# Deautoconvolution: a new decomposition approach versus TIGRA and local regularization

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**Abstract**

Solving an autoconvolution equation is a nonlinear ill-posed inverse problem. Besides standard methods for general nonlinear problems several customized methods for deautoconvolution are available. Recently, a new decomposition approach for solving ill-posed quadratic equations, e.g. autoconvolutions, has been proposed.

In this article we compare the new approach to the TIGRA method of Ramlau and to the local regularization method of Dai and Lamm. Numerical tests show that the new method yields better approximations to the unknown true solution than existing methods in comparable computation time.

## 1 Introduction

Autoconvolution equations appear in different fields of mathematics and natural sciences. Examples can be found in stochastics, where the probability density of the sum of two independent and identically distributed random variables is the autoconvolution of the density of the two random variables,

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and in physics, where the method of appearance potential spectroscopy leads to autoconvolution equations [1].

In this article we consider the simplest form of autoconvolution problems. We search for approximate solutions of

$$F(x) = y^0 \tag{1.1}$$

where the autoconvolution operator for functions $x$ on $\mathbb{R}$ is formally defined as

$$[F(x)](s) = \int_{\mathbb{R}} x(s-t)x(t)\,\mathrm{d}t.$$

If now $x$ is zero almost everywhere outside $[0,1]$ and its restriction to $[0,1]$ belongs to $L^2(0,1)$ then $F(x)$ is well-defined and zero outside $[0,2]$, so we may consider $F$ as an operator $F : L^2(0,1) \to L^2(0,2)$ with

$$\big(F(x)\big)(s) := \begin{cases} \int_0^s x(t)x(s-t)\,\mathrm{d}t, & s \in (0,1), \\ \int_{s-1}^1 x(t)x(s-t)\,\mathrm{d}t, & s \in [1,2). \end{cases} \tag{1.2}$$

The exact right-hand side $y^0$ in (1.1) is not accessible due to measurement errors and ill-posedness of deautoconvolution problems forces us to appropriately handle this data error. Available measurement data will be denoted by $y^\delta$ and we assume that the data error is bounded in $L^2(0,2)$ by some nonnegative noise level $\delta$, that is,

$$\|y^\delta - y^0\| \le \delta.$$

Ill-posedness of autoconvolution equations is investigated in [4, 5, 8].

In addition to the original autoconvolution problem (1.1) more sophisticated problems can be found in its vicinity. Of particular interest to the authors is a complexvalued version of autoconvolution which also has an additional kernel function in the integral. Such autoconvolution-type problems appear in the SD-SPIDER method for characterizing ultra-short laser pulses [7, 12]. As a first step in this direction we look at algorithms for the original problem.

Autoconvolution problems are of quadratic structure, that is, we may write $F(x)$ as $B_F(x,x)$ with a symmetric bounded bilinear mapping $B_F : L^2(0,1) \times L^2(0,1) \to L^2(0,2)$, see Section 2 for details. But standard algorithms for solving ill-posed nonlinear equations (see [3,11]) do not care about

this structure and thus maybe do not yield best possible results. Only few numerical methods adapted to autoconvolution or other quadratic problems can be found in the literature. Basically two methods are available:

- TIGRA method by Ramlau [13] (nonlinear Tikhonov regularization with gradient-based minimization),
- local regularization by Dai and Lamm [2] (data smoothing and explicit inversion).

A third approach based on decomposition of the autoconvolution operator $F$ into a well-posed quadratic and an ill-posed linear part has been proposed by the second author in [6].

The aim of this article is to further develop the new decomposition approach and to compare it to the TIGRA method and to local regularization.

In the next section we present the new decomposition approach. Therefor we summarize the results from [6] and provide an extension of the method described there. In Section 3 we outline the TIGRA method and the local regularization approach. The fourth section contains numerical tests and their results, and in Section 5 we discuss those results and look at the time complexity of all three methods.

## 2 Regularization of quadratic mappings by decomposition

In [6] the second author proposed a decomposition approach for solving quadratic equations. As already mentioned in the introduction a mapping $F$ is quadratic if there is a symmetric bounded bilinear mapping $B_F$ such that $F(x) = B_F(x, x)$ for all $x$. In case of autoconvolution we have

$$\big(B_F(x,u)\big)(s) := \int_{\mathbb{R}} x(t)u(s-t)\,\mathrm{d}t, \quad s \in (0,2). \tag{2.1}$$

In this article we only summarize the principle ideas of [6] and provide a concrete solution of an issue only broached in [6].

Choosing an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ in $X$ and defining mappings $Q : X \to \ell^2(\mathbb{N})$ and $A : \mathcal{D}(A) \subseteq \ell^2(\mathbb{N}) \to Y$ by

$$\big(Q(x)\big)_{j+\frac{i(i-1)}{2}} := \begin{cases} \sqrt{2}\langle x, e_i\rangle\langle x, e_j\rangle, & j < i, \\ \langle x, e_i\rangle^2, & j = i \end{cases} \tag{2.2}$$

for $(i,j)$ in $\mathbb{N}\times\mathbb{N}$ with $1\le j\le i$ and $x$ in $X$ and by

$$Az := \sum_{i=1}^{\infty}\left(\sum_{j=1}^{i-1}\sqrt{2}z_{j+\frac{i(i-1)}{2}}B_F(e_i,e_j)+z_{i+\frac{i(i-1)}{2}}B_F(e_i,e_i)\right) \tag{2.3}$$

for all $z$ which yield a convergent series one easily verifies

$$F = AQ. \tag{2.4}$$

Note that $(i,j)\mapsto j+\frac{i(i-1)}{2}$ defines a bijection between $\{(i,j)\in\mathbb{N}\times\mathbb{N}: 1\le j\le i\}$ and $\mathbb{N}$.

The mapping $Q$ is a quadratic isometry, that is, $\langle Q(x),Q(u)\rangle = \langle x,u\rangle^2$ for all $x$ and $u$. In [6] the interested reader finds more details on quadratic isometries. The most important fact in view of inverse problems is that $Q$ is in a slightly generalized sense continuously invertible. In other words, solving the equation $Q(x)=z$ is a well-posed problem. In addition the minimization problem

$$\|Q(x)-z\|\to\min_{x\in X} \tag{2.5}$$

has a solution and each global minimizer attains the form $\sqrt{\lambda}x$ where $\lambda$ is the largest eigenvalue of a certain Hilbert-Schmidt operator and $x$ is a corresponding normalized eigenelement. With respect to the basis $(e_i)_{i\in\mathbb{N}}$ used for defining $Q$ the mentioned Hilbert-Schmidt operator has the symmetric matrix representation $C_z\in\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ given by

$$(C_z)_{i,j} = \begin{cases} \frac{1}{\sqrt{2}}z_{j+\frac{i(i-1)}{2}}, & j<i,\\ z_{j+\frac{i(i-1)}{2}}, & j=i. \end{cases} \tag{2.6}$$

The mapping $A$ from (2.3) is a densely defined linear operator. Due to ill-posedness the corresponding equation $Az=y^\delta$ has to be regularized. Note that $A$ might be unbounded. Regularization of unbounded linear operators is handled in [10], for instance.

As we have seen, the decomposition approach results in two steps for solving a quadratic equation (1.1): at first solve the ill-posed linear equation $Az=y^\delta$ with the help of some regularization method for linear equations, then solve the minimization problem (2.5) by finding the largest eigenvalue and a corresponding eigenelement of a certain Hilbert-Schmidt operator.

The problem with this decomposition technique is, that the regularized solution $z_\alpha^\delta$ of $Az=y^\delta$ with regularization parameter $\alpha$ typically lies in

$(\ker A)^\perp$, the orthogonal complement of the nullspace, and we cannot guarantee that $(\ker A)^\perp$ is a subset of $\mathcal{R}(Q)$, the range of $Q$. This problem was already encountered in [6] and two rough sketches for handling this issue were given there. In the following we give more details on one of them.

The basic idea is to use a kind of iterated Tikhonov regularization. The method calculates a regularized solution $x_\alpha^\delta$ to (1.1) as follows:

0. Choose $x_0 = 0$ and $k = 0$.
1. Set $\bar{z}_k = Q(x_k)$.
2. Calculate $z_k^\delta$ as the minimizer of $z \mapsto \|Az - y^\delta\|^2 + \alpha\|z - \bar{z}_k\|^2$, where $\alpha$ is chosen according to some parameter choice rule (standard Tikhonov regularization).
3. Calculate $x_{k+1}$ as a minimizer of $x \mapsto \|Q(x) - z_k^\delta\|$.
4. If $\|x_{k+1} - x_k\|$ is small enough set $x_\alpha^\delta = x_{k+1}$ and stop. Otherwise increase $k$ by one and go to step 1.

Note that this is not the classical iterated Tikhonov method since the reference element in the penalty is not the previous iterate but the projection of the previous iterate onto the range of $Q$. In addition the regularization parameter is chosen in each iteration, which is not the case for the classical method. The element $x_1$ is the regularized solution one obtains from the decomposition approach if standard Tikhonov regularization is applied to $Az = y^\delta$. The idea behind the proposed algorithm is to gradually bring the intermediate values $z_k^\delta$ close to the range of $Q$. Thus, the projection onto $\mathcal{R}(Q)$ has not too much negative influence. To our regret at the moment we do not have a convergence proof for this method. All numerical experiments indicate convergence of the $x_k$ and yield satisfactory regularized solutions. Very good numerical results in comparison to existing methods motivated us to publish the technique and to postpone a deeper theoretical investigation to a future article.

## 3 Other regularization methods

In this section we outline the TIGRA and the local regularization method for solving equation (1.1) approximately with given noisy data $y^\delta$. Throughout this section we use the abbreviations $X := L^2(0,1)$ and $Y := L^2(0,2)$.

### 3.1 TIGRA by Ramlau

In [13] Ramlau proposes a method for regularizing ill-posed bilinear mappings and other twice differentiable mappings. This so called TIGRA method can also be applied to general quadratic mappings, especially to autoconvolution problems. All in all, the TIGRA method is a sophisticated realization of Tikhonov regularization for nonlinear equations. The idea is to apply the steepest descent method to the Tikhonov minimization problem

$$T_\alpha^\delta(x) := \|F(x) - y^\delta\|^2 + \alpha\|x - \bar{x}\|^2 \to \min_{x \in X} \tag{3.1}$$

with regularization parameter $\alpha$ and a priori information $\bar{x}$. The important observation is that for sufficiently large $\alpha$ the steepest descent method converges to a global minimizer of the Tikhonov functional. Choosing this minimizer as initial guess for minimizing the functional with smaller $\alpha$ again yields convergence to a global minimizer. Following this idea one obtains Tikhonov regularized solutions for a decreasing sequence of regularization parameters.

The investigation and realization of the TIGRA method in [13] involves several parameters and requires many more or less restrictive assumptions. As the author of [13] remarks, some assumptions have to be neglected for numerical realization. In our numerical experiments we use the following algorithm.

0. Choose $\alpha_0$ large enough and $q$ in $(0, 1)$.
1. Compute $x_0$ as a global minimizer of $x \mapsto \|F(x) - y^\delta\|^2 + \alpha_0\|x\|^2$ (steepest descent with starting point $t \mapsto 1$, the constant function, see below).
2. For each $k$ in $\mathbb{N}$ compute $x_k$ as the minimizer of $x \mapsto \|F(x) - y^\delta\|^2 + q^k\alpha_0\|x\|^2$ (steepest descent with starting point $x_{k-1}$, see below).

Note that the TIGRA method is not able to compute a regularized solution for only one fixed regularization parameter $\alpha$ (except for very large $\alpha$). Instead, it always produces regularized solutions for a sequence of parameters. From this sequence a suitable parameter can be chosen by well-known parameter choice rules, in our case the discrepancy principle and the quasioptimality criterion (see below).

The steepest descent method for computing $x_k$ uses step direction

$$d(x) := -2\big(F'[x]^*(F(x) - y^\delta) + q^k\alpha_0 x\big) \tag{3.2}$$

at position $x$. The corresponding step length is chosen as follows

1. Choose $\rho$ and $\gamma$ such that

$$\rho \in \big(0, \tfrac{1}{6}\big), \qquad \gamma \in \left(0, \frac{1-6\rho}{2}\right].$$

2. Set

$$\alpha = q^k \alpha_0, \qquad K = \max\left\{2\|y^\delta\|, \frac{4\rho}{\sqrt{1-2\rho}}\right\}, \qquad \nu = 1 - 6\rho - \gamma$$

and

$$r(\alpha) = \frac{1}{2+2\sqrt{2}} \min\left\{\sqrt{\frac{2\nu\alpha}{3}}, \frac{2\nu\alpha}{3K}\right\}.$$

Then set

$$\begin{aligned}
\kappa_1 &= 2r(\alpha) + K, \\
\kappa_2 &= 3\rho\alpha + Kr(\alpha) + r(\alpha)^2, \\
\kappa_3 &= \alpha^2 r(\alpha)^2 + \alpha\|y^\delta\|^2 + 2\alpha \max\big\{\alpha r(\alpha)^2, \|y^\delta\|^2\big\}
\end{aligned}$$

and

$$\kappa = 2(\kappa_1\kappa_2 + \kappa_3).$$

Further, set

$$M = 2\kappa_1^2 + 4\kappa_2 + 2\alpha + 12\kappa_1\kappa + 12\kappa^2$$

and

$$c = 4K^2 + 4\alpha + 24\rho\alpha + 8K + 4.$$

See [13] for details on all these constants.

3. Compute $t_{\min}$ as minimizer of

$$T_\alpha^\delta\big(x + td(x)\big) \to \min_{t>0}. \tag{3.3}$$

This is a polynomial in $t$ of degree 4.

4. Set the step length

$$\min\left\{\frac{\gamma\alpha}{\|d(x)\|^2}, \frac{4\gamma\alpha\big(T_\alpha^\delta(x) - T_\alpha^\delta\big(x + t_{\min}d(x)\big)\big)}{c\|d(x)\|^2}, \frac{1}{M}, 1\right\}. \tag{3.4}$$

The steepest descent iteration is stopped if the current iterate $x$ satisfies

$$\|d(x)\| \leq \frac{\gamma\alpha}{4+4\sqrt{2}} \min\left\{\sqrt{\frac{2\nu}{3}}, \frac{2\nu}{3K}\right\} \min\{q\alpha, \sqrt{q\alpha}\}. \tag{3.5}$$

For starting the steepest descent iteration for $x_0$ we use a constant function as starting point. Following [13] the starting point is not importance since the Tikhonov functional is globally convex if $\alpha_0$ is large enough. But at the natural starting point zero the Tikhonov functional has Frechét derivative zero and thus the step direction is zero, too.

### 3.2 Local regularization by Dai and Lamm

In [2] Dai and Lamm present a method for deautoconvolution if data is only available on the interval $(0, 1+R)$ with $R$ between 0 and 1 instead of $(0, 2)$. Their method is design in such a way that $R$ plays the role of the regularization parameter. Based on an idea called local regularization the original equation (1.1) is transformed into

$$2x(t)\int_0^R\int_0^\rho x(s)\,\mathrm{d}s\,\mathrm{d}\rho + \int_0^R\int_\rho^t x(t+\rho-s)x(s)\,\mathrm{d}s\,\mathrm{d}\rho = \int_0^R y^\delta(t+\rho)\,\mathrm{d}\rho, \tag{3.6}$$

which shall hold for all $t \in (0, 1)$.

Discretization with normalized box functions in $X$ and normalized hat functions in $Y$, cf. also next section, and approximation of integrals by rectangular quadrature leads to the following steps for obtaining the discretized solution $\underline{x}^\delta = (x_1^\delta, \ldots, x_n^\delta)$ to (3.6) from discretized data $\underline{y}^\delta$ (see [2] for details):

0. Choose the regularization parameter $r$ from $\{1, \ldots, n\}$, see below.

1. Set
$$x_1^\delta = \left(\frac{3n}{2}\right)^{\frac{1}{4}} \sqrt{y_1^\delta}.$$

2. For $i = 2, \ldots, r$ set
$$x_i^\delta = \frac{1}{2x_1^\delta}\left(\sqrt{\frac{3}{2}} n y_i^\delta - \sum_{j=2}^{i-1} x_j^\delta x_{i+1-j}^\delta\right).$$

3. Set

$$c = 2\sum_{k=1}^{r}\sum_{l=1}^{k} x_l^\delta.$$

4. For $i = r+1, \ldots, n$ set

$$x_i^\delta = \frac{1}{c}\left(\sqrt{\frac{3n}{2}}\sum_{j=0}^{r-1} y_{i+j}^\delta - \sum_{k=1}^{r}\sum_{l=k+1}^{i-1} x_l^\delta x_{i+k-l}^\delta\right).$$

Note that this algorithm only works for $y_1^\delta > 0$ as already mentioned in [2]. The discretized regularization parameter $r$ can take values from $\{1, \ldots, n\}$. The corresponding continuous parameter is $R = \frac{r}{n}$, that is, the method uses data on $(0, 1 + \frac{r}{n})$ only.

The regularization parameter $r$ causes two problems. On the one hand there are only $n$ possible choices, which provides only coarse control of the regularization process. On the other hand the interpretation of $r$ is different from the usual interpretation of the regularization parameter $\alpha$ in other methods. As described in [2] the algorithm calculates the first $r$ elements of $\underline{x}$ without regularization and the remaining elements with regularization by using additional data of length $\frac{r}{n}$. Thus, we observe the following role of $r$: If $r$ attains its smallest value, then the algorithm yields the unregularized solution of (1.1). If $r$ attains its largest value, also no regularization takes places and $\underline{x}$ coincides with the unregularized solution of (1.1), too. The influence of $r$ is not monotone and thus $r$ cannot be transformed into a real positive regularization parameter showing the typical behavior. In our numerical experiments below we calculate regularized solutions for all $r$ and then choose the one satisfying the discrepancy principle or the discrete quasioptimality criterion as described above.

# 4 Numerical tests

## 4.1 Parameter choice and discretization

In our numerical test below we apply two different parameter choice rules for selecting a suitable regularization parameter $\alpha$. The discrepancy principle is an a posteriori rule, that is, it requires knowledge of the noise level $\delta$. Contrary, the quasi optimality principle is heuristic, thus, it works without knowing $\delta$, but the theoretical foundation is less strong.

Choosing $\alpha$ according to the discrepancy principle means that $x_\alpha^\delta$ satisfies

$$\delta \leq \|F(x_\alpha^\delta) - y^\delta\| \leq \tau\delta \tag{4.1}$$

with some fixed $\tau$ close to one. There are also other common versions of the discrepancy principle, for instance the sequential discrepancy principle [9].

The quasi optimality criterion in its discrete form chooses $\alpha = q^{k^*}\alpha_0$, where $q$ is between zero and one, $\alpha_0$ is an upper bound for the regularization parameter, and $k^*$ is the minimizer of

$$\left\|x_{q^k\alpha_0}^\delta - x_{q^{k-1}\alpha_0}^\delta\right\| \to \min_{k\in\mathbb{N}}. \tag{4.2}$$

For all three regularization methods we use the same discretization, namely we decompose the interval $(0,1)$ into $n$ subintervals of equal length and approximate an element $x$ from $L^2(0,1)$ by

$$x \approx \sum_{i=1}^{n} x_i e_i,$$

where $e_i$ is the normalized box function on $(\frac{i-1}{n}, \frac{i}{n})$. The coefficient vector $(x_1, \ldots, x_n)$ will be denoted by $\underline{x}$. Note that the set $\{e_1, \ldots, e_n\}$ is an orthonormal system in $L^2(0,1)$. The subspace spanned by these box functions will be denoted by $X_n$.

Simple computations show that the autoconvolution operator $F$ transforms an element from $X_n$ into a piecewise affine function on $(0,2)$ with equispaced grid points at $\frac{j}{n}$ for $j = 0, 1, \ldots, 2n$. In addition $(F(x))(0) = 0$ and $(F(x))(2) = 0$ for all $x$ from $X_n$. The $(2n-1)$-dimensional span of $F(X_n)$ will be denoted by $Y_{2n-1}$. Each element from $Y_{2n-1}$ can be written as a sum

$$y = \sum_{j=1}^{2n-1} y_j f_j,$$

where $f_j$ is the normalized hat function with center $\frac{j}{n}$ and width $\frac{2}{n}$. The corresponding coefficient vector $(y_1, \ldots, y_{2n-1})$ will be denoted by $\underline{y}$.

## 4.2 Implementation details

For our numerical tests we simulate data in the following way: We choose an exact solution $x^0$ in $L^2(0,1)$ and use numerical integration to obtain values for $F(x^0)$ on a very fine grid. From these values we form a coarser piecewise affine approximation of $F(x^0)$ with grid points $\frac{1}{n}, \frac{2}{n}, \ldots, \frac{2n-1}{n}$. Remember

that $(F(x^0))(0) = 0$ and $(F(x^0))(2) = 0$ is always satisfied. The resulting vector will be denoted by $\underline{y}^0$. Then we add noise in form of a vector $\underline{e}$ of elements independently drawn from a Gaussian distribution with mean zero and standard deviation one, that is, we set

$$\underline{y}^\delta = \underline{y}^0 + \delta \underline{e}.$$

For our experiments below we always provide the relative noise level

$$\delta_{\text{rel}} := \frac{\delta \|e\|}{\|y^0\|}.$$

Here $y^0$ and $e$ denote the functions in $L^2(0,2)$ corresponding to $\underline{y}^0$ and $\underline{e}$, respectively, as described above in Section 3.

For the discrepancy principle we use $\tau = 1.2$. For the quasioptimality criterion we use $q = 0.9$ and $q = 0.5$ for the TIGRA method and the decomposition approach, respectively, and choose $\alpha_0$ large enough to prevent the quasioptimality criterion from choosing $q\alpha_0$, that is, the minimum of (4.2) lies not at the boundary of the considered range of parameters. Hints on the choice of the regularization parameter $r$ for the local regularization approach by Dai and Lamm were given above.

Throughout our experiments we use the discretization level $n = 300$.

Step length selection in the TIGRA method requires the choice of two parameters $\rho$ and $\gamma$. We use $\rho = 0.05$ and $\gamma = 0.2$.

The iteration of the Tikhonov step in our decomposition method is stopped if the distance between two iterates $x_k$ and $x_{k+1}$ becomes smaller than $10^{-2}$. If this does not happen within the first 10 iterations, then the iteration is stopped, too.

It is important to note that the autoconvolution operator $F$ is not injective. If $x$ is a solution to (1.1) then also $-x$ is a solution. The three algorithms choose one of these two possibilities more or less by chance. For plotting our results we chose the sign by hand in such a way that easy comparison of the results is possible.

### 4.3 Test A: quadratic function

The exact solution $x^\dagger$ for our first numerical experiment is the quadratic function given by

$$x^\dagger(t) = 1 - 3(t - \tfrac{1}{2})^2, \quad t \in (0,1).$$

This function was also considered in [2] for testing the local regularization approach. We run all three methods with 1% and 5% relative noise. Exact and noisy data as well as corresponding reconstructions from TIGRA method and our decomposition approach are shown in Figure 1.

Since the local regularization approach did not provide any useful reconstruction we test our algorithms also with the noise model and noise level used in [2]. There the authors use pointwise relative noise, that is, each component $\underline{y}_k^\delta$ of the noisy data vector is drawn from a uniform distribution on the interval $[0.99\,\underline{y}_k^0, 1.01\,\underline{y}_k^0]$. This corresponds to an overall relative noise level of 0.46%. Data and results are shown in Figure 2. Note that the regularized solution corresponding to the regularization parameter chosen by the discrepancy principle shows high oscillations. Thus, we did not plot it in Figure 2 (d) to keep the scaling of the vertical axis.

### 4.4 Test B: periodic function

The exact solution $x^\dagger$ for our second numerical experiment is the periodic function given by

$$x^\dagger(t) = \sin(4\pi t), \quad t \in (0,1).$$

Since TIGRA method and our decomposition approach seem to work fine we choose a relative noise level of 10%. Data and reconstructions are shown in Figure 3.

As already observed in Test A above, local regularization does not yield any useful results for such a high noise level.

### 4.5 Test C: discontinuous function

The exact solution for our third and last numerical experiment is a piecewise constant function given by

$$x^\dagger(t) = \begin{cases} 0.5, & \text{if } t \in [0, 0.5), \\ 0.25, & \text{if } t \in [0.5, 0.8), \\ 0.75, & \text{if } t \in [0.8, 1], \end{cases} \quad t \in (0,1).$$

We use a relative noise level of 5%. Again local regularization is ruled out due to lacking regularization. Data and results for TIGRA method and decomposition technique are shown in Figure 3.

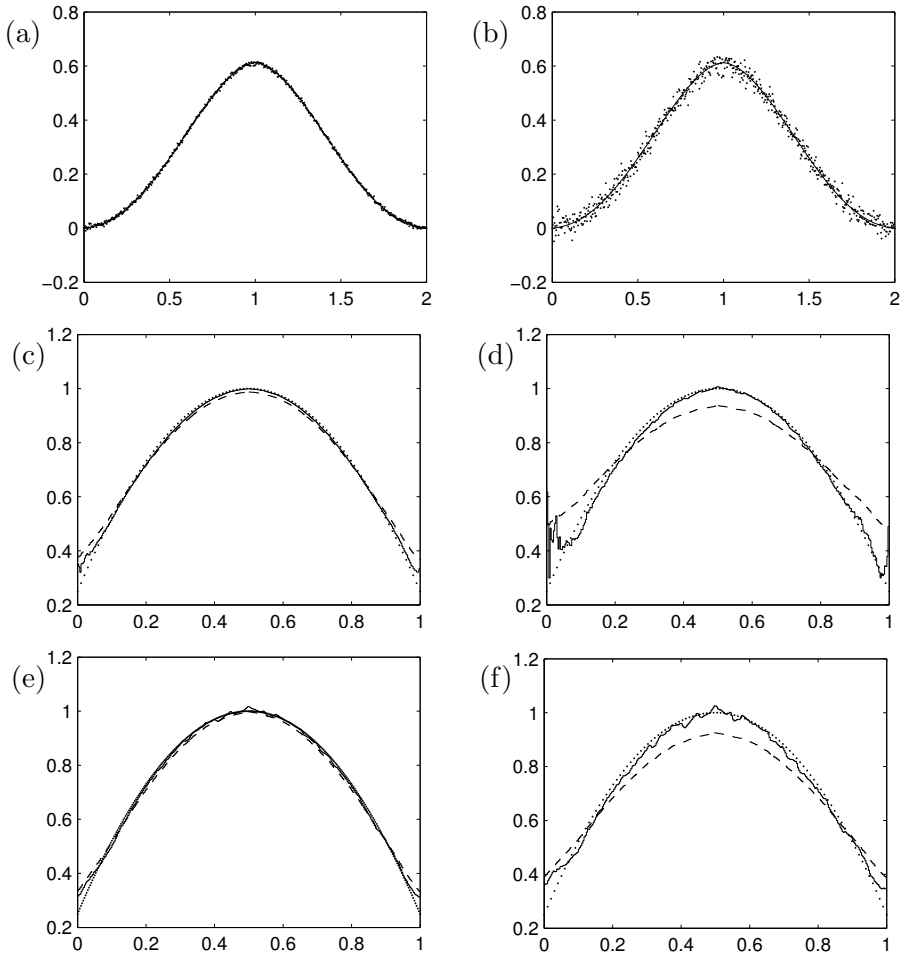


Figure 1: Test A with 1% (left column) and 5% (right column) relative noise, top row: exact data (solid) and noisy data (dots), middle row: exact solution (dotted line), results from decomposition approach with discrepancy principle (dashed line) and quasioptimality criterion (solid line), bottom row: same as middle row but with TIGRA method.

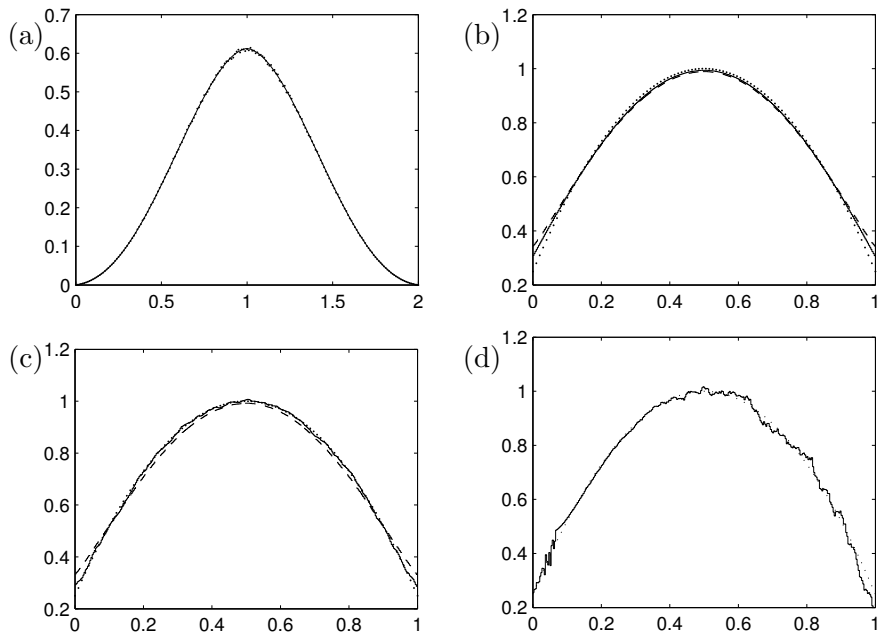


Figure 2: Test A with pointwise relative noise, (a) exact data (solid) and noisy data (dots), (b) exact solution (dotted line), results from decomposition approach with discrepancy principle (dashed line) and quasioptimality criterion (solid line), (c) same as (b) but with TIGRA method, (d) results from local regularization with quasioptimality criterion (solid line).

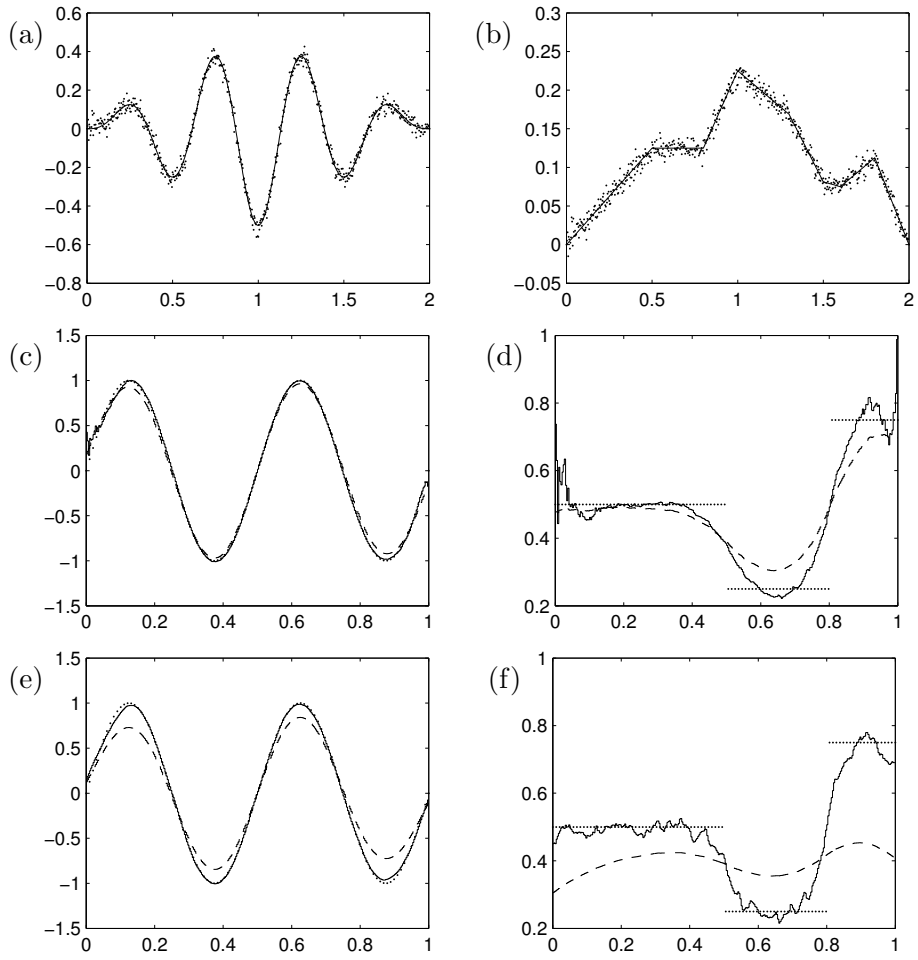


Figure 3: Test B with 10% relative noise (left column) and test C with 5% relative noise (right column), top row: exact data (solid) and noisy data (dots), middle row: exact solution (dotted line), results from decomposition approach with discrepancy principle (dashed line) and quasioptimality criterion (solid line), bottom row: same as middle row but with TIGRA method.

### 4.6 Regularization parameters and total errors

Table 1 shows the regularization parameters obtained by discrepancy principle and quasioptimality criterion for tests A, B, and C. For the decomposition approach multiple parameters have to be chosen, one in each iteration. We do not list them all but we provide the smallest (first line in corresponding cell) and the largest (second line in corresponding cell) regularization parameter for each test.

| Test | $\delta_{\text{rel}}$ | decomposition approach | | TIGRA | | local regularization | |
|---|---|---|---|---|---|---|---|
| | | $\alpha_{\text{d}}$ | $10^9 \cdot \alpha_{\text{q}}$ | $\alpha_{\text{d}}$ | $\alpha_{\text{q}}$ | $r_{\text{d}}$ | $r_{\text{q}}$ |
| A | 5% | 0.0286 | 1.9074 | 0.0828 | 0.0112 | - | 4 |
| | | 1.1810 | 1.9074 | | | | |
| | 1% | 0.0057 | 1.9074 | 0.0154 | 0.0035 | - | 4 |
| | | 0.2542 | 3.8147 | | | | |
| | 0.46% | 0.0026 | 3.8147 | 0.0171 | 0.0017 | 4 | 20 |
| | | 0.0922 | 15.259 | | | | |
| B | 10% | 0.0515 | 1.9074 | 0.1282 | 0.0102 | - | 4 |
| | | 1.8530 | 1.9074 | | | | |
| C | 5% | 0.0194 | 1.9074 | 0.0766 | 0.0040 | - | 9 |
| | | 0.4575 | 1.9074 | | | | |

Table 1: Regularization parameters $\alpha_{\text{d}}$ and $\alpha_{\text{q}}$ obtained by discrepancy principle and quasioptimality criterion, respectively.

Table 2 contains the deviations of the reconstructed solutions from the exact ones, measured in the $L^2(0,1)$-norm of the corresponding piecewise constant functions.

## 5 Comparison of the three methods

### 5.1 Conclusions from the numerical experiments

The major conclusion from our numerical experiments above is that our decomposition approach yields reconstructions comparable to the results of the TIGRA method. From Table 2 we see that in some cases the TIGRA method is slightly better and in other cases the decomposition technique shows slightly smaller errors. But these small deviations in the errors might also be caused by the parameter choice strategies.

| Test | $\delta_{\text{rel}}$ | decomposition approach | | TIGRA | | local regularization | |
|---|---|---|---|---|---|---|---|
| | | $\mu_{\text{d}}$ | $\mu_{\text{q}}$ | $\mu_{\text{d}}$ | $\mu_{\text{q}}$ | $\mu_{\text{d}}$ | $\mu_{\text{q}}$ |
| A | 5% | 0.0831 | 0.0429 | 0.0824 | 0.0336 | - | 0.9535 |
| | 1% | 0.0325 | 0.0168 | 0.0278 | 0.0154 | - | 0.3906 |
| | 0.46% | 0.0250 | 0.0160 | 0.0280 | 0.0092 | 0.2515 | 0.0240 |
| B | 10% | 0.0510 | 0.0425 | 0.1897 | 0.0335 | - | 17.5 |
| C | 5% | 0.0853 | 0.0680 | 0.2117 | 0.0665 | - | 0.4648 |

Table 2: Total reconstruction errors, $\mu_{\text{d}}$ denotes the error for reconstruction with discrepancy principle $\mu_{\text{q}}$ the error for reconstruction with quasioptimality criterion.

The local regularization approach does not give any useful result, except for very low noise levels. And even for such low noise levels that approach cannot keep up with the other algorithms. In [2] regularizing properties of local regularization are proven under various assumptions. But in view of our experiments the regularizing effect is too weak.

The computation time for the TIGRA method in all cases was much longer than for the decomposition technique because the TIGRA method requires many iterations till the stopping criterion is satisfied. Of course, computation times always depend on the concrete implementation and maybe further code optimization is possible. But we think that the choice of the step length and the stopping criterion, which differentiates between TIGRA method and steepest descent method, is the reason for the long computation time. In the next subsection we will compare the time complexity of all three methods and we will see that the TIGRA method should be faster than the decomposition technique, but only if the number of TIGRA iterations is lower than the squared discretization parameter $n$. The local regularization technique is extremely fast, but, as described above, often produces no useful results.

Concerning parameter choice strategies we observed that both the discrepancy principle and the quasioptimality criterion provide acceptable regularization parameters. The discrepancy principle yields parameters which seem to be somewhat too large, causing oversmoothing. In contrast, the quasioptimality criterion chooses the regularization parameter in some cases slightly too small. This is especially the case in combination with the decomposition approach, causing oscillations at the boundaries of the domain of the reconstructed function, see Figure 1 (d) and Figure 3 (d).

Finally we observed that for the decomposition technique the quasioptimality criterion chooses the regularization parameter extremely small in comparison to the discrepancy principle, but nevertheless the corresponding reconstructions seem to be regularized properly. Up to now we do not have a complete interpretation of this fact. But further theoretical investigation of the decomposition technique will bring light into this.

### 5.2 Computational complexity of the three methods

The TIGRA method requires numerical evaluation of $F$, which can be realized with $\mathcal{O}(n^2)$ elementary operations. Application of the Frechét derivative $F'[x]$ also requires $\mathcal{O}(n^2)$ operations. Step length selection lies below these counts. Thus each steepest descent step requires $\mathcal{O}(n^2)$ operations. The number of steepest descent iterations varies in a wide and does not only depend on the discretization level $n$ but also on $\alpha$ and $y^\delta$. Thus, per regularization parameter the TIGRA method requires

$$\mathcal{O}\big(\text{iterations} \cdot n^2\big) \tag{5.1}$$

elementary operations.

According to [2], for each regularization parameter $r$ the local regularization approach requires

$$\mathcal{O}(n^2 r - n r^2) \tag{5.2}$$

elementary operations if $r$ is much smaller than the discretization level $n$.

The discretization of the decomposition approach is described in [6]. Applying the quadratic isometry $Q$ to $x$ from the span of the $n$ box functions gives an $\ell^2(\mathbb{N})$-element with at most $\frac{n(n+1)}{2}$ nonzero components. The linear operator $A$ then transforms this finitely supported element into a piecewise affine function in the span of the hat functions used for discretization in $L^2(0,2)$. Thus, the Tikhonov step has to approximate the solution of a system of linear equations with system matrix $\underline{A}$ of dimension $(2n-1) \times \frac{n(n+1)}{2}$. Solving the Tikhonov minimization problem leads to a system with matrix $\underline{A}^T\underline{A}+\alpha\underline{I}$ of size $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$. Solving this system without further knowledge of the matrix' structure would require $\mathcal{O}(n^6)$ operations. But in case of autoconvolution the matrix $\underline{A}^T\underline{A} + \alpha\underline{I}$ is sparse and its structure allows to split the corresponding system of equations into $2n-1$ systems with square matrices of size $1, 2 \ldots, n, \ldots, 2, 1$. Thus, solving the Tikhonov problem can be realized with $\mathcal{O}(n^4)$ operations.

Inversion of $Q$ requires the computation of the largest eigenvalue and a corresponding eigenvector of a symmetric matrix of dimension $n \times n$. There

exist several algorithms for this purpose, the power method, the inverse power method or the Rayleigh quotient iteration, to name a few. Computational costs are lower than for the Tikhonov step.

Each iteration of the Tikhonov step requires the choice of a new regularization parameter and thus multiple Tikhonov problems have to be solved. Since all these Tikhonov problems have a similar structure computation time can be saved by using suitable matrix decompositions which do not depend on $\alpha$ (see [3, Section 9.3]). In addition we observed numerically that from iteration to iteration the parameter chosen by some fixed parameter choice rule varies only slightly. In fact, in all experiments we observed that the chosen parameter decreases. Thus, each chosen parameter seems to provide an upper bound for the parameter in the next iteration. To our regret we do not have a theoretical foundation of this observation.

Simplifying the situation slightly by neglecting the parameter choice per iteration, as can be justified by the numerical observation just described, the computational costs for the decomposition approach 'per parameter' are

$$\mathcal{O}\big(\text{iterations} \cdot n^4\big), \tag{5.3}$$

where the number of iterations in contrast to the TIGRA method is only about 10.

# 6 Conclusions and future work

In this article the presented a new decomposition technique for stable approximate solution of autoconvolution and other quadratic equations. The new method works in practice and can compete with existing approaches for deautoconvolution. The major idea is to split the nonlinear autoconvolution operator into a well-posed quadratic part and into an ill-posed linear part. The latter can be inverted by well-known regularization methods and inversion of the well-posed quadratic part reduces to finding an eigenvector corresponding to the largest eigenvalue of a symmetric matrix.

Up to now there is no complete proof that the proposed method is regularizing and convergent for decreasing noise level. But since it is a combination of well understood existing techniques a proof of such properties should be possible and we will try to find one in our future research.

As already described in the introduction, investigation of autoconvolution equations is only the first step towards the stable inversion of a complexvalued and kernel-based autoconvolution-type problem appearing in the

SD-SPIDER method for characterizing ultra-short laser pulses. Our decomposition technique allows for the necessary extensions to general quadratic mappings and also complexvalued functions should be manageable within this framework.

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