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Curious ill-posedness phenomena in the composition of non-compact linear operators

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Abstract

We consider the composition of operators with non-closed range in Hilbert spaces and how the nature of ill-posedness is affected by their composition. Specifically, we study the Hausdorff-, Cesàro-, integration operator, and their adjoints, as well as some combinations of those. For the composition of the Hausdorff- and the Cesàro-operator, we give estimates of the decay of the corresponding singular values. As a curiosity, this provides also an example of two practically relevant non-compact operators, for which their composition is compact.

1 Introduction

Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ denote infinite-dimensional real Hilbert spaces. In this note, we consider composite operators T, which are factorized as

$$T : \mathcal{X} \xrightarrow{\ T_1\ } \mathcal{Z} \xrightarrow{\ T_2\ } \mathcal{Y}\,,$$

where $T_1 : \mathcal{X} \to \mathcal{Z}$, $T_2 : \mathcal{Z} \to \mathcal{Y}$ and consequently $T = T_2 \circ T_1 : \mathcal{X} \to \mathcal{Y}$ are bounded and injective linear operators with non-closed range, which means that zero belongs to the spectrum of the operators T_1, T_2, and T. So the composite equation

$$T\,x \,=\, T_2\,(T_1\,x) \,=\, y \qquad (x \in \mathcal{X},\ y \in \mathcal{Y})\,, \tag{1}$$

but also the outer equation

$$T_2\,z \,=\, y \qquad (z \in \mathcal{Z},\ y \in \mathcal{Y}) \tag{2}$$

and the inner equation

$$T_1\,x \,=\, z \qquad (x \in \mathcal{X},\ z \in \mathcal{Z}) \tag{3}$$

represent ill-posed linear operator equations and can serve as models for inverse problems characterized by forward operators T_1, T_2, and T with non-closed dense ranges $\mathcal{R}(T) \subset \mathcal{Y}$, $\mathcal{R}(T_1) \subset \mathcal{Z}$, and $\mathcal{R}(T_2) \subset \mathcal{Y}$, respectively. This implies that the corresponding adjoint operators T^*, T_1^*, and T_2^* are also bounded and injective linear operators.

In this context, we recall the paper [17], where Nashed distinguished for such operator equations *ill-posedness of type I* when the forward operator is non-compact and, as alternative, *ill-posedness of type II* when the forward operator is compact. Unfortunately, only for type II the *strength and degree of ill-posedness* caused by the forward operator can be simply expressed by the decay rate of the associated singular values of this operator; see Definitions 1 and 2 below. For discussions about the degree of ill-posedness of equations (1) with non-compact operators that are ill-posed of type I in the sense of Nashed, we refer to the articles [9, 10, 16]. Here, however, we assume in

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the sequel that the composite operator $T : \mathcal{X} \to \mathcal{Y}$ in (1) is compact and possesses the singular system

$$\{\sigma_i(T) > 0, u_i \in \mathcal{X}, v_i \in Y\}_{i=1}^\infty ,$$

with decreasingly ordered singular values

$$\|T\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} = \sigma_1(T) \ge \sigma_2(T) \ge \ldots \ge \sigma_i(T) \ge \sigma_{i+1}(T) \ge \ldots$$

tending to zero as $i \to \infty$ and complete orthonormal systems $\{u_i\}_{i=1}^\infty$ in $\mathcal{X}$ and $\{v_i\}_{i=1}^\infty$ in $\mathcal{Y}$ obeying $Tu_i = \sigma_i(T)\, v_i$ as well as $T^* v_i = \sigma_i(T)\, u_i$, for all $i \in \mathbb{N}$.

Definition 1 (Mild, moderate, and severe ill-posedness)**.** *Let the bounded and injective linear operator $T : \mathcal{X} \to \mathcal{Y}$ be compact. Then we call the operator equation* (1)

- *mildly ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is slower than any polynomial rate.*
- *moderately ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is polynomial.*
- *severely ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is higher than any polynomial rate.*

Along the lines of [12] (see also [10]), one can define in more detail an interval and a degree of ill-posedness as follows.

Definition 2 (Interval and degree of ill-posedness)**.** *We call, for an ill-posed operator equations* (1) *with compact forward operator T, the well-defined interval of the form*

$$[\underline{\kappa}, \overline{\kappa}] = \left[\liminf_{i\to\infty} \frac{-\log(\sigma_i(T))}{\log(i)}\, , \limsup_{i\to\infty} \frac{-\log(\sigma_i(T))}{\log(i)}\right] \subset [0,\infty] \tag{4}$$

as interval of ill-posedness. If $\underline{\kappa}$ and $\overline{\kappa}$ from $[0,\infty]$ are both finite positive, then we have moderate ill-posedness, and if they even coincide as $\underline{\kappa} = \overline{\kappa} = \kappa$, then we call the equation ill-posed of degree $\kappa > 0$. Severe ill-posedness occurs if the interval degenerates as $\underline{\kappa} = \overline{\kappa} = \infty$, and vice versa mild ill-posedness is characterized by a degeneration as $\underline{\kappa} = \overline{\kappa} = 0$.

2 A selection of injective linear operators with non-closed range

To investigate the ill-posedness behaviour of composite operators $T = T_2 \circ T_1$ in the equation (1), we present a selection of bounded and injective compact and non-compact operators with non-closed range that can be exploited for T_1 and T_2 and for which also the adjoint operators are injective. We start with the *simple integration operator* $J : L^2(0,1) \to L^2(0,1)$ and its adjoint operator $J^* : L^2(0,1) \to L^2(0,1)$ defined as

$$[J\, x](s) := \int_0^s x(t)\, dt \qquad (0 \le s \le 1, \quad x \in L^2(0,1))\, , \tag{5}$$

and

$$[J^* x](t) := \int_t^1 x(s)\, ds \qquad (0 \le t \le 1, \quad x \in L^2(0,1))\, , \tag{6}$$

respectively. Both operators are *compact* and so is the self-adjoint specific *diagonal operator* $D : \ell^2 \to \ell^2$, which appears here as

$$[D\, y]_j := \frac{y_j}{j} \qquad (j = 1, 2, \ldots, \quad y \in \ell^2). \tag{7}$$

It is well-known for J and J^* and evident for D that the degree of ill-posedness is *one*. The singular system of J and J^* can be written down in an explicit manner, where we have $\sigma_i(J) \asymp i^{-1}$ for $i \to \infty$ as singular value asymptotics. The singular system of D is of the form $\{i^{-1}, e^{(i)}, e^{(i)}\}_{i=1}^{\infty}$, where $e^{(i)}$ denotes the i-th unit vector in ℓ^2.

The *Cesàro operator* $C : L^2(0,1) \to L^2(0,1)$ and its adjoint operator $C^* : L^2(0,1) \to L^2(0,1)$, which attain the form

$$[C\,x](s) := \frac{1}{s}\int_0^s x(t)\,dt \qquad (0 \le s \le 1, \quad x \in L^2(0,1)) \tag{8}$$

and

$$[C^* x](t) := \int_t^1 \frac{x(s)}{s}\,ds \qquad (0 \le t \le 1, \quad x \in L^2(0,1))\,, \tag{9}$$

respectively, are non-compact operators with non-closed range; see [1, 15]. A further interesting non-compact operator with non-closed range connecting the spaces $L^2(0,1)$ and ℓ^2 is the *Hausdorff moment operator* $A : L^2(0,1) \to \ell^2$ defined as

$$[A\,x]_j := \int_0^1 x(t)\,t^{j-1} dt \qquad (j = 1, 2, \dots, \quad x \in L^2(0,1))\,, \tag{10}$$

with the corresponding adjoint operator $A^* : \ell^2 \to L^2(0,1)$ of the form

$$[A^* y](t) := \sum_{j=1}^{\infty} y_j\,t^{j-1} \qquad (0 \le t \le 1, \quad y \in \ell^2), \tag{11}$$

and we refer for details to [5] (see also [4, 6, 11]).

For the last four non-compact operators, a degree or interval of ill-posedness in the sense of Definition 2 does not make sense. But if those operators occur as T_1 or T_2 in a composition $T = T_2 \circ T_1$, where T is compact, then they can substantially influence the degree of ill-posedness for T. This is also the case for non-compact *multiplication operators* $M : L^2(0,1) \to L^2(0,1)$ with non-closed range

$$[M\,x](t) := m(t)\,x(t) \qquad (0 \le t \le 1, \quad x \in L^2(0,1))\,, \tag{12}$$

for which the multiplier functions $m \in L^\infty(0,1)$ possess essential zeros in $(0,1)$. We refer in this context also to the papers [8, 13].

3 Can a non-compact operator in composition destroy the degree of ill-posedness of a compact operator?

In the past years, equations (1) with compact composite operators $T = T_2 \circ T_1$ have been studied under the assumption that T_1 is compact and T_2 is a non-compact operator with non-closed range. It had been an open question whether the non-compact operator T_2 can amend the degree of ill-posedness of the compact operator T_1 in the composition T.

The first studies in [3, 13, 14] investigated the case $T = M \circ J : L^2(0,1) \to L^2(0,1)$ with multiplication operators $T_2 := M : L^2(0,1) \to L^2(0,1)$ from (12) and the simple integration operator $T_1 := J : L^2(0,1) \to L^2(0,1)$ from (5). Indeed, all these studies indicated the asymptotics $\sigma_i(T) \asymp i^{-1}$ as $i \to \infty$, even for multiplier functions m with strong (exponential-type) zeros that occur in inverse problems of option pricing; see [7]. This means that along the lines of those studies the non-compact multiplication operator M does not destroy the degree of ill-posedness *one* of the compact operator J in such composition.

However, the situation changed when for $T_1 := J$ the multiplication operator M as T_2 was replaced with the non-compact Hausdorff moment operator $T_2 := A : L^2(0,1) \to \ell^2$ from (10). In the article [11], the assertion of the following proposition could be shown in the context of Corollary 2 and Theorem 3 ibid.

Proposition 1. *The operator $T = A \circ J : L^2(0,1) \to \ell^2$ with $A : L^2(0,1) \to \ell^2$ from* (10) *and $J : L^2(0,1) \to L^2(0,1)$ from* (5) *obeys for some positive constants $\underline{C}$ and $\overline{C}$ the inequalities*

$$\exp(-\underline{C}\, i) \le \sigma_i(T) \le \frac{\overline{C}}{i^{3/2}} \tag{13}$$

for sufficiently large indices $i \in \mathbb{N}$.

As a consequence of Proposition 1, the interval of ill-posedness for the composition $A \circ J$ is a subset of the interval $[\frac{3}{2}, \infty]$. This was the first example in the literature to demonstrate with respect to $T_1 := J$ the degree-destroying potential of a non-compact operator T_2 in such a composition. Unfortunately, by now it could not be cleared if $T = A \circ J$ really leads to an exponentially (severely) ill-posed problem or whether it leads to a moderate ill-posed problem. So it was exciting to replace the Hausdorff moment operator A as T_2 with the non-compact Cesàro operator $T_2 := C : L^2(0,1) \to L^2(0,1)$ from (8) in the composition $T = C \circ J$. The recent paper [2] has proven that we have, for such T, the asymptotics $\sigma_i(T) \asymp i^{-2}$ as $i \to \infty$, which means that the degree of ill-posedness is *two* for $T = C \circ J$. Hence, C increases in that composition the degree of ill-posedness of J just by one. Taking into account that $J^2 = M \circ T$ with the multiplication operator $M : L^2(0,1) \to L^2(0,1)$ and the multiplier function $m(t) = t$, one can see again that such multiplication operator does not amend the degree of ill-posedness, because the asymptotics $\sigma_i(J^2) \asymp i^{-2}$, as $i \to \infty$, is well-known; see for example [18].

4 The curious case that the composition of two non-compact operators is compact

It was surprising for the authors that also two *non-compact* operators T_1 and T_2 with non-closed range can generate a *compact* operator by composition, $T = T_2 \circ T_1$. Indeed, let $T_1 := C^* : L^2(0,1) \to L^2(0,1)$ from (9) and $T_2 := A : L^2(0,1) \to \ell^2$ from (10). Then we have such a situation as the next proposition indicates.

Proposition 2. *The operator $T : L^2(0,1) \to \ell^2$ defined as $T := A \circ C^*$ with the non-compact operators A from* (10) *and C^* from* (9) *is compact and even a Hilbert-Schmidt operator.*

Proof. We have that

$$T = A \circ C^* = D \circ A\,,$$

with the *compact* diagonal operator $D : \ell^2 \to \ell^2$ from (7). This can be seen by inspection of the j-th component of Tx, which can be written as

$$[A(C^*x)]_j = \int_0^1 \left(\int_t^1 \frac{x(s)}{s}\, ds \right) t^{j-1}\, dt\,.$$

Integration by parts yields moreover

$$[A(C^*x)]_j = \frac{1}{j} \int_0^1 x(t)\, t^{j-1}\, dt = \frac{1}{j}\, [Ax]_j = [D(Ax)]_j\,.$$

Since D is a compact operator, this property carries over to the composition $T = D \circ A$ of D with the bounded linear operator A. In the same manner, the Hilbert-Schmidt operator D with the Hilbert-Schmidt norm $\|D\|_{HS} = \sqrt{\sum_{i=1}^\infty \frac{1}{i^2}} < \infty$ leads to a Hilbert-Schmidt property of T by favour of the inequality $\|T\|_{HS} \le \|D\|_{HS}\, \|A\|_{\mathcal{L}(L^2(0,1),\ell^2)}$. □

Remark 1. We note that of course the same fact can also be formulated for the adjoint operator $T^* = C \circ A^* : \ell^2 \to L^2(0,1)$, where C from (8) and A^* from (11) are again non-compact operators with non-closed range, but $T^* = A^* \circ D$ is compact. □

Along the lines of the proof of [11, Theorem 3] one can prove the following theorem.

Theorem 1. *For the composition $T = D \circ A = A \circ C^* : L^2(0,1) \to \ell^2$ with the operators $A : L^2(0,1) \to \ell^2$ from (10), $D : \ell^2 \to \ell^2$ from (7) and $C^* : L^2(0,1) \to L^2(0,1)$ from (9), there exists a positive constant C such that*

$$\sigma_i(T) \le \frac{C}{i^{3/2}} \quad (i = 1, 2, \ldots), \tag{14}$$

and the degree of ill-posedness of T is at least 3/2.

Proof. A main tool for the proof is the system $\{L_j\}_{j=1}^{\infty}$ of shifted Legendre polynomials which represent a complete orthonormal system in the Hilbert space $L^2(0,1)$. This system is the result of the Gram-Schmidt orthonormalization process of the system $\{t^{j-1}\}_{j=1}^{\infty}$ of monomials. Consequently, we have

$$\text{span}(1, t, \ldots, t^{j-1}) = \text{span}(L_1, L_2, \ldots, L_j).$$

Hence, we have for $m \ge 2$ that $L_m \perp t^{j-1}$, for all $1 \le j < m$. As has been proven by [11, Proposition 3], we have for the Hilbert-Schmidt operator $T = D \circ A$

$$\sum_{i=n+1}^{\infty} \sigma_i^2(T) \le \|T(I - Q_n)\|_{HS}^2 \,,$$

where Q_n denotes the projection onto span$(L_1, ..., L_n)$. From that we derive here the estimates

$$\sum_{i=n+1}^{\infty} \sigma_i^2(T) \le \sum_{i=n+1}^{\infty} \|T(I - Q_n)L_i\|_{\ell^2}^2 = \sum_{i=n+1}^{\infty} \|T\,L_i\|_{\ell^2}^2 = \sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty} \langle D(AL_i), e^{(j)} \rangle_{\ell^2}^2 \,. \tag{15}$$

By exploiting the system of normalized functions

$$h_j(s) := \sqrt{2j+1}\, s^j \in L^2(0,1) \quad (j = 0, 1, 2, \ldots),$$

we can rewrite the terms of the form $\langle D(AL_i), e^{(j)} \rangle_{\ell^2}^2$ in (15) as

$$\langle D(AL_i), e^{(j)} \rangle_{\ell^2} = \frac{1}{j^2} \langle AL_i, e^{(j)} \rangle_{\ell^2} = \frac{1}{j^2} \left(\int_0^1 \frac{h_{j-1}(s)\, L_i(s)\, ds}{\sqrt{2j-1}} \right)^2 = \frac{1}{j^2(2j-1)} \langle h_{j-1}, L_i \rangle_{L^2(0,1)}^2 \,.$$

Taking into account $\|h_j\|_{L^2(0,1)} = 1$ and the orthogonality relations between h_j and L_i we derive now from (15) the estimate

$$\sum_{i=n+1}^{\infty} \sigma_i^2(T) \le \sum_{j=n+2}^{\infty} \frac{1}{j^2(2j-1)} \sum_{i=n+1}^{\infty} \langle h_{j-1}, L_i \rangle_{L^2(0,1)}^2 = \sum_{j=n+2}^{\infty} \frac{1}{j^2(2j-1)} \|(I - Q_n)h_{j-1}\|_{L^2(0,1)}^2 \,,$$

and with $\|(I - Q_n)h_{j-1}\|_{L^2(0,1)} \le 1$, we can further estimate as

$$\sum_{i=n+1}^{\infty} \sigma_i^2(T) \le \sum_{j=n+2}^{\infty} \frac{1}{j^2(2j-1)} \le C_1\, n^{-2}$$

for some constant $C_1 > 0$. We recall now from [11, Lemma 4] the fact that an estimate

$$\sum_{i=n+1}^{\infty} \sigma_i^2(T) \le C_1\, n^{-2\gamma} \;\; (n \in \mathbb{N}), \;\; \text{for} \;\; \gamma > 0 \;\; \text{and} \;\; C_1 > 0 \,,$$

implies the existence of a constant $C_2 > 0$ such that $\sigma_i^2(T) \le C_2\, i^{-(2\gamma+1)}$ $(i \in \mathbb{N})$. Applying this fact with $\gamma = 1$ yields the inequality (14), which completes the proof. □

Remark 2. The singular system of the compact operator $D : \ell^2 \to \ell^2$ mentioned above indicates the asymptotics $\sigma_i(D) \asymp i^{-1}$ as $i \to \infty$. From Theorem 1, however, we see that there is some constant $K > 0$ such that

$$\sigma_i(D \circ A)/\sigma_i(D) \le \frac{K}{i^{1/2}} \quad (i = 1, 2, \ldots).$$

Consequently, as for the composition $A \circ J$ (see Proposition 1) also here for $D \circ A$ the non-compact operator A has the power to increase the decay rate of the singular values of the respective compact operators by an exponent of at least 1/2. □

As in Proposition 1 (cf. [11, Corollary 3.6]) for the composition $A \circ J$, one can also here verify for the composition $D \circ A$ lower bounds of exponential type for the singular values. We make this explicit by the following theorem.

Theorem 2. *Consider the operator $T = A \circ C^* = D \circ A$ from Theorem 1. Then we have the lower bound*

$$\frac{C_0}{i} \exp(-2i) \le \sigma_i(T) \quad (i = 1, 2, \ldots), \tag{16}$$

with some constant $C_0 > 0$.

Proof. Consider the operator $TT^* : \ell^2 \to \ell^2$. Since AA^* is the infinite Hilbert matrix $\mathcal{H}$, we observe that

$$TT^* = D\mathcal{H}D$$

with D the diagonal operator $D = \operatorname{diag}((\frac{1}{i})_i)$ from (7). Let $P_N : \ell^2 \to \ell^2$ be the projection onto the first N components and let $\mathcal{H}_N = P_N \mathcal{H} P_N$ be the first $n \times n$-segment of the Hilbert matrix. Then the estimate

$$\sigma_N(P_N TT^* P_N) \le \|P_N\|^2 \sigma_N(TT^*) = \sigma_N(TT^*)$$

holds. On the other hand, P_N commutes with D. Now let $D_N = P_N D P_N$ be the $N \times N$-segment of D, which means that $D_n = \operatorname{diag}((1/i)_{i=1,N})$. Under such setting we consequently have

$$P_N TT^* P_N = D_N \mathcal{H}_N D_N.$$

It is well-known from [21] that there is a constant $C > 0$ in the context of an estimate from above for the norm of the inverse of the finite Hilbert matrix $\mathcal{H}_N$ as

$$\|\mathcal{H}_N^{-1}\| \le C \exp(4N).$$

This gives

$$\begin{aligned}\sigma_N(P_N TT^* P_N) &= \frac{1}{\|(D_N \mathcal{H}_N D_N)^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}} = \frac{1}{\|D_N^{-1} \mathcal{H}_N^{-1} D_N^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}} \\ &\ge \frac{1}{\|D_N^{-1}\|^2 \|\mathcal{H}_N^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}} \ge \frac{1}{N^2 C \exp(4N)}\end{aligned}$$

and yields the claimed result (16) by taking into account that $\sigma_N(T)^2 = \sigma_N(TT^*)$. □

Remark 3. Table 1 gives an overview of known estimates for the singular values of the composition of certain operators. By inspecting the estimates (13) as well as (14) and (16), it is a really challenging question whether the compositions $A \circ J$ and $D \circ A$ may lead to moderately ill-posed problems, although the character of the Hausdorff moment operator A seems to be severely ill-posed as the paper [19] indicates. If the answer is *yes*, then the moderate decay of the singular values of J and D has the power to stop in such compositions the severe ill-posedness character of A expressed by an exponential decay of the corresponding multiplier function in the spectral decomposition of AA^* (infinite Hilbert matrix). □

$\circ$	A	C	M	
J	$e^{-Ci} \lesssim \sigma_i \lesssim i^{-\frac{3}{2}}$	$\sigma_i \sim i^{-2}$	$\sigma_i \sim i^{-1}$	
C^*	$i^{-1}e^{-2i} \lesssim \sigma_i \lesssim i^{-\frac{3}{2}}$			

Table 1: Overview of known bounds for the singular values of compositions of certain operators.

By the above example of a compactification of two non-compact operators by composition, the following issue is raised that seems to be trivial only at first glimpse: When A is a non-compact operator between Hilbert spaces, is the selfadjoint operator A^*A also non-compact? Clearly, A is non-compact if and only if A^* is (by Schauder's theorem), however, as we have seen this does not necessarily imply non-compactness of the composition. Using polar decomposition, the following lemma can be shown:

Lemma 1. *Let $A : H_1 \to H_2$ be a bounded linear operator between Hilbert spaces H_1, H_2. Then*

$$A \text{ is compact} \Leftrightarrow A^* \text{ is compact} \Leftrightarrow A^*A \text{ is compact.} \tag{17}$$

Proof. As mentioned, the first equivalence is Schauder's theorem (see, e.g. [20, Thm 4.19]), and since compact operators form an ideal, we only have to show that if A^*A is compact, then A^* is compact. Compactness of A^*A implies compactness of the square root $\sqrt{A^*A}$, as can be shown by a spectral decomposition. Now the polar decomposition $A^* = \sqrt{A^*A}P$, e.g., [20, Thm 12.35], with a bounded (unitary) operator P implies compactness of A^*. □

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