New studies on the degree of ill-posedness

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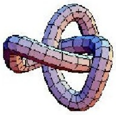

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Outline

1. Introduction
2. Impact of non-compact operators on the degree of ill-posedness in compositions with compact operators
3. Some mystery and a curiosity of the Hausdorff moment operator in compositions
4. Measuring ill-posedness of non-compact operators by spectral theorem and decreasing rearrangements

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Let X and Y be infinite dimensional separable Hilbert spaces.
We consider **operator equations** modelling inverse problems.

Although this seems like the more trivial case, today our focus is on **ill-posedness phenomena** for the ill-posed **linear case**

$$\boxed{A\,x\,=\,y \qquad (x\in X,\; y\in Y) \qquad (*)}$$

characterized by **bounded linear operators** $A \in \mathcal{L}(X, Y)$,
for which the range $\mathcal{R}(A)$ is a **non-closed** subset of Y, or
in other words, the Moore-Penrose inverse $A^\dagger$ is **unbounded**.

For simplicity, we suppose in the sequel that A is **injective**.

Nashed's ill-posedness concept applied to linear problems $(*)$ in Hilbert spaces

$\triangleright$ M. Z. NASHED: A new approach to classification and regularization of ill-posed operator equations. In: H. W. Engl and C. W. Groetsch (Eds.), Inverse and Ill-posed Problems (Sankt Wolfgang, 1986), volume 4 of Notes Rep. Math. Sci. Engrg., pp. 53–75. Academic Press, Boston, MA, 1987.

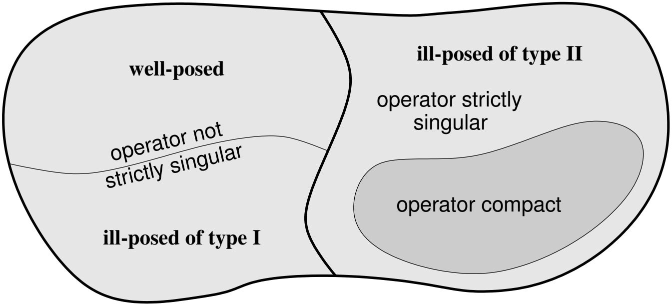
Definition

The ill-posed linear operator equation $(*)$ is called
– **ill-posed of type I** if the range $\mathcal{R}(A)$ contains an infinite dimensional closed subspace, and alternatively
– **ill-posed of type II** if A is compact.

Please allow two small digressions on the next slides:

More diverse ill-posedness types in Banach spaces

▷ J. FLEMMING, B.H. AND I. VESELIĆ: On ℓ^1-regularization in light of Nashed's ill-posedness concept. *Computational Methods in Applied Mathematics* **15** (2015), pp. 279–289.



Operator $A \in \mathcal{L}(X, Y)$ in Banach spaces X and Y is **strictly singular** if $\mathcal{R}(A)$ does not contain an infinite dimensional closed subspace.

T. KATO 1958: Is every strictly singular operator **compact**? In Hilbert spaces: Yes!

In Banach spaces: No! Embedding operator from ℓ^p $_{(1 \le p < 2)}$ to ℓ^2 is strictly singular.

In consideration of the 30th anniversary of the article

▷ B.H. AND O. SCHERZER: Factors influencing the ill-posedness of nonlinear problems. *Inverse Problems* **10** (1994), pp. 1277–1297.

a little sideways look at ill-posed **nonlinear** operator equations

$$\boxed{F(x) = y \qquad (x \in \mathcal{D}(F) \subseteq X,\ y \in Y) \qquad (**)}$$

Definition

The equation $(**)$ is called **locally well-posed** at the solution point $x^\dagger \in \mathcal{D}(F)$ if there is a ball $B_r(x^\dagger)$ around $x^\dagger$ with radius $r > 0$ such that for each sequence $\{x_n\}_{n=1}^\infty \subset B_r(x^\dagger) \cap \mathcal{D}(F)$

$$\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\|_Y = 0 \implies \lim_{n\to\infty} \|x_n - x^\dagger\|_X = 0$$

holds true. Otherwise $(**)$ is called **locally ill-posed** at $x^\dagger$.

In the sequel, we return to the linear case $(*)$ in Hilbert spaces:

Ill-posedness of type II (compact case)

Definition

For an injective compact operator $A : X \to Y$ between infinite dimensional Hilbert spaces X and Y and decreasingly ordered singular values $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ with $\sigma_n(A) \to 0$ as $n \to \infty$ we call the operator equation $(*)$

- **mildly ill-posed** whenever the sequence $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ decays slower than any polynomial rate.
- **moderately ill-posed** whenever the sequence $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ decays polynomially. If $\sigma_n(A) \asymp n^{-\kappa}$ as $n \to \infty$ with some $\kappa > 0$, then we call $(*)$ moderately ill-posed **of degree** κ**.**
- **severely ill-posed** whenever the sequence $\{\sigma_n(A)\}_{n\in\mathbb{N}}$ decays faster than any polynomial rate.

For combinations of polynomial and logarithmic terms in the decay rates, it can helpful to define:

Definition

We introduce the **interval of ill-posedness** as

$$[\underline{\kappa},\overline{\kappa}] := \left[\liminf_{n\to\infty} \frac{-\log(\sigma_n(A))}{\log(n)}\,,\, \limsup_{n\to\infty} \frac{-\log(\sigma_n(A))}{\log(n)}\right] \subset [0,\infty].$$

If the well-defined $\underline{\kappa}$ and $\overline{\kappa}$ from $[0,\infty]$ are both finite positive, then we have **moderate ill-posedness**, and if they even coincide as $\underline{\kappa}=\overline{\kappa}=\kappa$, then the equation $(*)$ is ill-posed of degree $\kappa>0$. **Severe ill-posedness** occurs if the interval degenerates as $\underline{\kappa}=\overline{\kappa}=\infty$, and vice versa **mild ill-posedness** is characterized by a degeneration as $\underline{\kappa}=\overline{\kappa}=0$.

▷ B.H. AND U. TAUTENHAHN: On ill-posedness measures and space change in Sobolev scales. *Z. Anal. Anwendungen* **16** (1997), pp. 979–1000.

Example: Fractional integration of order $\kappa > 0$

We have for operator equation $(*)$ with $X = Y = L^2(0,1)$ and $A = J^\kappa$ defined as

$$[J^\kappa x](s) := \int_0^s \frac{(s-t)^{\kappa-1}}{\Gamma(\kappa)} x(t)\, dt \qquad (s \in [0,1])$$

that $\sigma_n(J^\kappa) \asymp n^{-\kappa}$ as $n \to \infty$ and $(*)$ is ill-posed of degree κ.

$\triangleright$ R. Ramlau, Ch. Koutschan and B.H.: On the singular value decomposition of n-fold integration operators. In: *Inverse Problems and Related Topics* (Eds.: J. Cheng, S. Lu and M. Yamamoto). Springer Nature, Singapore, 2020 , pp. 237–256.

Example: *d*-dimensional multivariate integration

We have for operator equation $(*)$ with $X = Y = L^2((0,1)^d)$ and $A = J_d$ defined as

$$[J_d x](s_1, ..., s_d) := \int_0^{s_1} ... \int_0^{s_d} x(t_1, ..., t_d)\, dt_d ... dt_1 \qquad ((s_1, ..., s_d) \in (0,1)^d)$$

that the equation $(*)$ is ill-posed of degree **one** and **independent of d**, because

$$\sigma_n(J_d) \asymp \frac{[\log(n)]^{d-1}}{n} \text{ as } n \to \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{-\log(\frac{[\log(n)]^{d-1}}{n})}{\log(n)} = 1.$$

Hence the interval of ill-posedness degenerates to the single point $\kappa = 1$.

$\triangleright$ B.H. and H.-J. Fischer: A note on the degree of ill-posedness for mixed differentiation on the d-dimensional unit cube. *J. Inverse Ill-Posed Probl.* **31** (2023), pp. 949–957.

In contrast: Ill-posedness degree $\kappa = \frac{p}{d} > 0$ strongly **depends on d** for embedding operator $\mathcal{E}_d : H^p((0,1)^d) \to L^2((0,1)^d)$ with $\sigma_n(\mathcal{E}_d) \asymp n^{-\frac{p}{d}}$ as $n \to \infty$.

Ill-posedness characterization in literature: rough selection

▷ G. WAHBA: Ill-posed problems: Numerical and statistical methods for mildly, moderately and severely ill-posed problems with noisy data. Technical Report No. 595. Madison, University of Wisconsin, 1980.

▷ J. BAUMEISTER: *Stable Solution of Inverse Problems*. Vieweg, Braunschw., 1987.

▷ A. K. LOUIS: *Inverse und schlecht gestellte Probleme*. Teubner, Stuttgart, 1989.

▷ H. W. ENGL, M. HANKE AND A. NEUBAUER: *Regularization of Inverse Problems*. Kluwer, Dordrecht, 1996.

▷ O. SCHERZER, M. GRASMAIR, H. GROSSAUER, M. HALTMEIER, F. LENZEN: *Variational Methods in Imaging*. Springer, New York, 2009.

▷ T. SCHUSTER, B. KALTENBACHER, B.H. AND K. S. KAZIMIERSKI: *Regularization Methods in Banach Spaces*. Walter de Gruyter, Berlin/Boston, 2012.

▷ S. LU AND S. V. PEREVERZEV: *Regularization Theory for Ill-Posed Problems*. Walter de Gruyter, Berlin/Boston, 2013.

▷ B.H. AND R. PLATO: On ill-posedness concepts, stable solvability and saturation. *J. Inverse Ill-Posed Probl.* **26** (2018), pp. 287–297.

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Impact of non-compact operators on the degree of ill-posedness in compositions with compact operators

Ill-posedness of type I (non-compact case)

In his seminar paper, Nashed states 1987:

"An equation involving a bounded non-compact operator with non-closed range is **less ill-posed** than an equation with a compact operator with infinite-dimensional range."

However, this claim is problematic, and finding appropriate approaches for measuring the strength of type I ill-posedness is **a never ending story.** For example, it was tried in:

▷ B.H. AND G. FLEISCHER: Stability rates for linear ill-posed problems with compact and non-compact operators. *Z. Anal. Anwendungen* **18** (1999), pp. 267–286.

▷ B.H. AND S. KINDERMANN: On the degree of ill-posedness for linear problems with non-compact operators. *Methods and Applications of Analysis* **17** (2010), pp. 445–461.

▷ P. MATHÉ, B.H. AND M. T. NAIR: Regularization of linear ill-posed problems involving multiplication operators. *Appl. Anal.* **101** (2022), pp. 714–732.

▷ F. WERNER AND B.H.: A unified concept ... under the auspices of the spectral theorem. Paper in preparation, 2024.

Compositions of both ill-posedness types

We consider for Hilbert spaces X, Y, Z the ill-posed equation

$$A\,x \;=\; y \qquad (x \in X,\; y \in Y) \qquad (*)$$

with a linear **compact composite operator**

$$A: X \;\xrightarrow{D}\; Z \;\xrightarrow{B}\; Y\,,$$

where $A = B \circ D$ is a composition of the **compact** linear operator D having infinite dimensional range and the bounded **non-compact** linear operator B having non-closed range, too.

Then the **inner problem**

$$D\,x = z\,,$$

is ill-posed of **type II** due to the compactness of D, whereas the **outer problem**

$$B\,z = y$$

is ill-posed of **type I**, since B is non-compact.

General question:

Can the non-compact operator B with non-closed range in $A = B \circ D$ 'destroy' the ill-posedness degree of the compact D?

Compared to the compact original operator D, the ill-posedness degree of $A = B \circ D$ can **only grow** or **stay at the same level**, because we have the upper estimate

$$\sigma_n(A) \le \|B\|_{\mathcal{L}(Z,Y)}\, \sigma_n(D) \qquad (n \in \mathbb{N})\,.$$

The following lower estimate the conditional stability result of Theorem 2.1 from

▷ B.H. AND P. MATHÉ: The degree of ill-posedness of composite linear ill-posed problems with focus on the impact of the non-compact Hausdorff moment operator. *ETNA* **57** (2022), pp. 1–16.

Theorem 1

Suppose that there exists an index function $\Psi : (0,\infty) \to (0,\infty)$ such that for $0 < \delta \le \|A\|_{\mathcal{L}(X,Y)}$ the conditional stability estimate

$$\sup\{\, \|Dx\|_Z : \; \|Ax\|_Y \le \delta, \; \|x\|_X \le 1\} \le \Psi(\delta)$$

holds. Then we have

$$\Psi^{-1}(\sigma_n(D)) \le \sigma_n(A) \qquad (n \in \mathbb{N})\,.$$

Multiplication operators in $L^2(0,1)$ may be harmless

Setting $X = Y = Z = L^2(0,1)$ we have, for the **integration operator** $[Jx](s) := \int_0^s x(t)dt$ and **non-compact** classes of **multiplication operators** $[Mx](t) := m(t)x(t)$ with multiplier functions $m \in L^\infty(0,1)$ having essential zeros, that

$$\sigma_n(M \circ J) \asymp \sigma_n(J) \asymp n^{-1} \qquad \text{as} \quad n \to \infty.$$

The non-compact $B := M$ **does not 'destroy'** the singular value decay rate of $D := J$ by the composition $A = M \circ J$.

▷ M. Freitag and B.H.: Analytical and numerical studies on the influence of multiplication operators for the ill-posedness of inverse problems. *J. Inv. Ill-Posed Problems* **13** (2005), pp. 123–148.

▷ B.H. and L. von Wolfersdorf: Some results and a conjecture on the degree of ill-posedness for integration operators with weights. *Inverse Problems* **21** (2005), pp. 427–433.

▷ B.H. and L. von Wolfersdorf: A new result on the singular value asymptotics of integration operators with weights. *Journal of Integral Equations and Applications* **21** (2009), pp. 281–295.

Cesàro operator in $\mathrm{L}^2(0,1)$ is expected powerful

We consider now with $X=Y=Z=L^2(0,1)$ the composition of the **compact integration operator** $[Jx](s):=\int_0^s x(t)dt$ and the **(continuous) Cesàro operator** $C: L^2(0,1) \to L^2(0,1)$ as

$$[C\,x](s) := \frac{1}{s} \int_0^s x(t)\,dt \qquad (0 < s \le 1),$$

which is comprehensively characterized by

▷ A. BROWN, P. R. HALMOS AND A. L. SHIELDS: Cesàro operators. *Acta Sci. Math. (Szeged)* **26** (1965), pp. 125–137.

The operator C is **non-compact** and has a **non-closed range.**

However, the Cesàro operator $B := C$ has the power to **amend (increase)** in the composition $A = C \circ J$ the singular value decay rate of the compact operator $D := J$.

For the **Hilbert-Schmidt operator** $A = C \circ J : L^2(0,1) \to L^2(0,1)$, we have

$$[Ax](s) := \frac{1}{s}\int_0^s (s-t)\,x(t)dt = \int_0^s \frac{s-t}{s}\,x(t)dt \qquad (0 < s \le 1)\,,$$

which is connected to the **twofold integration operator** J^2 with $\sigma_n(J^2) \asymp \frac{1}{n^2}$ by

$J^2 = M \circ A$ for the multiplication operator $[Mx](s) = s\,x(s)$ $(0 \le s \le 1)$ in $L^2(0,1)$.

This implies: $\sigma_n(J^2) \le \|M\|_{\mathcal{L}(L^2(0,1))}\,\sigma_n(A) \le \sigma_n(A) \le \|C\|_{\mathcal{L}(L^2(0,1))}\,\sigma_n(J) \quad (n \in \mathbb{N})$.

Hence, there exist positive constants K_1 and K_2 such that

$$\frac{K_1}{n^2} \le \sigma_n(A) \le \frac{K_2}{n} \qquad (n \in \mathbb{N}).$$

By the Hilbert-Schmidt-type inequality $\sum_{i=n+1}^{\infty} \sigma_i^2(A) \le \sum_{i=n+1}^{\infty} \|Ae_i\|^2_{L^2(0,1)}$ for

orthonormal systems $\{e_i\}_{i=1}^{\infty}$ derived from shifted Legendre polynomials we show in

▷ Y. DENG, H.-J. FISCHER AND B.H.: The degree of ill-posedness for some composition governed by the Cesàro operator. arXiv:2401.11411, Jan. 2024.

Theorem 2

The Cesàro operator C **raises** the ill-posedness degree of J by **one** such that

$$\sigma_n(A) = \sigma_n(C \circ J) \asymp \frac{1}{n^2} \quad \text{as} \quad n \to \infty\,.$$

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Some mystery and a curiosity of the Hausdorff moment operator in compositions

We recall the **Hausdorff moment** operator $H : L^2(0,1) \to \ell^2$

$$[H\,x]_j := \int_0^1 t^{j-1} x(t)dt \qquad (j = 1, 2, ...)\,.$$

The subsequent propositions is from:

▷ D. GERTH, B.H., C. HOFMANN AND S. KINDERMANN: The Hausdorff moment problem in the light of ill-posedness of type I. *Eurasian Journal of Mathematical and Computer Applications* **9** (2021), pp. 57–87.

Proposition

$H : L^2(0,1) \to \ell^2$ is a bounded, injective and **non-compact** linear operator with **non-closed range**. The adjoint operator $H^* : \ell^2 \to L^2(0,1)$ attains the form

$$[H^* y](t) = \sum_{j=1}^{\infty} y_j\, t^{j-1} \qquad (0 \le t \le 1).$$

We have $H = \mathbb{L}\, Q$ with an isometry $Q : L^2(0,1) \to \ell^2$ and a lower triangular operator $\mathbb{L} : \ell^2 \to \ell^2$ being the lower Cholesky factor of the infinite Hilbert matrix $\mathbb{H} = \left(\frac{1}{i+j-1}\right)_{i,j=1}^{\infty} : \ell^2 \to \ell^2$. This means that $\quad \mathbb{L}\,\mathbb{L}^* = \mathbb{H} = H\,H^*$.

In our framework we set now $B := H$ and $D := J$. Hence, we consider the compact composition $A = H \circ J : L^2(0,1) \to \ell^2$

Proposition

There is a positive constant C_0 such that

$$\sup\{\,\|Jx\|_{L^2(0,1)} :\ \|H(Jx)\|_{\ell^2} \le \delta,\ \|x\|_{L^2(0,1)} \le 1\} \le \frac{C_0}{\log(1/\delta)}\,.$$

This proposition yields with Theorem 1 by setting $X = Z = L^2(0,1)$, $Y = \ell^2$ and $\Psi(\delta) = \frac{C_0}{\log(1/\delta)}$ the following

Corollary 1

There exists a positive constant $\underline{C}$ such that

$$\exp(-\underline{C}\,n) \le \sigma_n(H \circ J) \quad (n \in \mathbb{N}).$$

Again based on Hilbert-Schmidt-type estimates we find from:

▷ B.H. AND P. MATHÉ: The degree of ill-posedness of composite linear ill-posed problems with focus on the impact of the non-compact Hausdorff moment operator. *ETNA* **57** (2022), pp. 1–16.

Theorem 3

For $A = H \circ J$ there exist positive constants $\underline{C}$ and $\overline{C}$ such that

$$\exp(-\underline{C}\, n) \le \sigma_n(A) \le \frac{\overline{C}}{n^{3/2}} \quad (n \in \mathbb{N}).$$

The non-compact Hausdorff moment operator H is able to increase in a composition the degree of ill-posedness of J at least by $1/2$. However, the **gap** between lower and upper bounds for $\sigma_n(A)$ **is too large**.

Open question (Hausdorff mystery)

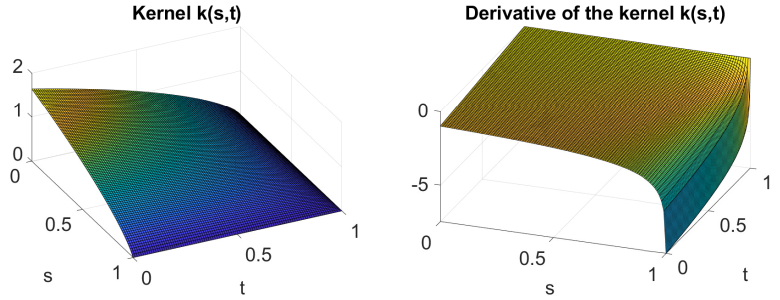
Is the linear operator equation $(*)$ with forward operator $A = H \circ J$ **moderately or severely ill-posed**?

By now there is no final unveiling of this mystery!

Arguments pro moderate ill-posedness:

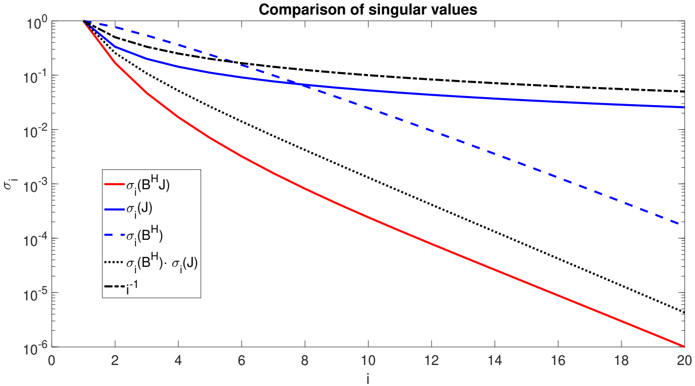
For the Hilbert-Schmidt operator $A = H \circ J$ we have

$$[A^*Ax](s) = \int_0^1 k(s,t)\,x(t)\,dt \quad (0 \le s \le 1) \quad \text{with} \quad k(s,t) = \sum_{j=1}^{\infty} \frac{(1-s^j)(1-t^j)}{j^2}\,.$$



Kernel k is smooth, but partial derivative $\frac{\partial k}{\partial s}$ has a pole at $s = 1$.

Arguments against moderate ill-posednes:



Semi-logarithmic plot of singular values of $n \times n$-matrices with $n = 10^4$ supporting points representing **discretization matrices** of the operators A, $B^{(H)} := H$ and J.

Singular values of $\mathbf{A = H \circ J}$ decay exponentially in the numerical experiments.

Is numerics reaching its limits here to evaluate the degree of ill-posedness for the infinite dimensional problem?

Kindermann's curiosity observation on Hausdorff operator

Stefan Kindermann observed that the non-compact operators C and H, both with non-closed range, have the property that they can lead to compact compositions.

The adjoint $C^* : L^2(0,1) \to L^2(0,1)$ of the Cesàro operator C defined as

$$[C^*x](t) = \int_t^1 \frac{x(s)}{s}\, ds \qquad (0 \le t \le 1)$$

is also non-compact. We have in composition with operator $H : L^2(0,1) \to \ell^2$ that

$$[H(C^*x)]_j = \int_0^1 \left(\int_t^1 \frac{x(s)}{s}\, ds \right) t^{j-1}\, dt = \frac{1}{j} \int_0^1 x(t)\, t^{j-1}\, dt = \frac{1}{j}\, [Hx]_j \quad (j \in \mathbb{N})\,.$$

Evidently, $H \circ C^* : L^2(0,1) \to \ell^2$ is a **compact operator**. By now, as in the case $H \circ J$, a decision **moderate versus severe ill-posedness** has not yet been made.

For more details see:

▷ S. KINDERMANN AND B.H.: Curious ill-posedness phenomena in the composition of non-compact linear operators. arXiv:2401.14701v1, Jan. 2024.

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Measuring ill-posedness of non-compact operators by spectral theorem and decreasing rearrangements

One more option for progress in evaluating the **type I ill-posedness** for non-compact operators $A : X \to Y$ is to use the **Halmos spectral theorem**, which says that there exists a locally compact space Ω, a semi-finite measure μ on Ω, an a.e. non-negative function $\lambda \in L^\infty(\Omega, \mu)$, and a unitary mapping $U : L^2(\Omega, \mu) \to X$ such that

$$U^* A^* A U = M_\lambda ,$$

with the multiplication operator $M_\lambda : L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ defined as

$$[M_\lambda \xi](\omega) := \lambda(\omega) \cdot \xi(\omega) \qquad (\mu - \text{a.e. on } \Omega, \quad \xi \in L^2(\Omega, \mu)).$$

The essential range $\mathrm{essran}(\lambda)$ of the **multiplier function** λ coincides with the spectrum $\mathrm{spec}(A^* A)$ of the self-adjoint operator $A^* A$.

Main idea for case $\mu(\Omega) = \infty$:
Inspection of the decay rate at infinity of the **decreasing rearrangement** λ^* of λ.

See for further details:

▷ P. MATHÉ, B.H. AND M. T. NAIR: Regularization of linear ill-posed problems involving multiplication operators. *Appl. Anal.* **101** (2022), pp. 714–732.

▷ F. WERNER AND B.H.: A unified concept ... under the auspices of the spectral theorem. Paper in preparation, 2024.

A prominent role is due to the **distribution function**

$$\Phi_\lambda(\epsilon) := \mu\left(\{\omega \in \Omega : \lambda(\omega) > \epsilon\}\right) \quad (\epsilon > 0)\,.$$

Assumption

(a) $\mu(\Omega) = \infty$.

(b) It holds $0 \le \Phi_\lambda(\epsilon) < \infty$ for all $\epsilon > 0$ and $\lim_{\epsilon \to 0} \Phi_\lambda(\epsilon) = \infty$.

Then **decreasing rearrangement** λ^* of λ with $\lim_{t\to\infty} \lambda^*(t) = 0$

$$\lambda^*(t) = \Phi_\lambda^{-1}(t) := \inf\{\tau > 0 : \Phi_\lambda(\tau) \le t\} \quad (0 \le t < \infty)$$

is well-defined and allows for decay inspection at the infinity in order to evaluate the degree of ill-posedness.

Example: Severe ill-posedness of the infinite Hilbert matrix $\mathbb{H} = H\,H^* : \ell^2 \to \ell^2$

For $\Omega = [0, \infty)$ with the Lebsgue measure μ on $\mathbb{R}$ we take from

▷ M. ROSENBLUM: On Hilbert matrix II. *Proc. Amer. Math. Soc* **9** (1958), pp. 581-585.

the multiplier function

$$\lambda(\omega) = \frac{\pi}{\cosh(\pi\omega)} \quad (\omega \in [0, \infty) \text{ a.e.}).$$

This leads to the distribution function

$$\Phi_\lambda(\epsilon) \approx \frac{1}{\pi} \log\left(\frac{2\pi}{\epsilon}\right) \quad \text{for sufficiently small } \epsilon > 0\,,$$

which gives the asymptotics of decreasing rearrangement $\lambda^* = \Phi_\lambda^{-1}$ as

$$\lambda^*(t) \asymp \exp(-t) \quad \text{as} \quad t \to \infty\,.$$

Inspection indicates kind of severe ill-posedness for the Hausdorff moment problem.

However, no statement about the degree of ill-posedness of the composition $A = H \circ J$ can be concluded from this example.

Uncharted territory at the end:

Example: Sinusoidal behaviour of multiplier function

Consider $\Omega = [0, \infty)$ and the associated Lebesgue measure μ on $\mathbb{R}$ with $\mu(\Omega) = \infty$. Multiplier function $\lambda(\omega) = \sin^2(\omega)$ $(\omega \in [0, \infty))$ is due to a non-compact operator A^*A and leads to $\Phi_\lambda(\epsilon) = \begin{cases} \infty & \text{for} \quad 0 < \epsilon < 1, \\ 0 & \text{for} \quad \epsilon \geq 1 \end{cases}$ as distribution function, where item (b) of Assumption fails. Decreasing rearrangement λ^* also fails. No idea of degree!

Example: Non-compact diagonal operator

Consider $\Omega = \mathbb{N}$ and the counting measure μ with $\mu(\Omega) = \infty$. Multiplier function

$$\lambda(n) = \begin{cases} n^{-2} & \text{if} \quad n = k^2 \ \text{for some} \ k \in \mathbb{N}, \\ 1 & \text{else} \end{cases}$$

is due to a non-compact diagonal operator $A^*A : \ell^2 \to \ell^2$ and leads to $\Phi_\lambda(\epsilon) = \infty$ for all $\epsilon > 0$ as distribution function, where item (b) of Assumption fails. No idea of degree!