
1.0	131.120	6.411	8.688	6.411	6.411	$\times 10^{-4}$
2.0	68.985	7.588	11.251	1.412	2.746	$\times 10^{-3}$

setup costs for the diffusion preconditioner $\widehat{P}_L$ (and its AMG version) are clearly independent of all parameters of the stochastic discretization. The setup timings listed in Table 5.9 agree with this observation.

We note that the infinite-dimensional weak formulations (3.2) and (2.7) are equivalent, as was observed in section 3. However, this is not the case for the discrete weak formulations (3.3) and (2.13) since $v \cdot \exp(-a^{(M)}) \notin X_h \otimes S_d$ for $v \in X_h \otimes S_d$. This observation is consistent with the results in Table 5.10, where the maximal difference of the chaos coefficients $u_{i,j}^{(\mathrm{cd})}$ and $u_{i,j}^{(\mathrm{diff})}$ determined by the Galerkin equations (3.3) and (2.13), respectively, is presented.

6. Conclusions.

In this study, we have tested iterative solvers for Galerkin discretizations of a steady-state diffusion problem with log-transformed random diffusion coefficients. Specifically, we have focused on a reformulated version of this problem in terms of a convection-diffusion equation with stochastic convective velocity. We have introduced and analyzed two block-diagonal preconditioners for the associated

Galerkin equations, one based on the diffusive part of the operator and the other based on a convection-diffusion operator where the log-transformed diffusion coefficient has been replaced by its mean value. Spectral inclusion bounds obtained for the preconditioned Galerkin matrix are insensitive to the characteristic mesh size of the spatial discretization and only slightly sensitive to the degree of the chaos polynomials used in the stochastic discretization and the standard deviation of the log-transformed diffusion coefficient. Numerical tests showed that the mean-based preconditioner can outperform the diffusion preconditioner for problems where a mean convective part occurs. In addition, GMRES in combination with a simple block-diagonal preconditioner employed for the iterative solution of the convection-diffusion formulation far outperformed the CG method used with a more elaborate Kronecker product preconditioner to solve the associated diffusion formulation. In conclusion, the availability of a robust iterative solver together with cheap matrix-vector products make the convection-diffusion formulation of the log-transformed random diffusion problem extremely appealing.

Acknowledgment. We thank Ingolf Busch at TU Bergakademie Freiberg for providing software to carry out the gradient computations in section 3.3.

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