Oversmoothing Tikhonov regularization in Banach spaces

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Abstract

This paper develops a Tikhonov regularization theory for nonlinear ill-posed operator equations in Banach spaces. As the main challenge, we consider the so-called oversmoothing state in the sense that the Tikhonov penalization is not able to capture the true solution regularity and leads to the infinite penalty value in the solution. We establish a vast extension of the Hilbertian convergence theory through the use of invertible sectorial operators from the holomorphic functional calculus and the prominent theory of interpolation scales in Banach spaces. Applications of the proposed theory involving ℓ^1, Bessel potential spaces, and Besov spaces are discussed.

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1 Introduction

Tikhonov regularization is a celebrated and powerful method for solving a wide class of nonlinear ill-posed inverse problems of the type: Given $y \in Y$, find $x \in D(F)$ such that

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear operator with domain $D(F)$ acting between two Banach spaces X and Y. Here, $D(F)$ is a closed and convex subset of X, and the forward

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operator F is assumed to be weakly sequentially continuous. Moreover, for simplicity, we suppose that the inverse problem (1.1) admits a unique solution $x^\dagger \in D(F)$. In the presence of a small perturbation $y^\delta \in Y$ satisfying

$$\|y^\delta - y\|_Y \leq \delta, \quad \delta \in (0, \delta_{\max}], \tag{1.2}$$

for a fixed constant $\delta_{\max} > 0$, the Tikhonov regularization method proposes a stable approximation of the true solution $x^\dagger$ to (1.1) by solving the following minimization problem:

$$\begin{cases} \text{Minimize} \quad T_\kappa^\delta(x) := \|F(x) - y^\delta\|_Y^\nu + \kappa \|x\|_V^m, \\ \text{subject to} \quad x \in \mathcal{D} := D(F) \cap V. \end{cases} \tag{1.3}$$

In the setting of (1.3), $\kappa > 0$ and $\nu, m \geq 1$ are real numbers and denote, respectively, the Tikhonov regularization parameter and the exponents to the norms of the misfit functional and of the penalty functional. Furthermore, V is a Banach space that is continuously embedded into X with a strictly finer topology ($\exists z \in X : \|z\|_V = \infty$) such that the sub-level sets of the penalty functional $\|x\|_V^m$ are weakly sequentially pre-compact in X. Thanks to this embedding $V \hookrightarrow X$, the penalty $\|\cdot\|_V$ is stabilizing. Therefore, by the presupposed conditions on the forward operator $F : D(F) \subseteq X \to Y$, we obtain the existence and stability of solutions x_κ^δ to (1.3) for all $\kappa > 0$ (cf. [36, Section 4.1] and [35, Section 3.2]).

Throughout this paper, the Tikhonov regularization parameter κ is specified based on the following variant of the discrepancy principle: For a prescribed constant $C_{DP} > 1$, we choose $\kappa = \kappa_{\mathrm{DP}} > 0$ in (1.3) such that

$$\|F(x_{\kappa_{\mathrm{DP}}}^\delta) - y^\delta\|_Y = C_{DP}\delta. \tag{1.4}$$

In this paper, let $\delta_{\max}$ be sufficiently small, and we assume that for all $\delta \in (0, \delta_{\max}]$ and all $y^\delta \in Y$ fulfilling (1.2), there exist a parameter $\kappa_{\mathrm{DP}} = \kappa_{\mathrm{DP}}(\delta, y^\delta) > 0$ and a solution $x_{\kappa_{\mathrm{DP}}}^\delta$ to (1.3) for the regularization parameter $\kappa = \kappa_{\mathrm{DP}}$ such that (1.4) holds. If F is linear, then the condition

$$\|y^\delta\|_Y > C_{DP}\delta$$

is sufficient for the existence of κ_{DP} in the discrepancy principle (1.4). However, due to possibly occurring duality gaps of the minimization problem (1.3), the solvability of (1.4) may fail to hold for nonlinear operators F. Sufficient conditions for the existence of κ_{DP} can be found in [3, Theorem 3.10]. The existence of κ_{DP} is assured in general whenever the minimizers to (1.3) are uniquely determined for all $\kappa > 0$.

Unfortunately, our present techniques do not allow for a generalization of the discrepancy principle (1.4) by replacing the equality with an inequality. Our argumentations for the derivation of (3.18) and (3.22) are based on the equality condition (1.4).

The classical Tikhonov regularization theory relies on the fundamental assumption that the true solution $x^\dagger$ lies in the underlying penalization space V. Under this requirement, the Tikhonov regularization method (1.3) has been widely explored by many authors and seems to have reached an advanced and satisfactory stage of mathematical development. In the real application, however, the assumption $x^\dagger \in V$ often fails to hold since the (unknown) solution regularity generally cannot be predicted a priori from the mathematical model. In other words, the so-called *oversmoothing* state

$$T^\delta_\kappa(x^\dagger) = \infty \tag{1.5}$$

is highly possible to occur. For this reason, our present paper considers the critical circumstance (1.5), which makes the analysis of (1.3) becomes highly challenging and appealing at the same time. In particular, the fundamental minimizing property $T^\delta_\kappa(x^\delta_\kappa) \leq T^\delta_\kappa(x^\dagger)$ for any solutions to (1.3), used innumerably in the classical regularization theory, becomes useless owing to (1.5).

Quite recently, motivated by the seminal paper [32] for linear inverse problems, the second author and Mathé [20] studied the Tikhonov regularization method for nonlinear inverse problems with oversmoothing penalties in *Hilbert scales*. Their work considers the quadratic case $\nu = m = 2$ for (1.3) and Hilbert spaces X, Y, and V. Benefiting from the variety of link conditions in Hilbert scales, they constructed auxiliary elements through specific *proximal operators* associated with an auxiliary quadratic-type Tikhonov functional, which can be minimized in an explicit manner. This idea leads to a convergence result for (1.3) with a power-type rate. Unfortunately, [20] and all its subsequent extensions [15,19,21,23] cannot be applied to the Banach setting or to the non-quadratic case $m, \nu \neq 2$. First steps towards very special Banach space models for oversmoothing regularization have been taken recently in [14] and [31, Section 5].

Building on a profound application of sectorial operators from the holomorphic functional calculus (cf. [34]) and the celebrated theory of interpolation scales in Banach spaces (cf. [30,37]), our paper develops two novel convergence results (Theorems 1 and 2) for the oversmoothing Tikhonov regularization problem (1.1)-(1.5). They substantially extend under comparable conditions the recent results for the Hilbert scale case to the general Banach space setting. Furthermore, we are able to circumvent the technical assumption on $x^\dagger$ being an interior point of $D(F)$ (see [20]) by introducing an alternative invariance assumption, which serves as a remedy in the case of $x^\dagger \notin \mathrm{int}(D(F))$. We should underline that our theory is established under a two-sided nonlinear assumption (2.1) on the forward operator. More precisely, (2.1) specifies that for all $x \in D(F)$ the norm $\|F(x) - F(x^\dagger)\|_Y$ is bounded from below and above by some factors of $\|x - x^\dagger\|_U$ for a certain Banach space U, whose topology is weaker than X. This condition is motivated by the Hilbertian case [20, Assumption 2] and seems to be reasonable as it may characterize the degree of ill-posedness of the underlying inverse problem (1.1). On this basis, Theorem 1 proves a convergence rate result for (1.1)-(1.5) in the case where V is governed by an invertible ω-sectorial operator $A : D(A) \subset X \to X$ with a sufficiently small angle ω such that

$$D(A) = V \quad \text{with norm equivalence} \quad \|\cdot\|_V \sim \|A\cdot\|_X. \tag{1.6}$$

The condition (1.6) and the proposed invertible ω-sectorial property allow us to apply the exponent laws and moment inequality (Lemma 3), which are together with the holomorphic functional calculus (Lemma 4) the central ingredients for our proof. It turns out that our arguments for our first result can be refined by the theory of interpolation Banach scales with appropriate decomposition operators for the corresponding scales. This leads to our final result (Theorem 2) which essentially generalizes Theorem 1. In particular, Theorem 2 applies to the case where the underlying space V cannot be described by an invertible sectorial operator satisfying (1.6). This occurs (see Lemma 6), for instance, if the Banach space X is reflexive (resp. separable), but V is non-reflexive (resp. non-separable). Applications and examples with various Banach spaces, including ℓ^1, Bessel potential spaces, and Besov spaces, are presented and discussed in the final section.

This paper is organized as follows. In the upcoming subsection, we recall some basic definitions and well-known facts regarding interpolation couples and sectorial operators. The main results of this paper (Theorems 1 and 2) and all their mathematical requirements are stated in Section 2. The proofs for these two results are presented, respectively, in Sections 3 and 4. The final section discusses various applications of our theoretical findings, including those arising from inverse elliptic coefficient problems.

Lastly, we mention that it would be desirable to replace the version (1.4) of the discrepancy principle used throughout this paper by the *sequential discrepancy principle* (see, e.g., [23, Algorithm 4.7]), but to this more flexible principle there are not even convergence rates results for the oversmoothing case in the much simpler Hilbert scale setting (cf. [20] where also only (1.4) applies). Therefore, such extension is reserved for our future work.

1.1 Preliminaries

We begin by introducing terminologies and notations used in this paper. The space of all linear and bounded operators from X to Y is denoted by

$$B(X,Y) = \{A : X \to Y \text{ is linear and bounded}\},$$

endowed with the operator norm $\|A\|_{X\to Y} := \sup_{\|x\|_X=1} \|Ax\|_Y$. If $X = Y$, then we simply write $B(X)$ for $B(X,X)$. The notation X^* stands for the dual space of X. The domain and range of a linear operator $A : D(A) \subset X \to Y$ is denote by $D(A)$ and $\mathrm{rg}(A)$, respectively. A linear operator $A : D(A) \subset X \to X$ is called closed if its graph $\{(x, Ax),\ x \in D(A)\}$ is closed in $X \times X$. If $A : D(A) \subset X \to X$ is a linear and closed operator, then

$$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda\mathrm{id} - A : D(A) \to X \text{ is bijective and } (\lambda\mathrm{id} - A)^{-1} \in B(X)\}$$

and

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

denote, respectively, the resolvent set and the spectrum of A. For every $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda\mathrm{id} - A)^{-1} \in B(X)$ is referred to as the resolvent operator of A. If X is a Hilbert space, a densely defined operator $A : D(A) \subset X \to X$ is called self-adjoint if $A^*x = Ax$ for all $x \in D(A)$ and $D(A) = D(A^*)$. Moreover, if X is a Hilbert space and $(Ax, x)_X > 0$ for all $x \in D(A) \setminus \{0\}$, then we say that $A : D(A) \subset X \to X$ is positive definite. If there exists a constant $c > 0$, independent of a and b, such that $c^{-1}a \leq b \leq ca$, we write $a \sim b$. Throughout this paper, we also make use of the set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For two given Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$, we call $(\mathcal{X}_1, \mathcal{X}_2)$ an interpolation couple if and only if there exists a locally convex topological space $\mathcal{U}$ such that the embeddings

$$\mathcal{X}_1 \hookrightarrow \mathcal{U} \quad \text{and} \quad \mathcal{X}_2 \hookrightarrow \mathcal{U}$$

are continuous. In this case, both $\mathcal{X}_1 \cap \mathcal{X}_2$ and $\mathcal{X}_1 + \mathcal{X}_2$ are well-defined Banach spaces. We say that $\mathcal{X}_3$ is an intermediate space in $(\mathcal{X}_1, \mathcal{X}_2)$ if the embeddings

$$\mathcal{X}_1 \cap \mathcal{X}_2 \hookrightarrow \mathcal{X}_3 \hookrightarrow \mathcal{X}_1 + \mathcal{X}_2$$

are continuous. Furthermore, for $s \in [0, 1]$, we write

$$\mathcal{X}_3 \in J_s(\mathcal{X}_1, \mathcal{X}_2) \quad \Longleftrightarrow \quad \exists c \geq 0 \ \forall x \in \mathcal{X}_1 \cap \mathcal{X}_2 : \ \|x\|_{\mathcal{X}_3} \leq c\|x\|_{\mathcal{X}_1}^{1-s}\|x\|_{\mathcal{X}_2}^{s}.$$

For a given interpolation couple $(\mathcal{X}_1, \mathcal{X}_2)$ and $s \in [0, 1]$, $[\mathcal{X}_1, \mathcal{X}_2]_s$ denotes the complex interpolation between $\mathcal{X}_1$ and $\mathcal{X}_2$. For the convenience of the reader, we provide the definition of $[\mathcal{X}_1, \mathcal{X}_2]_s$ in the appendix. By a well-known result [18, Proposition B.3.5], we know that

$$[\mathcal{X}_1, \mathcal{X}_2]_s \in J_s(\mathcal{X}_1, \mathcal{X}_2) \quad \forall s \in [0, 1]. \tag{1.7}$$

On the other hand, for $s \in (0, 1)$ and $q \in [1, \infty]$, $(\mathcal{X}_1, \mathcal{X}_2)_{s,q}$ stands for the real interpolation between $\mathcal{X}_1$ and $\mathcal{X}_2$ (see appendix for the precise definition). Similarly to (1.7) (see [30, Corollary1.2.7]), it holds that

$$(\mathcal{X}_1, \mathcal{X}_2)_{s,q} \in J_s(\mathcal{X}_1, \mathcal{X}_2) \quad \forall s \in (0, 1) \quad \forall q \in [1, \infty]. \tag{1.8}$$

Lemma 1 (see [18, Theorem B.2.3.]). *Let $(\mathcal{X}_1, \mathcal{X}_2)$ and $(\mathcal{Y}_1, \mathcal{Y}_2)$ be interpolation couples. If $T \in B(\mathcal{X}_1, \mathcal{Y}_1) \cap B(\mathcal{X}_2, \mathcal{Y}_2)$, then $T \in B((\mathcal{X}_1, \mathcal{X}_2)_{\tau,q}, (\mathcal{Y}_1, \mathcal{Y}_2)_{\tau,q})$ for all $\tau \in (0, 1)$ and $q \in [1, \infty]$. Moreover,*

$$\|T\|_{(\mathcal{X}_1,\mathcal{X}_2)_{\tau,q} \to (\mathcal{Y}_1,\mathcal{Y}_2)_{\tau,q}} \leq \|T\|_{\mathcal{X}_1 \to \mathcal{Y}_1}^{1-\tau}\|T\|_{\mathcal{X}_2 \to \mathcal{Y}_2}^{\tau}.$$

As described in the introduction, our theory is realized through the use of sectorial operators and holomorphic functional calculus. For the sake of completeness, let us recall some terminologies and well-known results regarding sectorial operators. For $\omega \in (0, \pi)$, let

$$S_\omega := \{z \in \mathbb{C}\backslash\{0\} \mid |\arg z| < \omega\}$$

denote the symmetric sector around the positive axis of aperture angle 2ω. If $\omega = 0$, then we set $S_\omega = (0, +\infty)$.

Definition 1 (cf. [18,26]). *Let $\omega \in (0, \pi)$. A linear and closed operator $A : D(A) \subset X \to X$ is called ω-sectorial if the following conditions hold:*

(i) the spectrum $\sigma(A)$ is contained in $\overline{S_\omega}$.

(ii) $\mathrm{rg}(A)$ *is dense in* X.

(iii) $\forall\, \varphi \in (\omega, \pi)\ \exists\, C_\varphi > 0\ \forall\, \lambda \in \mathbb{C}\backslash\overline{S_\varphi} : \|\lambda R(\lambda, A)\|_{X\to X} \le C_\varphi$.

If $A : D(A) \subset X \to X$ *is* ω*-sectorial for all* $\omega \in (0, \pi)$*, then it is called* 0*-sectorial.*

If X is a Hilbert space, and $A : D(A) \subset X \to X$ is positive definite and self-adjoint, then it is a 0-sectorial operator (see, e.g., [34, Theorem 3.9.]). Moreover, a large class of second elliptic operators in general function spaces is also ω-sectorial [34, Section 8.4] for some $0 \le \omega < \frac{\pi}{2}$. Note that (ii) and (iii) imply that every ω-sectorial operator is injective (cf. [18]). In the following, let $\omega \in (0, \pi)$ and $A : D(A) \subset X \to X$ be a ω-sectorial operator. For each $\varphi \in (\omega, \pi)$, we introduce the function space

$$\mathcal{H}(S_\varphi) := \{f : S_\varphi \mapsto \mathbb{C} \,\text{is holomorphic} \mid \exists C, \beta > 0\, \forall z \in S_\varphi : |f(z)| \le C \min\{|z|^\beta, |z|^{-\beta}\}\}.$$

We enlarge this algebra to

$$\mathcal{E}(S_\varphi) := \mathcal{H}(S_\varphi) \oplus \mathrm{Span}\{1\} \oplus \mathrm{Span}\{\eta\} \tag{1.9}$$

with $\eta(z) := (1 + z)^{-1}$. Given a function $f \in \mathcal{E}(S_\varphi)$ with $f(z) = \psi(z) + c_1 + c_2\eta(z)$ for $c_1, c_2 \in \mathbb{C}$, we define

$$G_A(f) := f(A) := \psi(A) + c_1\mathrm{id} + c_2(\mathrm{id} + A)^{-1} \in B(X) \tag{1.10}$$

with $\psi(A)$ defined by the Cauchy-Dunford integral

$$\psi(A) := \frac{1}{2\pi i} \int_{\Gamma_{\omega'}} \psi(z) R(z, A) dz, \tag{1.11}$$

where $\Gamma_{\omega'} = \partial S_{\omega'}$ denotes the boundary of the sector $S_{\omega'}$ that is oriented counterclockwise and $\omega' \in (\omega, \varphi)$. Note that the above integral is absolute convergent. Furthermore, by the Cauchy integral formula for vector-valued holomorphic functions, it admits the same value for all $\omega' \in (\omega, \varphi)$. Details of such construction can be found in [18].

Lemma 2 (see [18, Lemma 2.2.3])**.** *Let* $\varphi \in (0, \pi)$*. A holomorphic function* $f : S_\varphi \to \mathbb{C}$ *belongs to* $\mathcal{E}(S_\varphi)$ *if and only if* f *is bounded and has finite polynomial limits at* 0 *and* ∞*, i.e., the limits* $f_0 := \lim_{S_\varphi \ni z \to 0} f(z)$*,* $f_\infty := \lim_{S_\varphi \ni z \to 0} f(z^{-1})$ *exist in* $\mathbb{C}$*, and*

$$\lim_{S_\varphi \ni z \to 0} \frac{|f(z) - f_0|}{|z|^\beta} = \lim_{S_\varphi \ni z \to 0} \frac{|f(z^{-1}) - f_\infty|}{|z^{-1}|^\beta} = 0$$

for some $\beta > 0$*.*

There is a standard way to extend the functional calculus $G_A : \mathcal{E}(S_\varphi) \to B(X)$ to a larger algebra of functions on the sector S_φ (see [18]) with a larger range containing unbounded operators in X. In particular, since A is injective, one can define *fractional power* A^s for all $s \in \mathbb{R}$ by the extended functional calculus (see [18]). If $A : D(A) \subset X \to X$ is also invertible, then for any $s \ge 0$, the possibly unbounded operator $A^s : D(A^s) \subset X \to X$ is an invertible operator with inverse $A^{-s} \in B(X)$ (see [18, Proposition 3.2.3]). Therefore, $D(A^{-s}) = X$, and *the fractional power domain space* $D(A^s)$ is a Banach space endowed with the norm $\|\cdot\|_{D(A^s)} := \|A^s \cdot\|_X$. We collect the properties of the fractional power operator A^s that will be used below.

Lemma 3 (see [18, Propositions 3.2.1, 3.2.3 and 6.6.4.]). *If $A : D(A) \subset X \to X$ is an invertible ω-sectorial operator for some $\pi > \omega \geq 0$, then the following assertions hold:*

(i) *For $s_1, s_2 \in \mathbb{R}$ with $s_1 \geq s_2$, the embedding $D(A^{s_1}) \hookrightarrow D(A^{s_2})$ is continuous, and*

$$t^{s_1} A^{s_1} x = (tA)^{s_1} x \quad \forall t > 0 \text{ and } x \in D(A^{s_1}). \tag{1.12}$$

(ii) *For $s, t \in \mathbb{R}$, it holds that $A^{t+s}x = A^t A^s x$ for all $x \in D(A^\tau)$ with $\tau = \max\{t, s, t+s\}$.*

(iii) *If $\beta \in (0, \pi/\omega)$, then A^β is invertible $\omega\beta$-sectorial and for all $s > 0$, we have $(A^\beta)^s x = A^{\beta s} x$ for all $x \in D(A^{\beta s})$.*

(iv) (Moment inequality) *For all $a \geq 0$, $s \geq 0$, and $-a \leq 0 < r \leq s$, there exists a constant $L > 0$ such that*

$$\|A^r x\|_X \leq L \|A^s x\|_X^{\frac{r+a}{a+s}} \|A^{-a} x\|_X^{\frac{s-r}{a+s}} \quad \forall x \in D(A^s). \tag{1.13}$$

If $A : D(A) \subset X \to X$ is an invertible ω-sectorial operator for some $\pi > \omega \geq 0$, we define the following Banach space:

$$X_A^s := \begin{cases} (D(A^s), \|A^s \cdot \|_X) & s \geq 0, \\ \text{Completion of } X \text{ under the norm } \|A^s \cdot \|_X & s < 0. \end{cases} \tag{1.14}$$

If X is a reflexive Banach space, then the adjoint operator $A^* : D(A^*) \subset X^* \to X^*$ is also an invertible ω-sectorial operator [18, Proposition 2.1.1 (d) and (j)]. Moreover, for all $s \geq 0$ and $x \in X$, it holds that

$$\|A^{-s} x\|_X = \|x\|_{(X_A^{*,s})^*}, \tag{1.15}$$

where $X_A^{*,s}$ denotes the fractional power domain $D((A^*)^s)$ (see [2, Chapter V, Theorem 1.4.6] for more details).

Lemma 4 (see [18, Theorem 2.3.3], [17, Lemma 2.2 and 2.3]). *Let $0 \leq \omega < \pi$, $A : D(A) \subset X \to X$ be an invertible ω-sectorial operator, and $g \in \mathcal{E}(S_\varphi)$ for $\varphi \in (\omega, \pi)$. Then, for any $s \in \mathbb{R}$, it holds that*

$$A^s g(tA) x = g(tA) A^s x \quad \forall t > 0 \text{ and } x \in D(A^s). \tag{1.16}$$

Moreover,

$$C_g := \sup_{t>0} \|g(tA)\|_{X \to X} < \infty \tag{1.17}$$

and the mapping $t \mapsto f(tA)$ is continuous from $(0, \infty)$ to $B(X)$. If, in addition, the mapping $z \mapsto \psi(z) := z^s g(z)$ is of class $\mathcal{E}(S_\varphi)$ for some $s \in \mathbb{R}$, then it holds that

$$\mathrm{rg}(g(tA)) \subset D(A^s) \quad \forall t > 0 \quad \text{if } s > 0 \tag{1.18}$$

and

$$(tA)^s g(tA) x = \psi(tA) x \quad \text{for all } t > 0 \text{ and all } x \in X. \tag{1.19}$$

2 Main results

We begin by formulating the required two-sided nonlinear mathematical property for the forward operator $F : D(F) \subseteq X \to Y$:

Assumption 1 (Two-sided nonlinear structure)**.** *There exist a Banach space $U \supsetneq X$ and two numbers $0 < c_U \le C_U < \infty$ such that*

$$c_U \|x - x^\dagger\|_U \le \|F(x) - F(x^\dagger)\|_Y \le C_U \|x - x^\dagger\|_U \quad \forall x \in D(F). \tag{2.1}$$

Moreover, there exists a neighborhood $B^\dagger \subset X$ of $x^\dagger$ such that the operator $F : D(F) \subseteq X \to Y$ is continuous in $B^\dagger \cap D(F)$.

If the norm of the pre-image space is weakened to $\|\cdot\|_U$, i.e., if we consider $U = X$, then the left-hand inequality of (2.1) implies that (1.1) is *locally well-posed* at $x^\dagger$ (see [22]). Of course, we do not consider the case $U = X$ in (2.1) since the operator equation (1.1) is supposed to be *locally ill-posed*. Let us also note that the pre-image space characterizes the ill-posedness for the problem under the condition (2.1) (see also [20] for further discussions). In view of Assumption 1, (1.4) yields the following result:

Lemma 5. *Let Assumption 1 be satisfied and let the regularization parameter $\kappa_{\mathrm{DP}} > 0$ be chosen according to the discrepancy principle* (1.4)*. Then,*

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_U \le \frac{C_{DP}+1}{c_U}\delta$$

holds for all $\delta \in (0, \delta_{\max}]$.

Proof. In view of (1.2), (1.4), and (2.1),

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_U \le \frac{1}{c_U}\|F(x^\delta_{\kappa_{\mathrm{DP}}}) - F(x^\dagger)\|_Y \le \frac{1}{c_U}(\|F(x^\delta_{\kappa_{\mathrm{DP}}}) - y^\delta\|_Y + \|y^\delta - y\|_Y) \le \frac{(C_{DP}+1)\delta}{c_U}$$

holds for all $\delta \in (0, \delta_{\max}]$ and all data y^δ obeying (1.2). □

Now we are ready to formulate the first main theorem. Its proof, the structure of which is analog to the proof of the theorem in [20], will be given in Section 3.

Theorem 1. *Suppose that* (1.6) *and Assumption 1 hold with an invertible ω-sectorial operator $A : D(A) \subset X \to X$ for some angle $0 \le \omega < \pi$ and $U = X_A^{-a}$ for some $a \ge 0$. Furthermore, let the regularization parameter $\kappa_{\mathrm{DP}} > 0$ be chosen according to the discrepancy principle* (1.4)*, and assume that there exists an $f \in \mathcal{E}(S_\varphi)$ with $\varphi \in (\omega, \pi)$ such that for every $s \in (0,1)$ the mappings $z \mapsto z^{-(a+s)}(f(z)-1)$ and $z \mapsto z^s f(z)$ are of class $\mathcal{E}(S_\varphi)$. If the solution $x^\dagger$ of* (1.1) *belongs to*

$$M_{\theta,E} := \{x \in D(A^\theta) \mid \|A^\theta x\|_X \le E\}$$

for some $0 < \theta < 1$ *and* $E > 0$ *and satisfies either*

$$x^\dagger \in int(D(F)) \quad \text{or} \quad \exists\, t_0 \in (0,\infty)\ \forall\, 0 < t \le t_0: \ f(tA)x^\dagger \in D(F), \tag{2.2}$$

then there exists a constant $c > 0$*, independent of* δ*, such that the error estimate*

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_X \le c\,\delta^{\frac{\theta}{a+\theta}} \tag{2.3}$$

holds for all sufficiently small $\delta > 0$*.*

Remark 1. *The condition* $x^\dagger \in int(D(F))$ *means that every* $x \in X$ *with a sufficiently small distance* $\|x - x^\dagger\|_X$ *belongs readily to* $D(F)$*. If* $\omega = 0$*, then the functions* $f(z) = e^{-z^{a+1}}$ *or* $f(z) = (z^{a+1}+1)^{-1}$ *satisfy all the requirements of Theorem 1 by Lemma 2. Moreover, if* $\omega \in (0,\pi)$ *and* $a \ge 0$ *are sufficiently small, then these two functions also satisfy all the requirements of Theorem 1.*

If both X and V are Hilbert spaces, then the condition (1.6) with an invertible 0-sectorial operator $A : D(A) \subset X \to X$ is valid if the embedding $V \hookrightarrow X$ is, in addition, dense (see Section 5.1 for more details). We underline that (1.6) is the main restriction of Theorem 1 that could fail to hold in the practice, as the following lemma demonstrates:

Lemma 6. *Let* X *be a reflexive (resp. separable) Banach space and* V *non-reflexive (resp. non-separable). Then, there exists no linear and closed operator* $A : D(A) \subset X \to X$ *satisfying* (1.6)*.*

Proof. Let us first consider the case where X is reflexive and V is non-reflexive. We recall the prominent Eberlein-Šmulian theorem that a Banach space is reflexive if and only if every bounded sequence contains a weakly converging subsequence.

Suppose that there exists a linear and closed operator $A : D(A) \subset X \to X$ satisfying (1.6). Let us consider the linear mapping

$$P : D(A) \to X \times X, \quad x \mapsto (x, Ax).$$

By definition, $P(D(A)) \subset X \times X$ is a closed subspace because $A : D(A) \subset X \to X$ is closed. Thus, since $X \times X$ is reflexive, $P(D(A))$ endowed with the norm $\|(x, Ax)\|_{P(D(A))} = \|x\|_X + \|Ax\|_X$ is a reflexive Banach space (see [1, Theorem 1.22]). On the other hand, thanks to (1.6) and $V \hookrightarrow X$, both norms $\|\cdot\|_V$ and $\|\cdot\|_X + \|A\cdot\|_X$ are equivalent. Thus, the Eberlein-Šmulian theorem leads to a contradiction that $\{V = D(A), \|\cdot\|_V\}$ is reflexive.

Let us next consider the case where X is separable and V is not separable. Suppose again that there exists a linear and closed operator $A : D(A) \subset X \to X$ satisfying (1.6). Then, as before, since $P(D(A)) \subset X \times X$ is a closed subspace, and $X \times X$ is separable, [1, Theorem 1.22] implies that $P(D(A))$ is a separable Banach space. By the definition of separable spaces and since both norms $\|\cdot\|_V$ and $\|\cdot\|_X + \|A\cdot\|_X$ are equivalent, we obtain a contradiction that $\{V = D(A), \|\cdot\|_V\}$ is separable. □

Lemma 6 motivates us to extend Theorem 1 to the case where (1.6) cannot be realized by an invertible ω-sectorial operator A. This assumption is primarily required for the application of Lemma 3. Therefore, it provides us with an illuminating hint of how to generalize the previous result by the theory of interpolation scales.

Assumption 2 (Interpolation scales)**.** *There exist a Banach space U satisfying Assumption 1, a family of Banach spaces $\{X_s\}_{s\in[0,1]}$, and a family of decomposition operators $\{P_t\}_{0<t\leq t_0} \subset B(U)$ with $t_0 \in (0,\infty)$ such that*

(i) *The embeddings $V \hookrightarrow X \hookrightarrow U$ are continuous.*

(ii) *$X_0 = X$, $X_1 = V$, and the embedding $X_s \hookrightarrow X_t$ is continuous for all $0 \leq t \leq s \leq 1$.*

(iii) *There exits a constant $a \geq 0$ such that for all $s \in (0,1]$ and $r \in [0,s)$ it holds that*

$$X_r \in J_{\frac{s-r}{a+s}}(X_s, U) \quad \Longleftrightarrow \quad \|x\|_{X_r} \leq L\|x\|_{X_s}^{\frac{r+a}{a+s}}\|x\|_U^{\frac{s-r}{a+s}} \quad \forall\, x \in X_s. \tag{2.4}$$

(iv) *For any $s \in [0,1]$ and $t \in (0,t_0]$, it holds that $P_tX_s \subset X_s$ with*

$$C_P := \sup_{0<t\leq t_0} \|P_t\|_{X_s\to X_s} < \infty, \tag{2.5}$$

and for any $x \in X$, the mapping $t \to P_tx$ is continuous from $(0,t_0]$ into X.

(v) *For all $0 < s < 1$, there exists a constant $C_{Proj} \geq C_P + 1$ such that for all $0 < t \leq t_0$*

$$\|P_t - \mathrm{id}\|_{X_s\to U} \leq C_{Proj}t^{a+s} \quad \text{and} \quad \|P_t\|_{X_s\to V} \leq C_{Proj}t^{s-1} \tag{2.6}$$

hold true with a as in (iii)*.*

The first three conditions (i)-(iii) generalize the assumption of Theorem 1 concerning the existence of an invertible ω-sectorial operator $A : D(A) \subset X \to X$. More precisely, by Lemma 3 and $X_s = X_A^s$, this assumption implies (i)-(iii), but not vice versa. On the other hand, (iv)-(v) weaken the assumption of Theorem 1 regarding the existence of the holomorphic function $f \in \mathcal{E}(S_\varphi)$. Indeed, as shown in Section 3, the linear operator $P_tx := f(tA)x$ satisfies the properties (iv)-(v). However, in general, the existence of a family of linear operators $\{P_t\}_{0<t\leq t_0} \subset B(U)$ fulfilling (iv)-(v) does not imply the existence of a holomorphic function $f \in \mathcal{E}(S_\varphi)$ satisfying the assumption of Theorem 1.

Intuitively, the decomposition property (2.6) gives a quantitative characterization of the approximation of elements in V to an element $x \in X_s$. Indeed, (2.6) shows that both the "distance" between x and $P_tx \in V$ in weaker norm and the smoothness of P_tx in V can be controlled by $\|x\|_{X_A^s}$. Similar properties have been utilized to verify variational source conditions for inverse PDEs problems in Hilbert spaces (see, e.g., [5,6]).

Let us finally state the regularity assumption for the true solution $x^\dagger$ to (1.1). Here, in place of the exponent A^θ of the sectorial operator A, we modify the smoothness condition of Theorem 1 by using the scale of the Banach space X and the corresponding family $\{P_t\}_{1<t\leq t_0}$ of decomposition operators from Assumption 2.

Assumption 3 (Solution smoothness)**.** *There exist $\theta \in (0,1)$ and $E > 0$ such that*

$$x^\dagger \in M_{\theta,E} := \{x \in X \mid \|x\|_{X_\theta} \leq E\}, \tag{2.7}$$

where $\{X_\theta\}_{\theta\in[0,1]}$ is as in Assumption 2. Assume that one of the following conditions holds true:

$$x^\dagger \in \ int(D(F)) \quad \text{or} \quad P_t x^\dagger \in D(F) \quad \forall\, 0 < t \leq t_0, \tag{2.8}$$

where $\{P_t\}_{0<t\leq t_0}$ is as in Assumption 2.

Now all assumptions are complete to formulate the second main theorem. Its proof will be given in Section 4

Theorem 2. *Let Assumptions 1–3 be satisfied and and let the regularization parameter $\kappa_{\mathrm{DP}} > 0$ be chosen according to the discrepancy principle* (1.4)*. Then, there exists a constant $c > 0$, independent of δ, such that the error estimate*

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_X \leq c\, \delta^{\frac{\theta}{a+\theta}}$$

holds for all sufficiently small δ.

3 Proof of Theorem 1

Let $\delta \in (0, \delta_{\max}]$ be arbitrarily fixed. In view of the moment inequality (Lemma 3) for $s = \theta$ and $r = 0$, it holds that

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_X \leq L\|A^\theta(x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger)\|_X^{\frac{a}{a+\theta}} \|A^{-a}(x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger)\|_X^{\frac{\theta}{a+\theta}}.$$

Accordingly, we have $U = D(A^{-a})$ and $\|\cdot\|_U = \|A^{-a}\cdot\|_X$, which yields that

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_X \leq L\|A^\theta(x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger)\|_X^{\frac{a}{a+\theta}} \|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_U^{\frac{\theta}{a+\theta}}.$$

Therefore, Lemma 5 implies that

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger\|_X \leq L\|A^\theta(x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger)\|_X^{\frac{a}{a+\theta}} \left(\frac{C_{DP}+1}{c_U}\delta\right)^{\frac{\theta}{a+\theta}}. \tag{3.1}$$

In conclusion, Theorem 1 is valid, once we can show the existence of a constant $\hat{E} > 0$, independent of δ, such that

$$\|A^\theta(x^\delta_{\kappa_{\mathrm{DP}}} - x^\dagger)\|_X \leq \hat{E}. \tag{3.2}$$

Step 1. Let us define the approximate elements

$$x_t := \begin{cases} f(tA)x^\dagger & t > 0, \\ x^\dagger & t = 0. \end{cases} \tag{3.3}$$

In the following, we prove that there exists a constant $C_{ap} > 0$, depending only on f, such that

$$\|A^{-a}(x_t - x^\dagger)\|_X \le C_{ap} E t^{\theta+a} \quad \forall t > 0, \tag{3.4}$$

$$\|Ax_t\|_X \le C_{ap} E t^{\theta-1} \quad \forall t > 0, \tag{3.5}$$

$$\|A^\theta(x_t - x^\dagger)\|_{X_\theta} \le (1 + C_{ap})E \quad \forall t > 0, \tag{3.6}$$

$$\|x_t - x^\dagger\|_X \le C_{ap} E t^\theta \quad \forall t > 0. \tag{3.7}$$

In the following, let $t > 0$ be arbitrarily fixed. Since $x^\dagger \in D(A^\theta)$ and $A^\theta x^\dagger \in X = D(A^{-\theta})$, applying (1.16) and Lemma 3 (ii), we obtain that

$$A^{-a}(f(tA)x^\dagger - x^\dagger) = A^{-a}(f(tA) - \mathrm{id})A^{-\theta}A^\theta x^\dagger = A^{-(a+\theta)}(f(tA) - \mathrm{id})A^\theta x^\dagger. \tag{3.8}$$

As a consequence,

$$\begin{aligned} \|A^{-a}(x_t - x^\dagger)\|_X &= \|A^{-(a+\theta)}(f(tA) - \mathrm{id})A^\theta x^\dagger\|_X \\ \underbrace{=}_{(1.12)} t^{a+\theta}\|(tA)^{-(a+\theta)}(f(tA) - \mathrm{id})A^\theta x^\dagger\|_X \underbrace{=}_{(1.19)\&\psi(z):=\frac{(f(z)-1)}{z^{\theta+a}}} &t^{\theta+a}\|\psi(tA)A^\theta x^\dagger\|_X \\ &\le t^{\theta+a}\|\psi(tA)\|_{X\to X}\|A^\theta x^\dagger\|_X \le t^{\theta+a}C_\psi\|A^\theta x^\dagger\|_X, \end{aligned} \tag{3.9}$$

where we used (1.17) with $g = \psi$. Notice that the function ψ belongs to $\mathcal{E}(S_\varphi)$ due to our assumptions on f. Similarly, one has by (1.16), and the Lemma 3 (ii) that

$$\begin{aligned} \|Ax_t\|_X =& \|Af(tA)A^{-\theta}A^\theta x^\dagger\|_X \underbrace{=}_{(1.18)\text{ with } g=f} \|A^{1-\theta}f(tA)A^\theta x^\dagger\|_X \\ &\underbrace{=}_{(1.12)} t^{\theta-1}\|(tA)^{1-\theta}f(tA)A^\theta x^\dagger\|_X \underbrace{=}_{(1.19)\text{ with } g=\widetilde{\psi}} t^{\theta-1}\|\widetilde{\psi}(tA)A^\theta x^\dagger\|_X, \end{aligned}$$

where $\widetilde{\psi}(z) := z^{1-\theta}f(z)$. Due to our assumption on f, $\widetilde{\psi}$ belongs also to $\mathcal{E}(S_\varphi)$. Then, by setting $g = \widetilde{\psi}$ in (1.17), we obtain from the above inequality that

$$\|Ax_t\|_X \le t^{\theta-1}\|\widetilde{\psi}(tA)\|_{X\to X}\|A^\theta x^\dagger\|_X \le t^{\theta-1}C_{\widetilde{\psi}}\|A^\theta x^\dagger\|_X. \tag{3.10}$$

Next, a combination of (1.16) and (1.17) yields

$$\|A^\theta x_t\|_X = \|f(tA)A^\theta x^\dagger\|_X \le \|f(tA)\|_{X\to X}\|A^\theta x^\dagger\|_X \le C_f\|A^\theta x^\dagger\|_X,$$

which implies

$$\|A^\theta(x_t - x^\dagger)\|_X \le (1 + C_f)\|A^\theta x^\dagger\|_X. \tag{3.11}$$

On the other hand, the moment inequality (1.13) with $s = \theta$ and $r = 0$, we have

$$\|x_t - x^\dagger\|_X \le L\|A^\theta(x_t - x^\dagger)\|_X^{\frac{a}{a+\theta}}\|A^{-a}(x_t - x^\dagger)\|^{\frac{\theta}{a+\theta}} \tag{3.12}$$

$$\underbrace{\le}_{(3.9)\&(3.11)} L(1 + C_f)^{\frac{a}{a+\theta}}C_\psi^{\frac{\theta}{a+\theta}}t^\theta\|A^\theta x^\dagger\|_X. \tag{3.13}$$

In view of (3.9)-(3.13), the claims (3.4)-(3.6) follow due to $\|A^{\theta}x\|_X \le E$ and by choosing $C_{ap} > 0$ to be large enough.

Step 2. In this step, we construct an important auxiliary element and study its basic properties. Owing to (2.2), (3.3), and (3.7), the approximate element x_t belongs to $D(F)$ as long as t is small enough. Thus, thanks to (3.7) and Lemma 4, Assumption 1 yields that the mapping $t \mapsto \|F(x_t) - F(x^{\dagger})\|_Y$ is continuous at all sufficiently small t and converges to zero as $t \downarrow 0$. For this reason, we may reduce $\delta \in (0, \delta_{\max}]$ (if necessary) and find a positive real number $t_{\text{aux}}(\delta) > 0$ satisfying

$$x_{t_{aux}(\delta)} \in D(F) \quad \text{and} \quad (C_{DP} - 1)\delta = \|F(x_{t_{aux}(\delta)}) - F(x^{\dagger})\|_Y \tag{3.14}$$

with $C_{DP} > 1$ as in (1.4). In all what follows, we simply write $x_{\text{aux}}(\delta) := x_{t_{\text{aux}}(\delta)}$ for the auxiliary element. By (3.14) and (2.1), we obtain that

$$\|A^{-a}(x_{\text{aux}}(\delta) - x^{\dagger})\|_X = \|x_{\text{aux}}(\delta) - x^{\dagger}\|_U \le \frac{1}{c_U}\|F(x_{\text{aux}}(\delta)) - F(x^{\dagger})\|_Y = \frac{C_{DP} - 1}{c_U}\delta. \tag{3.15}$$

Applying (3.6) and (3.15) to (3.12) yields that

$$\|x_{\text{aux}}(\delta) - x^{\dagger}\|_X \le L(1 + C_{ap})^{a/(a+\theta)} E^{a/(a+\theta)} \left(\frac{C_{DP} - 1}{c_U}\delta\right)^{\frac{\theta}{a+\theta}}. \tag{3.16}$$

In addition, the auxiliary element also satisfies

$$(C_{DP} - 1)\delta \underbrace{=}_{(3.14)} \|F(x_{\text{aux}}(\delta)) - F(x^{\dagger})\|_Y \underbrace{\le}_{(2.1)} C_U \|A^{-a}(x_{\text{aux}}(\delta) - x^{\dagger})\|_X \underbrace{\le}_{(3.4)} C_{ap} E C_U t_{\text{aux}}(\delta)^{\theta + a},$$

which gives a low bound for $t_{\text{aux}}(\delta)$ as follows:

$$t_{\text{aux}}(\delta) \ge \left(\frac{C_{DP} - 1}{C_{ap} E C_U}\delta\right)^{\frac{1}{a+\theta}}.$$

Making use of this lower bound and due to $0 < \theta < 1$, we eventually obtain

$$\|A x_{\text{aux}}(\delta)\|_X \underbrace{\le}_{(3.5)} C_{ap} E t_{\text{aux}}(\delta)^{\theta - 1} \le (C_{ap} E)^{\frac{a+1}{a+\theta}} \left(\frac{C_{DP} - 1}{C_U}\delta\right)^{\frac{\theta - 1}{a+\theta}}. \tag{3.17}$$

Step 3. In this step, we prove (3.2). According to (3.6), the auxiliary element $x_{\text{aux}}(\delta)$ satisfies $\|A^{\theta}(x_{\text{aux}}(\delta) - x^{\dagger})\|_X \le (1 + C_{ap})E$ for all $\delta \in (0, \delta_{\max}]$. Therefore, we see that (3.2) is valid if we are able to prove that the existence of constant $E' > 0$, independent of $x^{\dagger}, x^{\delta}_{\kappa_{\text{DP}}}, \delta$, and κ, such that

$$\|A^{\theta}(x^{\delta}_{\kappa_{\text{DP}}} - x_{\text{aux}}(\delta))\|_X \le E'.$$

Since $x^\delta_{\kappa_{\rm DP}}$ is a minimizer for the Tikhonov regularization problem (1.3) and $x_{\rm aux}(\delta) \in \mathcal{D} = D(F) \cap V$ (due to (3.14), (1.6), and (3.17)),we have

$$
\begin{aligned}
(C_{DP}\delta)^\nu + \kappa \|x^\delta_{\kappa_{\rm DP}}\|^m_V \underbrace{=}_{(1.4)} & \|F(x^\delta_{\kappa_{\rm DP}}) - y^\delta\|^\nu_Y + \kappa \|x^\delta_{\kappa_{\rm DP}}\|^m_V \\
& \leq \|F(x_{\rm aux}(\delta)) - y^\delta\|^\nu_Y + \kappa \|x_{\rm aux}(\delta)\|^m_V \\
& \underbrace{\leq}_{(3.14)} ((C_{DP}-1)\delta + \delta)^\nu + \kappa \|x_{\rm aux}(\delta)\|^m_V,
\end{aligned}
$$

which affirms that

$$
\|x^\delta_{\kappa_{\rm DP}}\|_V \leq \|x_{\rm aux}(\delta)\|_V \quad \underbrace{\Longrightarrow}_{(1.6)} \quad \|Ax^\delta_{\kappa_{\rm DP}}\|_X \leq C_A \|Ax_{\rm aux}(\delta)\|_X \tag{3.18}
$$

with $C_A := \max\{\|{\rm id}\|_{D(A)\to V}, \|{\rm id}\|_{V \to D(A)}\}^2$. The above inequality, along with the triangle inequality, implies

$$
\|A(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X \leq \|Ax^\delta_{\kappa_{\rm DP}}\|_X + \|Ax_{\rm aux}(\delta)\|_X \leq (1 + C_A)\|Ax_{\rm aux}(\delta)\|_X. \tag{3.19}
$$

Now, applying (1.13) with $s = 1$ and $r = \theta$, we obtain

$$
\begin{aligned}
\|A^\theta(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X & \leq L \|A(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X^{\frac{\theta+a}{a+1}} \|A^{-a}(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X^{\frac{1-\theta}{a+1}} \\
& \underbrace{\leq}_{(3.19)} L(C_A+1)^{\frac{\theta+a}{a+1}} \|Ax_{\rm aux}(\delta)\|_X^{\frac{\theta+a}{a+1}} \|A^{-a}(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X^{\frac{1-\theta}{a+1}}.
\end{aligned} \tag{3.20}
$$

The first factor in the right-hand side of (3.20) can be estimated by (3.17) as follows:

$$
\|Ax_{\rm aux}(\delta)\|_X^{\frac{\theta+a}{a+1}} \leq C_{ap} E \left(\frac{C_{DP}-1}{C_U} \delta \right)^{\frac{\theta-1}{a+1}}, \tag{3.21}
$$

whereas the second factor can be estimated as follows

$$
\begin{aligned}
\|A^{-a}(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X \leq & \|A^{-a}(x^\delta_{\kappa_{\rm DP}} - x^\dagger)\|_X + \|A^{-a}(x_{\rm aux}(\delta) - x^\dagger)\|_X \\
= & \|x^\delta_{\kappa_{\rm DP}} - x^\dagger\|_U + \|A^{-a}(x_{\rm aux}(\delta) - x^\dagger)\|_X \\
& \underbrace{\leq}_{(2.1)\&(3.15)} \frac{1}{c_U} \|F(x^\delta_{\kappa_{\rm DP}}) - F(x^\dagger)\|_Y + \frac{C_{DP}-1}{c_U}\delta \underbrace{\leq}_{(1.2)\&(1.4)} \frac{2C_{DP}}{c_U}\delta.
\end{aligned} \tag{3.22}
$$

In conclusion, (3.20)-(3.22) yield

$$
\|A^\theta(x^\delta_{\kappa_{\rm DP}} - x_{\rm aux}(\delta))\|_X \leq L(C_A+1)^{\frac{\theta+a}{a+1}} C_{ap} E \left(\frac{C_{DP}-1}{C_U} \right)^{\frac{\theta-1}{a+1}} \left(\frac{2C_{DP}}{c_U} \right)^{\frac{1-\theta}{a+1}} =: E'.
$$

This completes the proof. □

4 Proof of Theorem 2

We now generalize the arguments used in the previous section for the proof of Theorem 2. Our goal is to prove the existence of a constant $\hat{E}_* > 0$, independent of δ, such that

$$\|x^\delta_{\kappa_{\rm DP}} - x^\dagger\|_V \le \hat{E}_* \tag{4.1}$$

holds true for all sufficiently small δ. This estimate implies the claim of Theorem 2, since (4.1) together with Lemma 5 and (2.4) for $r = 0$ and $s = \theta$ implies

$$\|x^\delta_{\kappa_{\rm DP}} - x^\dagger\|_X \le L\hat{E}_*^{\frac{a}{a+\theta}} \left(\frac{C_{DP}+1}{c_U}\delta\right)^{\frac{\theta}{a+\theta}}. \tag{4.2}$$

Let $\delta \in (0, \delta_{\max}]$ be arbitrarily fixed. We define

$$\hat{x}_t := \begin{cases} P_t x^\dagger & t \in (0, t_0], \\ x^\dagger & t = 0. \end{cases} \tag{4.3}$$

According to (2.6) with $s = \theta$ and (2.7), it holds that

$$\|\hat{x}_t - x^\dagger\|_U \le C_{Proj} t^{a+\theta} E \quad \text{and} \quad \|\hat{x}_t\|_V \le C_{Proj} t^{1-\theta} E. \tag{4.4}$$

On other hand, Assumption 2 (iv) ensures that

$$\|\hat{x}_t\|_{X_\theta} \le C_P \|x^\dagger\|_{X_\theta} \underbrace{\le}_{(2.7)} C_P E, \tag{4.5}$$

which implies

$$\|\hat{x}_t - x^\dagger\|_{X_\theta} \underbrace{\le}_{(2.7)} (C_P + 1)E \le C_{Proj} E. \tag{4.6}$$

A combination of (2.4), with $r = 0$ and $s = \theta$, (4.4), and (4.6) yields

$$\|\hat{x}_t - x^\dagger\|_X \le L\|\hat{x}_t - x^\dagger\|_{X_\theta}^{\frac{a}{a+\theta}} \|\hat{x}_t - x^\dagger\|_U^{\frac{\theta}{a+\theta}} \le LC_{Proj} t^\theta E. \tag{4.7}$$

In view of (2.8), (4.3), (4.4), and (4.7), $\hat{x}_t \in \mathcal{D} = D(F) \cap V$ holds true as long as t is small enough. Now, if necessary, we reduce δ to obtain an auxiliary element $\hat{x}_{\rm aux}(\delta) \in \mathcal{D}$ satisfying (3.14) with x_t replaced by $\hat{x}_t$. Proceeding as in **Step 2** of the proof of Theorem 1 (see (3.15) and (3.17)), by (4.4), it follows that

$$\|\hat{x}_{\rm aux}(\delta) - x^\dagger\|_U \le \frac{C_{DP}-1}{c_U}\delta, \tag{4.8}$$

$$\|\hat{x}_{\rm aux}(\delta)\|_V \le (C_{Proj}E)^{\frac{a+1}{a+\theta}} \left(\frac{C_{DP}-1}{C_U}\delta\right)^{\frac{\theta-1}{\theta+a}}. \tag{4.9}$$

By similar arguments for (3.18), we also obtain

$$\|x^\delta_{\kappa_{\mathrm{DP}}}\|_V \le \|\hat{x}_{\mathrm{aux}}(\delta)\|_V, \tag{4.10}$$

and consequently (2.4) with $r = \theta$ and $s = 1$ implies that

$$\begin{aligned}\|x^\delta_{\kappa_{\mathrm{DP}}} - \hat{x}_{\mathrm{aux}}(\delta)\|_{X_\theta} \le & L2^{\frac{\theta+a}{a+1}} \|\hat{x}_{\mathrm{aux}}(\delta)\|_V^{\frac{\theta+a}{a+1}} \|x^\delta_{\kappa_{\mathrm{DP}}} - \hat{x}_{\mathrm{aux}}(\delta)\|_U^{\frac{1-\theta}{a+1}} \\ & \underbrace{\le}_{(4.9)} L2^{\frac{\theta+a}{a+1}} C_{Proj} E \left(\frac{C_{DP}-1}{C_U}\delta\right)^{\frac{\theta-1}{a+1}} \|x^\delta_{\kappa_{\mathrm{DP}}} - \hat{x}_{\mathrm{aux}}(\delta)\|_U^{\frac{1-\theta}{a+1}}. \end{aligned} \tag{4.11}$$

Similar as in (3.22), applying (2.1) and (4.8) results in

$$\|x^\delta_{\kappa_{\mathrm{DP}}} - \hat{x}_{\mathrm{aux}}(\delta)\|_U \le \frac{2C_{DP}}{c_U}\delta.$$

Thus, inserting the above inequality into (4.11), we conclude that the desired estimate (4.1) holds with

$$\hat{E}_* = L2^{\frac{\theta+a}{a+1}} C_{Proj} E \left(\frac{C_{DP}-1}{C_U}\right)^{\frac{\theta-1}{a+1}} \left(\frac{2C_{DP}}{c_U}\right)^{\frac{1-\theta}{a+1}}.$$

This completes the proof. □

5 Applications

In this section, we present some applications of the abstract theoretical results from Theorems 1 and 2. The first three applications present different possible choices and settings for the governing Tikhonov-penalties: Hilbertian case, ℓ^1-penalties, and Besov-penalties. While these three examples are somewhat still too abstract, we present a more practical application of our theory in Section 5.4 regarding radiative problems [10, 12, 25]. We also believe that our theory is applicable to the Tikhonov regularization method for nonlinear electromagnetic inverse or design problems arising for instance in the context of ferromagnetics [29, 41], superconductivity [42, 43], electromagnetic shielding [44], and many others. Such problems suffer from oversmoothing phenomena, mainly due to the low regularity and the lack of compactness properties in the associated function space for the electromagnetic fields. Another related application with oversmoothing character can be found in elastic full waveform inversion. The application of our theory to all these problems require however investigations with different techniques that would go beyond the scope of our present paper and will be considered in our upcoming research.

5.1 Hilbertian penalties

Let us first consider the case where both X and V are Hilbert spaces, and the embedding $V \hookrightarrow X$ is dense and continuous. By the Riesz representation theorem, there exists an isometric isomorphism

$$B_V : V \to V^*, \quad \langle B_V u, v\rangle_{V^*\times V} = (u,v)_V \quad \forall v \in V, \tag{5.1}$$

which induces an unbounded operator $B_X : D(B_X) \subset X \to X$ by

$$B_X u = B_V u \quad \text{with } D(B_X) = \{u \in V \mid B_V u \in X\}. \tag{5.2}$$

Lemma 7 (see [38, Theorems 2.1 and 2.34])**.** *If both X and V are Hilbert spaces such that the embedding $V \hookrightarrow X$ is dense and continuous, then the operator $B_X : D(B_X) \subset X \to X$ defined by* (5.2) *is an invertible positive definite and self-adjoint operator. Moreover, its square root $A = B_X^{1/2} : D(A) \subset X \to X$ of B_X fulfills*

$$D(A) = V \quad \textit{with norm equivalence} \quad \|\cdot\|_V \sim \|A\cdot\|_X.$$

The above-defined operator $A : D(A) \subset X \to X$ is also an invertible, positive definite, and self-adjoint operator. Therefore, (1.6) is true for this case. Moreover, there exits $\lambda_0 > 0$ such that $\sigma(A) \subset [\lambda_0, \infty)$, and for any $\omega \in (0,\pi)$, the following estimate holds

$$\|R(z,A)\|_{X\to X} \le \frac{1}{\operatorname{dist}(z,\sigma(A))} \le \frac{1}{|z|\sin\omega} \quad \forall z \in \mathbb{C}\backslash\overline{S_\omega} \quad \forall \omega \in (0,\pi). \tag{5.3}$$

On the other hand, it is well-known that there exists an spectral resolvent $\{E_\lambda\}_{\lambda\ge\lambda_0}$ for $(A, D(A))$. Let $\mathcal{B}$ be the algebra of all Borel measurable function over $[\lambda_0,\infty)$, and $\mathcal{B}_\infty$ be the sub-algebra of $\mathcal{B}$ consists of all essentially bounded functions. Then, for every $f \in \mathcal{B}_\infty$, we can define an algebra homomorphism from $\mathcal{B}$ into closed operators on X by

$$f(A) := \int_{\lambda_0}^{\infty} f(\lambda) dE_\lambda x \quad \forall x \in D(f(A)) = \{x \in X \mid \int_{\lambda_0}^{\infty} |f(\lambda)|^2 d\|E_\lambda x\|^2 < \infty\},$$

satisfying $\|f(A)\|_{X\to X} \le \|f\|_\infty$. Furthermore, a simplified analogue of Lemma 4 is obtained as follows:

(a) If $f \in \mathcal{B}_\infty$ and $s \ge 0$, then $A^s f(A)x = f(A)A^s x$ for all $x \in D(A^s)$.

(b) If $f \in \mathcal{B}_\infty$, then $\sup_{t>0} \|f(tA)\|_{X\to X} \le \|f\|_\infty$.

(c) If $f \in \mathcal{B}_\infty$ and $\lambda \mapsto \lambda^s f(\lambda)$ belongs to $\mathcal{B}_\infty$ for some $s > 0$, then $\|A^s f(A)\|_{X\to X} < \infty$.

Therefore, in the Hilbertian setting, Theorem 1 remains true if we replace the condition $f \in \mathcal{E}(S_\varphi)$ by $f \in \mathcal{B}_\infty$ being continuous. This can be seen as a generalization of Theorem 1 because $\mathcal{B}_\infty$ is larger than $\mathcal{E}(S_\varphi)$. As an instance, we can choose the following non-holomorphic function

$$f(\lambda) = \begin{cases} 1 & \text{if } \lambda \in [0,1], \\ 2-\lambda & \text{if } \lambda \in [1,2], \\ 0, & \text{if } \lambda \ge 2. \end{cases}$$

We underline that the generalization of Theorem 1 in the Hilbert case using a continuous but not necessarily holomorphic function $f \in \mathcal{B}_\infty$ is important since the choice of f influences the invariance requirement in (2.2).

5.2 ℓ^1-penalties

In this section, we consider the case when $V = \ell^1$ but the solution $x^\dagger$ to (1.1) lies in $X = \ell^p(w)$ (see Definition 2) with $1 < p < \infty$. Thus, in view of Lemma 6, (1.6) fails to hold such that Theorem 1 does not apply to this case. In the case of $V = \ell^1$, one typically sets $m = 1$ for the exponent of the penalty functional (cf. [14]), whereas the exponent $\nu \geq 1$ for the misfit functional may vary depending on the mathematical model.

Definition 2. *Let $w : \mathbb{N}_0 \to (0, \infty)$ be a function. For any $1 \leq p < \infty$, we define*

$$\ell^p(w) := \left\{ \{x(n)\}_{n=0}^\infty \mid \|x\|_{\ell^p_w} := (\sum_{n\in\mathbb{N}_0} w(n)^{1-p}|x(n)|^p)^{1/p} < \infty \right\} \tag{5.4}$$

and for $p = \infty$

$$\ell^\infty(w) := \left\{ \{x(n)\}_{n=0}^\infty \mid \|x\|_{\ell^\infty(w)} := \sup_{n\in\mathbb{N}_0} |w(n)^{-1}x(n)| < \infty \right\}.$$

If $p = 1$, then $\ell^p(w)$ identical to ℓ^1. Also, we would like to mention that $\ell^p(w)$ is reflexive if $1 < p < \infty$. In the following lemma, we verify Assumption 2 for Theorem 2.

Proposition 1. *Let $V = \ell^1$ and $w : \mathbb{N}_0 \to (0, \infty)$ such that $w(n_1) \leq w(n_2)$ for all $n_1 < n_2$ with $\lim_{n\to\infty} w(n) = \infty$. Furthermore, suppose that there exists a non-negative, decreasing function $f \in C^1[0, \infty)$ such that $f(0) = 1$, and*

$$\sum_{n\in\mathbb{N}_0} w(n) f(\tau w(n)) \leq C_w \tau^{-1} \quad \forall \tau \in (0, \tau_0] \tag{5.5}$$

holds with some real numbers $\tau_0 > 0$ and $C_w > 0$. If $1 < p < \infty$ and $X = \ell^p(w)$, then Assumption 2 holds with $U = \ell^\infty(w)$, $a = \frac{1}{p-1}$,

$$X_s := \ell^{p_s}(w) \quad \textit{with} \quad p_s := \frac{p}{1 + s(p-1)} \quad s \in [0, 1],$$

and

$$(P_t x)(n) := f(t^{a+1}w(n))x(n) \quad \forall x \in \ell^p(w) \tag{5.6}$$

for all $t \in (0, t_0]$ with $t_0 := \tau_0^{\frac{1}{1+a}}$.

Proof. For any $q \geq 1$, it holds that

$$\|x\|_{\ell^\infty(w)} \leq \frac{1}{w(0)} \|x\|_{\ell^q(w)} \quad \forall x \in \ell^q(w), \tag{5.7}$$

since $w(n) \geq w(0)$ for all $n \geq 0$. Furthermore, if $1 \leq q_1 \leq q_2$, then the embedding $\ell^{q_1}(w) \hookrightarrow \ell^{q_2}(w)$ is continuous since

$$\begin{aligned} \|x\|_{\ell^{q_2}(w)} &= (\sum_{n=0}^\infty w|w^{-1}x|^{q_2})^{1/q_2} = (\sum_{n=0}^\infty w|w^{-1}x|^{q_1}|w^{-1}x|^{q_2-q_1})^{1/q_2} \leq \|x\|_{\ell^{q_1}(w)}^{\frac{q_1}{q_2}} \|x\|_{\ell^\infty(w)}^{1-\frac{q_1}{q_2}} \\ &\leq w(0)^{q_1/q_2-1}\|x\|_{\ell^{q_1}(w)} \quad \forall x \in \ell^{q_1}(w), \end{aligned} \tag{5.8}$$

where we have used (5.7) with $q = q_1$. Both (5.7) and (5.8) verify the conditions (i)-(ii) of Assumption 2. For any $s \geq 0$ and $r \in [0, s]$, let us choose $q_1 = p_s$ and $q_2 = p_r$ in (5.8), which yields due to $p = 1 + \frac{1}{a}$ that

$$\|x\|_{\ell^{p_r}(w)} \leq \|x\|_{\ell^{p_s}(w)}^{\frac{1+r(p-1)}{1+s(p-1)}} \|x\|_{\ell^\infty(w)}^{\frac{(p-1)(s-r)}{1+s(p-1)}} = \|x\|_{\ell^{p_s}(w)}^{\frac{a+r}{a+s}} \|x\|_{\ell^\infty(w)}^{\frac{s-r}{a+s}} \quad \forall x \in \ell^{p_s}(w) = X_s. \tag{5.9}$$

This verifies the condition (iii) of Assumption 2.

Obviously, from (5.6) and the property that $\max_{\tau \in [0,\infty)} |f(\tau)| = 1$, it follows that

$$|(P_t x)(n)| \leq |x(n)| \quad \forall\, t \in (0, t_0],\ n \in \mathbb{N}_0, \text{ and } x \in \ell^{p_s}(w) = X_s,$$

which implies that $\|P_t x\|_{X_s \to X_s} \leq 1 =: C_p$ for all $t \in (0, t_0]$ and all $s \in [0, 1]$. From the continuity of f and $\max_{\tau \in [0,\infty)} |f(\tau)| = 1$, it follows by (5.6) that for every $x \in X = \ell^p(w)$, the mapping $t \mapsto P_t x$ is continuous from $(0, t_0]$ into X. Therefore, the requirement (2.5) is satisfied.

On the other hand, for every $s \in (0, 1)$, it follows from the (right) differentiability of f at 0 that there exists a constant $C^*(s)$, only depending on s, such that

$$\frac{1 - f(\tau^{p_s})}{\tau} = \frac{f(0) - f(\tau^{p_s})}{\tau} \leq C^*(s) \quad \forall \tau > 0,$$

since $p_s > 1$. By plugging $\tau = t^{a+s} w(n)^{\frac{1}{p_s}}$ in the above inequality and using $p_s = \frac{a+1}{a+s}$, we obtain

$$1 - f(t^{a+1} w(n)) \leq C^*(s) t^{a+s} w(n)^{\frac{1}{p_s}} \quad \forall\, n \in \mathbb{N}_0.$$

Hence, for all $x \in \ell^{p_s}(w) = X_s$ and $n \in \mathbb{N}_0$, it holds that

$$\begin{aligned} w(n)^{-1} |(1 - f(t^{a+1} w(n))) x(n)| &\leq C^*(s) t^{a+s} w(n)^{\frac{1}{p_s} - 1} |x(n)| \\ &\leq C^*(s) t^{a+s} (\sum_{n \in \mathbb{N}} w(n)^{1-p_s} |x(n)|^{p_s})^{1/p_s}, \end{aligned}$$

which ensures that

$$\|(\mathrm{id} - P_t) x\|_{\ell^\infty(w)} \leq C^*(s) t^{a+s} \|x\|_{\ell^{p_s}(w)}.$$

On the other hand, Hölder's inequality implies that

$$\begin{aligned} \|P_t x\|_{\ell^1} &= \sum_{n \in \mathbb{N}_0} f(t^{a+1} w(n)) |x(n)| = \sum_{n \in \mathbb{N}_0} f(t^{a+1} w(n)) w(n)^{1 - \frac{1}{p_s}} w(n)^{\frac{1}{p_s} - 1} |x(n)| \\ &\underbrace{\leq}_{1 - \frac{1}{p_s} = \frac{1-s}{a+1}} \left(\sum_{n \in \mathbb{N}_0} w(n) f(t^{a+1} w(n))^{\frac{a+1}{1-s}} \right)^{\frac{1-s}{a+1}} \left(\sum_{n \in \mathbb{N}_0} w(n)^{1-p_s} |x_n|^{p_s} \right)^{1/p_s} \\ &\underbrace{\leq}_{f \leq 1} \left(\sum_{n \in \mathbb{N}_0} w(n) f(t^{a+1} w(n)) \right)^{\frac{1-s}{a+1}} \left(\sum_{n \in \mathbb{N}_0} w(n)^{1-p_s} |x_n|^{p_s} \right)^{1/p_s} \\ &\leq C_w^{\frac{1-s}{a+1}} t^{s-1} \|x\|_{\ell^{p_s}(w)}, \end{aligned}$$

where we have used the growth rate (5.5) with $\tau = t^{a+1}$. In conclusion, the last condition (2.6) holds true. □

5.3 Besov-penalties

For any $s \in \mathbb{R}$ and $1 < p < \infty$, we define the Bessel potential space

$$H^s_p(\mathbb{R}^d) := \{u \in \mathcal{S}(\mathbb{R}^d)' \mid \|u\|^p_{H^s_p(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} e^{i\xi x}\langle\xi\rangle^s \hat{u}(\xi)d\xi|^p dx < \infty\},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, $\hat{u} := \mathcal{F}(u)$, $\mathcal{F} : \mathcal{S}(\mathbb{R}^d)' \to \mathcal{S}(\mathbb{R}^d)'$ is the Fourier transform, and $\mathcal{S}(\mathbb{R}^d)'$ denotes the tempered distribution space (see, e.g., [38]). If s is a non-negative integer, $H^s_p(\mathbb{R}^d)$ is identical to the classical Sobolev space $W^{s,p}(\mathbb{R}^d;\mathbb{C})$. In particular, $L^p(\mathbb{R}^d) = H^0_p(\mathbb{R}^d)$ is the space of complex-valued p-integrable functions. Throughout this subsection, let us define the operator $A_p : H^1_p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ given by

$$A_p u := \mathcal{F}^{-1}(\langle\xi\rangle\hat{u}(\xi)) := \sqrt{(I-\Delta_p)}\, u,$$

where $-\Delta_p : H^2_p(\mathbb{R}^d) \subset L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ denotes the Laplace operator on $L^p(\mathbb{R}^d)$. By a well-known result [17, Theorem 8.2.1], it is an invertible 0-sectorial operator. Moreover, for any holomorphic function $f \in \mathcal{E}(S_\varphi)$ with $0 < \varphi < \pi$, the operator $f(A_p) \in B(L^p(\mathbb{R}^d))$ admits the characterization:

$$f(A_p)u = \mathcal{F}^{-1}(f(\langle\xi\rangle)\hat{u}(\xi)) \quad \forall\, u \in L^p(\mathbb{R}^d), \tag{5.10}$$

i.e., $f(A_p)$ can be characterized by the *Fourier multiplier* associated with the function $\mathbb{R}^d\backslash\{0\} \ni \xi \mapsto f(\langle\xi\rangle)$ (see e.g. [18, Proposition 8.2.3]). In all what follows, we equip $H^s_p(\mathbb{R}^d)$ with the norm of $D(A^s_p)$. Moreover, its fractional power domain space $X^s_{A_p}$ can be characterize by Bessel potential spaces as follows:

$$X^s_{A_p} = H^s_p(\mathbb{R}^d) \quad \forall\, s \in \mathbb{R} \tag{5.11}$$

(see e.g. [18, Section 8.3]).

Let $\phi_0 \in C^\infty_0(\mathbb{R}^n)$ be such that $\phi_0(\xi) = 1$ for $|\xi| \le 1$ and $\phi_0(\xi) = 0$ for $|\xi| \ge 2$. Moreover, $\phi_j(\xi) := \phi_0(2^{-j}\xi) - \phi_0(2^{-j+1}\xi)$ for $j \in \mathbb{N}$. Then, we define

$$S_j : \mathcal{S}(\mathbb{R}^d)' \to C^\infty(\mathbb{R}^d), \quad S_j u := \mathcal{F}^{-1}[\phi_j(\xi)\hat{u}(\xi)] \quad \forall u \in \mathcal{S}(\mathbb{R}^d)' \quad \forall j \in \mathbb{N}_0.$$

For $s \in \mathbb{R}$, $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^d)$ is defined by

$$B^s_{p,q}(\mathbb{R}^d) := \{u \in \mathcal{S}(\mathbb{R}^d)' \mid \|u\|_{B^s_{p,q}(\mathbb{R}^d)} < \infty\},$$

where

$$\|u\|_{B^s_{p,q}(\mathbb{R}^d)} := \begin{cases} \left(\sum_{j=0}^{\infty} 2^{jsq}\|S_j u\|^q_{L^p(\mathbb{R}^d)}\right)^{\frac{1}{q}} & q < \infty, \\ \sup_{j\ge 0}\{2^{js}\|S_j u\|_{L^p(\mathbb{R}^d)}\} & q = \infty. \end{cases}$$

Other definitions of equivalent norms in Besov spaces can be found in [37].

Proposition 2. *Let* $a_0 \geq 0$, $a_1 \in (0,1)$, $1 \leq q_1 \leq \infty$, *and* $1 < p < \infty$. *Suppose that*

$$X = L^p(\Omega), \quad V = B^{a_1}_{p,q_1}(\mathbb{R}^d),$$

and there exists a function $f \in \mathcal{E}(S_\varphi)$ *for some* $0 < \varphi < \pi$ *such that the mappings* $z \mapsto z^{-(a_0+1)}(f(z)-1)$ *and* $z \mapsto zf(z)$ *are of class* $\mathcal{E}(S_\varphi)$. *Then, Assumption 2 holds with* $U := H^{-a_0}_p(\mathbb{R}^d)$,

$$X_s := B^{sa_1}_{p,q}(\mathbb{R}^d) \quad \forall s \in (0,1], \quad X_0 := X, \quad \text{and} \quad a := \frac{a_0}{a_1},$$

where $q := \max\{2,p\}$, *and*

$$P_t := f(t^{\frac{1}{a_1}} A_p) \quad \forall t > 0.$$

Remark 2. *If* $a_0 = 1$, *then the mappings*

$$z \mapsto f(z) := e^{-z^2}, \quad z \mapsto z^{-2}(f(z)-1), \quad z \mapsto zf(z)$$

are of class $\mathcal{E}(S_\varphi)$ *for any* $0 < \varphi < \frac{\pi}{4}$ *by Lemma 2. Thus, the function* f *satisfies all assumptions of Proposition 2. According to [18, Proposition 8.3.1.], it holds that*

$$(P_t u)(x) = \frac{e^{-\frac{2}{a_1}t}}{(4\pi t^{\frac{2}{a_1}})^{-\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{t^{2/a_1}}} u(y)dy \quad \forall u \in L^p(\Omega),$$

which gives an explicit expression of decomposition operators.

Proof. Due to the well-known results [37, Subsection 2.8.1. Remark 1 & 2], the embeddings $B^{a_1}_{p,q_1}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \hookrightarrow H^{-a_0}_p(\mathbb{R}^d)$, and $B^{s_1}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s_2}_{p,q}(\mathbb{R}^d)$ for any $s_1 > s_2$ are continuous. Thus, both (i) and (ii) of Assumption 2 are valid.

On the other hand, according to Theorem 2 in [37, Section 2.4.2], for any $-\infty < s_0, s_1 < \infty$, $s_0 \neq s_1$, $1 < p < \infty$, $1 \leq q_0, q \leq \infty$ and $\Theta \in (0,1)$, it holds for $s = (1-\Theta)s_0 + \Theta s_1$ that

$$(B^{s_0}_{p,q_0}(\mathbb{R}^d), H^{s_1}_p(\mathbb{R}^d))_{\Theta,q} = (H^{s_0}_p(\mathbb{R}^d), H^{s_1}_p(\mathbb{R}^d))_{\Theta,q} = B^{s}_{p,q}(\mathbb{R}^d) \tag{5.12}$$

with equivalent norms. In particular, (5.12) ensures that, for all $s \in (0,1]$ and $r \in [0,s)$,

$$(B^{a_1 s}_{p,q}(\mathbb{R}^d), H^{-a_0}_p(\mathbb{R}^d))_{\frac{s-r}{s+a},q} = B^{a_1 r}_{p,q}(\mathbb{R}^d), \tag{5.13}$$

which according to (1.8) yields

$$B^{a_1 r}_{p,q}(\mathbb{R}^d) \in J_{\frac{s-r}{s+a}}(B^{a_1 s}_{p,q}(\mathbb{R}^d), H^{-a_0}_p(\mathbb{R}^d)). \tag{5.14}$$

That is, $X_r \in J_{\frac{s-r}{s+a}}(X_s, U)$ for all $s \in (0,1]$ and $r \in (0,s)$, and $B^0_{p,q}(\mathbb{R}^d) \in J_{\frac{s}{a+s}}(X_s, U)$ for $s \in (0,1]$. Then, from the continuous embedding $B^0_{p,q}(\mathbb{R}^d) \hookrightarrow X$ it follows that $X \in J_{\frac{s}{a+s}}(X_s, U)$ for $s \in (0,1]$. In conclusion, Assumption 2 (iii) is valid.

By the arguments used in (3.9)-(3.10) and using (5.11) as well as the facts that $z \mapsto f(z)$, $z \mapsto zf(z)$, $z \mapsto z^{-a_0-1}(f(z)-1)$ are of class $\mathcal{E}(S_\varphi)$, we can infer that there exists a constant $C_{ap} > 0$ such that

$$\|(f(t^{\frac{1}{a_1}}A_p) - \mathrm{id})\|_{U\to U} = \|(f(t^{\frac{1}{a_1}}A_p) - \mathrm{id})\|_{X\to X} \le C_{ap} \quad \forall t > 0, \tag{5.15}$$

$$\|(f(t^{\frac{1}{a_1}}A_p) - \mathrm{id})x\|_{D(A_p)\to U} = \|A_p^{a_0+1}(f(t^{\frac{1}{a_1}}A_p) - \mathrm{id})x\|_{X\to X} \le C_{ap} t^{\frac{a_0+1}{a_1}} \quad \forall t > 0, \tag{5.16}$$

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{X\to D(A_p)} = \|A_p f(t^{\frac{1}{a_1}}A_p)\|_{X\to X} \le C_{ap} t^{-1/a_1} \, \forall t > 0, \tag{5.17}$$

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{X\to X} = \|f(t^{\frac{1}{a_1}}A_p)\|_{D(A_p)\to D(A_p)} \le C_{ap} \quad \forall t > 0. \tag{5.18}$$

Due to (5.12), the interpolation result

$$X_s = (D(A_p), U)_{\frac{1-a_1 s}{1+a_0}, q} = (D(A_p), X)_{1-s,q} \tag{5.19}$$

holds with equivalent norms. Selecting $\mathcal{X}_1 = \mathcal{X}_2 = D(A_p)$, $\mathcal{Y}_1 = \mathcal{Y}_1 = X$, $\tau = 1 - s$ in Lemma 1, we have by (5.19)-(5.18) that for all $s \in (0, 1)$

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{X_s\to X_s} \le C\|f(t^{\frac{1}{a_1}}A_p)\|_{(D(A_p),X)_{1-s,q}\to(D(A_p),X)_{1-s,q}} \le CC_{ap}. \tag{5.20}$$

Here and afterforward $C > 0$ denote a genetic constant associated with the embedding between equivalent Banach spaces. Hence, Assumption 2 (iv) holds.

Then, by taking $\mathcal{Y}_1 = \mathcal{Y}_2 = U$, $\mathcal{X}_1 = D(A_p)$, $\mathcal{X}_2 = U$, $\tau = \frac{1-a_1 s}{1+a_0}$ in Lemma 1 and recalling that $(U, U)_{\frac{1-a_1 s}{1+a_0}, q} = U$ holds with equivalent norms, we conclude that

$$\begin{aligned} \|f(t^{\frac{1}{a_1}}A_p) - \mathrm{id}\|_{X_s\to U} &\le C\|f(t^{\frac{1}{a_1}}A_p) - \mathrm{id}\|_{D(A_p)\to U}^{\frac{a_1 s + a_0}{1+a_0}} \|f(t^{\frac{1}{a_1}}A_p) - \mathrm{id}\|_{U\to U}^{\frac{1-a_1 s}{1+a_0}} \\ &\underbrace{\le}_{(5.15)-(5.16)} CC_{ap} t^{s+a}, \end{aligned} \tag{5.21}$$

which proves the first inequality in (v) of Assumption 2. Similarly, in view of the interpolation result

$$(D(A_p), X)_{1-a_1,q_1} = V \text{ with equivalent norm} \tag{5.22}$$

due to the second identity of (5.12) and (5.17)-(5.18), we obtain by Lemma 1 that

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{X\to V} \le C\|f(t^{\frac{1}{a_1}}A_p)\|_{X\to X}^{1-a_1} \|f(t^{\frac{1}{a_1}}A_p)\|_{X\to D(A_p)}^{a_1} \le CC_{ap} t^{-1}. \tag{5.23}$$

On the other hand, Lemma 1 in combination with (5.18) and the interpolation result (5.22) implies

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{V\to V} \le C\|f(t^{\frac{1}{a_1}}A_p)\|_{D(A_p)\to D(A_p)}^{1-a_1} \|f(t^{\frac{1}{a_1}}A_p)\|_{X\to X}^{a_1} \le CC_{ap} \tag{5.24}$$

for $V = (X, D(A_p))_{a_1,q_1}$. Choosing $\mathcal{X}_1 = X$, $\mathcal{X}_2 = V$ and $\mathcal{Y}_1 = \mathcal{Y}_2 = V$ in Lemma 1, we finally obtain

$$\|f(t^{\frac{1}{a_1}}A_p)\|_{X_s\to V} \le C\|f(t^{\frac{1}{a_1}}A_p)\|_{X\to V}^{1-s} \|f(t^{\frac{1}{a_1}}A_p)\|_{V\to V}^{s} \underbrace{\le}_{(5.23)-(5.24)} CC_{ap} t^{s-1}. \tag{5.25}$$

This proves the second inequality in Assumption 2 (v).

□

5.4 Inverse radiative problem

In this section, we apply our theory to an inverse radiative problem. Although convergence rates have been well investigated for the Tikhonov regularization method in the inverse elliptic or parabolic radiativity problems (see e.g. [5, 10, 12, 13, 25, 28]), all these convergence results are established under the assumption that unknown true solution has a finite penalty value. In the following, we focus on the case that the unknown radiativity fails to have a finite penalty value.

As a preparation, let us recall the notion of the Bessel potential space on a bounded domain. For $p \in (1, \infty)$ and a bounded domain $U \subset \mathbb{R}^d$ with a Lipschitz boundary ∂U, the space $H^s_p(U)$ with a possibly non-integer exponent $s \geq 0$ is defined as the space of all complex-valued functions $v \in L^p(U)$ satisfying $\hat{v}_{|U} = v$ for some $\hat{v} \in H^s_p(\mathbb{R}^d)$, endowed with the norm

$$\|v\|_{H^s_p(U)} := \inf_{\substack{\hat{v}_{|U}=v \\ \hat{v} \in H^s_p(\mathbb{R}^d)}} \|\hat{v}\|_{H^s_p(\mathbb{R}^d)}. \tag{5.26}$$

If s is a non-negative integer, then $H^s_p(U)$ coincides with the classical Sobolev space $W^{s,p}(U)$. The space $\mathring{H}^s_p(U)$ denotes the closure of $C^\infty_0(U)$ in $H^s_p(U)$, and the dual of $\mathring{H}^s_p(U)$ is denoted by $H^{-s}_q(U)$ with $q = \frac{p}{p-1}$.

Lemma 8. *Let $1 < p < \infty$. Then, it holds that*

$$\|fg\|_{H^{-1}_p(\Omega)} \leq \|f\|_{W^{1,\infty}(\Omega)} \|g\|_{H^{-1}_p(\Omega)} \quad \forall\, (f, g) \in W^{1,\infty}(\Omega) \times L^p(\Omega).$$

Proof. Let $f \in W^{1,\infty}(\Omega)$, $g \in L^p(\Omega)$, and $q = \frac{p}{p-1}$. For any $v \in \mathring{H}^1_q(\Omega)$, an interplay of Libniz's rule (see e.g. [38]) and Hölder's inequality yields

$$\|fv\|_{\mathring{H}^1_q(\Omega)} \leq \|f\|_{W^{1,\infty}(\Omega)} \|v\|_{\mathring{H}^1_q(\Omega)}.$$

This inequality together with the fact that

$$\|fg\|_{H^{-1}_p(\Omega)} = \sup_{\|v\|_{\mathring{H}^1_q(\Omega)}=1} \left| \int_\Omega fgv dx \right|$$

implies the desired result.

∎

Let us consider the following elliptic equation:

$$\begin{cases} -\nabla \cdot (a \nabla u) + (\chi_0 + \chi) u &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \partial\Omega. \end{cases} \tag{5.27}$$

In this setting, $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a bounded domain with a C^2-boundary $\partial\Omega$, $a \in C^1(\overline{\Omega})$ satisfying $\min_{x \in \overline{\Omega}} a(x) > 0$, $g \in W^{2-1/p,p}(\partial\Omega)$ (see [16]) and $f \in L^p(\Omega)$ with $d < p < \infty$.

Moreover, $\chi_0 \in L^\infty(\Omega)$ is a non-negative function. We are interested in recovering the unknown radiativity χ in the following admissible set:

$$K_R := \{\chi \in L^p(\Omega) \mid 0 \leq \chi(x) \leq R \text{ for a.e. } x \in \Omega\}, \quad R \in (0, \infty)$$

from the noisy data $u^\delta \in H^1_p(\Omega)$ of the true solution $u^\dagger$ satisfying

$$\|u^\delta - u^\dagger\|_{H^1_p(\Omega)} \leq \delta, \tag{5.28}$$

where $\delta > 0$ represents the noisy level. In the following, we summarize the well-posed result for the elliptic equation (5.27).

Lemma 9 (see [8])**.** *For every $\chi \in K_R$, the elliptic equation* (5.27) *admits a unique strong solution $u(\chi) \in H^2_p(\Omega)$ satisfying*

$$C_E := \sup_{\chi \in K_R} \|u(\chi)\|_{H^2_p(\Omega)} < \infty. \tag{5.29}$$

Moreover, the operator $B_\chi : \mathring{H}^1_p(\Omega) \to H^{-1}_p(\Omega)$, defined by

$$\langle B_\chi u, v\rangle_{H^{-1}_p(\Omega), \mathring{H}^1_q(\Omega)} := \int_\Omega a\nabla u \cdot \nabla v + (\chi_0 + \chi)uvdx \quad \forall (u, v) \in \mathring{H}^1_p(\Omega) \times \mathring{H}^1_q(\Omega) \tag{5.30}$$

is a topological isomorphism satisfying

$$C_B := \sup_{\chi \in K_R} \max\{\|B_\chi\|_{\mathring{H}^1_p(\Omega) \to H^{-1}_p(\Omega)}, \|B^{-1}_\chi\|_{H^{-1}_p(\Omega) \to \mathring{H}^1_p(\Omega)}\} < \infty. \tag{5.31}$$

In view of Lemma 9, if we set $X = L^p(\Omega)$, $Y = H^1_p(\Omega)$, and $F : D(F) \to Y$ by

$$F(\chi) := u(\chi) \quad \forall \chi \in D(F) := K_R, \tag{5.32}$$

then the inverse radiativity problem is equivalent to the operator equation (1.1). The associated Tikhonov regularization method reads as follows:

$$\begin{cases} \text{Minimize} \quad \|u(\chi) - u^\delta\|^p_{H^1_p(\Omega)} + \kappa\|\chi\|^p_{\mathring{H}^1_p(\Omega)}, \\ \text{subject to} \quad \chi \in K_R. \end{cases} \tag{5.33}$$

We underline that (5.33) is oversmoothing since in general the true solution $\chi^\dagger$ does not admit the regularity property in $\mathring{H}^1_p(\Omega)$, i.e., $\|\chi\|_{\mathring{H}^1_p(\Omega)} = \infty$. Let us demonstrate that Theorem 1 applies to the oversmoothing Tikhonov problem (5.33). To this end, we verify that all assumptions of Theorem 1 are satisfied. First of all, the existence of an invertible sectorial operator with efficient domain identical to $\mathring{H}^1_p(\Omega)$ is guaranteed by the following lemma:

Lemma 10 (see [7])**.** *Let $A_p : D(A_p) \subset L^p(\Omega) \to L^p(\Omega)$ be given by $A_p u := -\Delta u$ and $D(A) := \{u \in H^2_p(\Omega) \mid \gamma u = 0\}$. Then $A_p : D(A_p) \subset L^p(\Omega) \to L^p(\Omega)$ is a 0-sectorial operator such that its fractional power spaces $D(A^\theta_p)$ can be characterized as follows:*

$$D(A^\theta_p) = \begin{cases} H^{2\theta}_p(\Omega) & 0 \leq \theta < \frac{1}{2p}, \\ \{H^{2\theta}_p(\Omega) \mid \gamma u = 0\} & 1 \geq \theta > \frac{1}{2p} \text{ and } \theta \neq \frac{p+1}{2p}. \end{cases} \tag{5.34}$$

Setting $A := A_p^{\frac{1}{2}}$, Lemma 10 and Lemma 3 (iii) imply that $A : D(A) \subset L^p(\Omega) \to L^p(\Omega)$ is an invertible 0-sectorial operator with $D(A) = \mathring{H}_p^1(\Omega)$ with norm equivalence $\|\cdot\|_{\mathring{H}_p^1(\Omega)} \sim \|A\cdot\|_{L^p(\Omega)}$. In particular, (1.6) is satisfied with $V = \mathring{H}_p^1(\Omega)$. Moreover, the adjoint operator of it is exactly $A_q^{1/2} : D(A_q^{1/2}) \subset L^q(\Omega) \to L^q(\Omega)$ with $q = \frac{p}{p-1}$. Then, it follows from (1.15) and (5.34) that

$$\|A^{-1}x\|_{L^p(\Omega)} \sim \|x\|_{H_p^{-1}(\Omega)} \quad \forall x \in L^p(\Omega). \tag{5.35}$$

In view of (5.35), the upcoming lemma shows that Assumption 1 holds with the forward operator F given by (5.32), $a = 1$, and $U = H_p^{-1}(\Omega)$.

Lemma 11. *If $|u(\chi^\dagger)| > c_0$ holds in $\overline{\Omega}$ for some positive constant $c_0 > 0$, then there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|\chi - \chi^\dagger\|_{H_p^{-1}(\Omega)} \le \|u(\chi) - u(\chi^\dagger)\|_{H_p^1(\Omega)} \le C\|\chi - \chi^\dagger\|_{H_p^{-1}(\Omega)} \quad \forall\, \chi \in K_R.$$

Proof. Let $q = \frac{p}{p-1}$. By the definition (5.27), for each $\chi \in K_R$, the function $w := u(\chi) - u(\chi^\dagger)$ satisfies

$$\int_\Omega a\nabla w \cdot \nabla v + (\chi_0 + \chi)wvdx = \int_\Omega (\chi^\dagger - \chi)u(\chi^\dagger)vdx \quad \forall\, v \in \mathring{H}_q^1(\Omega),$$

which is equivalent to $B_\chi w = (\chi^\dagger - \chi)u(\chi^\dagger)$ where $B_\chi : \mathring{H}_p^1(\Omega) \to H_p^{-1}(\Omega)$ is defined as in (5.30). Then, Lemma 9 ensures that

$$\frac{1}{C_B}\|u(\chi^\dagger)(\chi - \chi^\dagger)\|_{H_p^{-1}(\Omega)} \le \|u(\chi) - u(\chi^\dagger)\|_{H_p^1(\Omega)} \le C_B\|u(\chi^\dagger)(\chi - \chi^\dagger)\|_{H_p^{-1}(\Omega)}. \tag{5.36}$$

On the other hand, Lemmas 8 and 9 ensure that

$$\begin{aligned}\|u(\chi^\dagger)(\chi - \chi^\dagger)\|_{H_p^{-1}(\Omega)} \le& \|u(\chi^\dagger)\|_{W^{1,\infty}(\Omega)}\|\chi - \chi^\dagger\|_{H_p^{-1}(\Omega)} \\ \le& C_m\|u(\chi^\dagger)\|_{H_p^2(\Omega)}\|\chi - \chi^\dagger\|_{H_p^{-1}(\Omega)},\end{aligned} \tag{5.37}$$

where $C_m > 0$ denotes the embedding constant associated with $H_p^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ as $p > d$. On the other hand, using the Leibniz rule and the condition $|u(\chi^\dagger)| > c_0$, we obtain that $u(\chi^\dagger)^{-1} \in H_p^2(\Omega)$. Then, again by Lemma 8,

$$\|\chi - \chi^\dagger\|_{H_p^{-1}(\Omega)} \le C_m\|u(\chi^\dagger)^{-1}\|_{H_p^2(\Omega)}\|u(\chi^\dagger)(\chi - \chi^\dagger)\|_{H_p^{-1}(\Omega)}. \tag{5.38}$$

The combination of (5.36)-(5.38) yields the desired result.

□

Now, it remains to verify the last assumption of Theorem 1 concerning the existence of the holomorphic function f with properties as in Theorem 1. Let us set $f(z) := e^{-z^2}$. This function obviously belongs to $\mathcal{E}(S_\varphi)$ for any $0 < \varphi < \frac{\pi}{4}$. In the following, let

$\varphi \in (0, \pi/4)$ be arbitrarily fixed. Since $e^{-z^2} = 1 - z^2 + o(z^4)$ as $z \to 0$, and the mapping $z \mapsto ze^{-z^2}$ belongs to $\mathcal{E}(S_\varphi)$, it follows by Lemma 2 that for any $s \in (0,1)$, the mappings $z \mapsto z^{-(1+s)}(f(z)-1)$ and $z \mapsto z^s f(z)$ are of class $\mathcal{E}(S_\varphi)$. Let us now verify the final condition (2.2) in Theorem 1. First, as $A := A_p^{\frac{1}{2}}$, we have that

$$f(tA) = e^{-(tA)^2} = e^{-t^2 A_p} \quad t > 0. \tag{5.39}$$

Furthermore, according to [33, Corollary 4.3 and Theorem 4.9], it holds that

$$\|e^{-A_p t}x\|_{L^\infty(\Omega)} \leq \|x\|_{L^\infty(\Omega)} \quad \forall\, (x,t) \in L^\infty(\Omega) \times (0,\infty)$$

and

$$e^{-tA_p}x \geq 0 \text{ a.e. in } \Omega \quad \text{for all non-negative}\, x \in L^p(\Omega) \text{ and all } t \geq 0.$$

As a consequence,

$$f(tA)K_R \subset K_R \quad \forall t \geq 0,$$

which yields the desired condition (2.2) for Theorem 1. Altogether, we have verified all requirements of Theorem 1 for the oversmoothing Tikhonov problem (5.33), leading to the following result:

Corollary 1. *Suppose that $\chi^\dagger \in K_R$ and $|u(\chi^\dagger)| > c_0$ for some positive constant c_0. If $\chi^\dagger \in H_p^\theta(\Omega)$ with $\theta \in (0, \frac{1}{p})$ or $\chi^\dagger \in \mathring{H}_p^\theta(\Omega)$ for some $\theta \in (\frac{1}{p}, 1)$, then every minimizer χ_κ^δ of (5.33) satisfies*

$$\|\chi_\kappa^\delta - \chi^\dagger\|_{L^p(\Omega)} = O(\delta^{\frac{\theta}{1+\theta}}) \quad \text{as } \delta \to 0.$$

Remark 3. *The application of the developed theory (Theorems 1 and 2) can be extended to more complicated nonlinear PDEs with low regularity arising in particular from applications in nonlinear electromagnetic inverse and optimal design problems [6, 29, 40–44].*

6 Appendix: real and complex interpolation

We follow [18, Appendix B] for the definition of the real and complex interpolation spaces. Let (X,Y) be an interpolation couple. For given $x \in X+Y$ and $t > 0$, we introduce

$$K(t,x) := K(t,x,X,Y) := \inf\{\|a\|_X + t\|b\|_Y \mid x = a+b, a \in X, b \in Y\}.$$

Moreover, for given $p \in [1,\infty]$, let $L_\star^p(0,\infty)$ denote the space of p-integrable functions on $(0,+\infty)$ with respect to the measure $\frac{1}{t}dt$. Now, we introduce for $\theta \in (0,1)$ and $p \in [1,\infty]$, the real interpolation space as follows:

$$(X,Y)_{\theta,p} := \{x \in X+Y \mid \text{the function}\, (0,\infty) \ni t \mapsto t^{-\theta}K(t,x) \in \mathbb{R} \text{ belongs to } L_\star^p(0,\infty)\},$$

endowed with the norm

$$\|x\|_{(X,Y)_{\theta,p}} := \begin{cases} \left(\int_0^\infty t^{-p\theta-1} K(t,x)^p dt\right)^{1/p} & p \in [1,\infty), \\ \sup_{t\in(0,\infty)}\{t^{-\theta}K(t,x)\} & p = \infty. \end{cases}$$

In the following, let S denote the vertical strip, i.e.,

$$S := \{z \in \mathbb{C} \mid 0 < \Re z < 1\},$$

where $\Re z$ denotes the real part of $z \in \mathbb{C}$.

Definition 3. *Let (X,Y) be an interpolation couple. The space $\mathcal{F}(X,Y)$ consists of all functions $f : S \to X+Y$ satisfying the following properties:*

1. *f is holomorphic in the interior of the strip, continuous and bounded up to its boundary with values in $X+Y$.*
2. *The function*
$$f_0(t) := f(it) \qquad \forall\, t \in \mathbb{R}$$
is bounded and continuous with values in X.
3. *The function*
$$f_1(t) := f(1+it) \qquad \forall\, t \in \mathbb{R}$$
is bounded and continuous with values in Y.

Moreover, the space $\mathcal{F}(X,Y)$ is endowed with the norm

$$\|f\|_{\mathcal{F}(X,Y)} = \max\{\sup_{t\in\mathbb{R}} \|f_0(t)\|, \sup_{t\in\mathbb{R}} \|f_1(t)\|\}.$$

For every $\theta \in [0,1]$, we define the complex interpolation space by

$$[X,Y]_\theta := \{f(\theta) \in X+Y \mid f \in \mathcal{F}(X,Y)\},$$

equipped with the norm

$$\|x\|_{[X,Y]_\theta} = \inf_{f\in\mathcal{F}(X,Y)\&f(\theta)=x} \|f\|_{\mathcal{F}(X,Y)}.$$

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