ABOUT A DEFICIT IN LOW ORDER CONVERGENCE RATES ON THE EXAMPLE OF AUTOCONVOLUTION

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ABSTRACT. We revisit in L^2-spaces the autoconvolution equation $x * x = y$ with solutions which are real-valued or complex-valued functions $x(t)$ defined on a finite real interval, say $t \in [0,1]$. Such operator equations of quadratic type occur in physics of spectra, in optics and in stochastics, often as part of a more complex task. Because of their weak nonlinearity deautoconvolution problems are not seen as difficult and hence little attention is paid to them wrongly. In this paper, we will indicate on the example of autoconvolution a deficit in low order convergence rates for regularized solutions of nonlinear ill-posed operator equations $F(x) = y$ with solutions $x^\dagger$ in a Hilbert space setting. So for the real-valued version of the deautoconvolution problem, which is locally ill-posed everywhere, the classical convergence rate theory developed for the Tikhonov regularization of nonlinear ill-posed problems reaches its limits if standard source conditions using the range of $F'(x^\dagger)^*$ fail. On the other hand, convergence rate results based on Hölder source conditions with small Hölder exponent and logarithmic source conditions or on the method of approximate source conditions are not applicable since qualified nonlinearity conditions are required which cannot be shown for the autoconvolution case according to current knowledge. We also discuss the complex-valued version of autoconvolution with full data on $[0,2]$ and see that ill-posedness must be expected if unbounded amplitude functions are admissible. As a new detail, we present situations of local well-posedness if the domain of the autoconvolution operator is restricted to complex L^2-functions with a fixed and uniformly bounded modulus function.

1. Introduction

Regularization theory for linear ill-posed operator equations in Hilbert spaces representing linear inverse problems seems to be almost complete, including results on convergence rates (cf. [8, Chapters 2-9] and more recently for example [29, 31, 36]). Moreover, there are now successful steps toward Banach space theory (cf., e.g., [37] and references therein). However, in the treatment of *nonlinear inverse problems* aimed at solving operator equations

$$F(x) = y \tag{1.1}$$

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with *nonlinear forward operators* $F : \mathcal{D}(F) \subseteq X \to Y$ and domain $\mathcal{D}(F)$ there is still much to do even in the Hilbert space setting. With focus on *autoconvolution* problems we will consider in this paper nonlinear equations (1.1), where X and Y are infinite dimensional separable *Hilbert spaces*, and we denote by the symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norms and the inner products, respectively, in both spaces.

It begins with the rarely clarified question of how ill-posedness is to be defined for nonlinear problems, whereas for linear problems ill-posedness is completely well-defined by the fact that the range of the linear forward operator is not closed in Y. In [23, Definition 2] a concept of *local well-posedness and ill-posedness* was suggested and we repeat here this idea:

Definition 1.1. *We call a nonlinear operator equation (1.1) locally well-posed at a solution point $x^\dagger \in \mathcal{D}(F)$ if there is a closed ball $\overline{B}_r(x^\dagger) := \{x \in X : \|x - x^\dagger\| \le r\}$ around $x^\dagger$ with radius $r > 0$ such that, for every sequence $\{x_n\}_{n=1}^\infty \subset \overline{B}_r(x^\dagger) \cap \mathcal{D}(F)$, the limit condition $\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\| = 0$ implies that $\lim_{n\to\infty} \|x_n - x^\dagger\| = 0$. Otherwise the equation is called locally ill-posed at $x^\dagger \in \mathcal{D}(F)$, which means that, for arbitrarily small radii $r > 0$, there exist sequences $\{x_n\}_{n=1}^\infty \subset \overline{B}_r(x^\dagger) \cap \mathcal{D}(F)$ such that $\lim_{n\to\infty} \|F(x_n) - F(x^\dagger)\| = 0$, but $\lim_{n\to\infty} \|x_n - x^\dagger\| = 0$ fails.*

Furthermore, in the past 25 years a general theory including convergence rates results was developed for variational (Tikhonov-type) regularization methods (cf., e.g., [8, Chapter 10] and [34, Chapter 3]) and iterative regularization methods (cf., e.g., [27]) applied to abstract ill-posed nonlinear equations (1.1). This general theory is mostly based on Gâteaux, Frechét or directional derivatives $F'(x)$ of F for elements x from some neighborhood of a solution $x^\dagger$ to (1.1). There is a collection of *nonlinearity conditions* which are relevant for that theory. In particular, starting from the paper [16], the *tangential cone condition*

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \le C\,\|F(x) - F(x^\dagger)\| \tag{1.2}$$

for some constant $0 < C < \infty$ and all $x \in \overline{B}_r(x^\dagger) \cap \mathcal{D}(F)$ is playing a prominent role, where the focus of iterative regularization methods is on constants $0 < C < 1$. But the verification of such qualified nonlinearity conditions is still missing or cannot be proven for large relevant classes of nonlinear inverse problems. The same can be said for weaker conditions of the form

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \le C\,\varphi(\|F(x) - F(x^\dagger)\|), \tag{1.3}$$

where φ is a concave *index function* $\varphi : (0,\infty) \to (0,\infty)$. As usual (cf. [20, 30]) we call φ an index function if it is strictly increasing with the limit condition $\lim_{t\to+0} \varphi(t) = 0$.

However, for wide classes of problems there are good chances to show at least a local Lipschitz condition of F' at $x^\dagger$ with the consequence that

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \le K\,\|x - x^\dagger\|^2 \tag{1.4}$$

holds for some constant $0 < K < \infty$ on $\overline{B}_r(x^\dagger) \cap \mathcal{D}(F)$. This is of great interest for the classical form of Tikhonov regularization for nonlinear ill-posed problems in Hilbert spaces, where instead of $y = F(x^\dagger)$ only noisy data $y^\delta \in Y$ with $\|y - y^\delta\| \le \delta$ with noise level $\delta > 0$ are available. Then stable approximate solutions x_α^δ to $x^\dagger$ are minimizers of the extremal problem

$$\|F(x) - y^\delta\|^2 + \alpha\,\|x - \overline{x}\|^2 \to \min, \quad \text{subject to} \quad x \in \mathcal{D}(F), \tag{1.5}$$

with *regularization parameter* $\alpha > 0$ and a prescribed reference element $\overline{x} \in X$. Whenever the limit conditions

$$\alpha \to 0 \qquad \text{and} \qquad \frac{\delta^2}{\alpha} \to 0 \tag{1.6}$$

hold for the regularization parameter one can show by using the concept of *$\overline{x}$-minimum-norm solutions* (see [8, Sect. 10.1]) that the regularized solutions x_α^δ converge (in the sense of subsequences, cf. [8, Theorem 10.3]) for $\delta \to 0$ with respect to the norm in X to such solutions $x^\dagger$ which have minimal distance to $\overline{x}$ under all solutions to (1.1) If, moreover, for convex domain $\mathcal{D}(F)$ and weakly sequentially closed operator F the *benchmark source condition*

$$x^\dagger = \overline{x} + \frac{1}{2} F'(x^\dagger)^* v \tag{1.7}$$

with the adjoint operator $F'(x^\dagger)^*$ of $F'(x^\dagger)$ and with a source element $v \in Y$ is satisfied and moreover the *smallness condition*

$$K\,\|v\| < 1 \tag{1.8}$$

is fulfilled, then the results of the seminal paper [9] on convergence rates for the Tikhonov regularization of nonlinear ill-posed problems apply and yield for an a priori choice $\alpha(\delta) \sim \delta$ of the regularization the convergence rate

$$\|x^\delta_{\alpha(\delta)} - x^\dagger\| \;=\; \mathcal{O}\left(\sqrt{\delta}\right) \qquad \text{as} \quad \delta \to 0. \tag{1.9}$$

If the set of $\overline{x}$-minimum-norm solutions to (1.1) is not unique, then it is an immediate consequence of the result (1.9) that only *one* such solution $x^\dagger \in \mathcal{D}(F)$ to (1.1) can satisfy the three conditions (1.4), (1.7) and (1.8), simultaneously.

The papers [19] and [3] have discussed consequences of nonlinearity conditions of the form (1.3) for Banach space regularization, but they also apply to the Hilbert space situation of Tikhonov regularization (1.5) under consideration here. In this situation, we obtain for a choice $\alpha = \alpha(\delta, y^\delta)$ of the regularization parameter by the sequential discrepancy principle (cf. [1, 21]) convergence rates

$$\|x^\delta_{\alpha(\delta,y^\delta)} - x^\dagger\| = \mathcal{O}\left(\sqrt{\varphi(\delta)}\right) \qquad \text{as} \quad \delta \to 0 \tag{1.10}$$

whenever (1.3) is satisfied for some concave index function φ together with the benchmark source condition (1.7), and no smallness condition is required. If the benchmark source condition fails, but the derivative $F'(x^\dagger) : X \to Y$ is an injective and bounded linear operator, then under (1.3) the *method of approximate source conditions* developed in [18] can be used together with *variational inequalities* combining solution smoothness and nonlinearity structure in one tool (cf. [22], [34, Chapt. 3], [11, Chapt. 12] and [15]). This yields convergence rates

$$\|x^\delta_{\alpha(\delta,y^\delta)} - x^\dagger\| = \mathcal{O}\left(\psi(\delta)\right) \qquad \text{as} \quad \delta \to 0, \tag{1.11}$$

which are lower than the rates in (1.10). Taking into account [3, Theorem 5.2] and [21, Theorem 2] it can be seen that the rate function ψ in (1.11) is an index function of the form

$$\psi(\delta) = d\left(\Psi^{-1}(\varphi(\delta))\right) \qquad \text{with} \qquad \Psi(R) := \frac{d(R)^2}{R},$$

essentially based on the decay rate of the concave decreasing and strictly positive distance function

$$d(R) := \min\{\|x^\dagger - \overline{x} - \frac{1}{2}F'(x^\dagger)^* w\| : w \in Y, \|w\| \le R\}, \qquad R > 0,$$

to zero as $R \to \infty$ which indicates for $x^\dagger$ the degree of violation with respect to (1.7). The rate (1.11) can be arbitrarily slow if $x^\dagger$ misses the benchmark source condition significantly what goes hand in hand with a very low decay of $d(R) \to 0$ as $R \to \infty$.

If the benchmark source condition (1.7) fails, but the Fréchet derivative $F'(x)$ exists for all $x \in \overline{B}_r(x^\dagger) \subset \mathcal{D}(F)$ and some $r > 0$, by extending the ideas of [16, 33, 38] two further alternatives for obtaining convergence rates to (1.5) have been presented in the paper [26] with focus on low order *Hölder source conditions* (see also [23, 38])

$$x^\dagger = \overline{x} + (F'(x^\dagger)^* F'(x^\dagger))^\nu \, w, \quad w \in X, \qquad 0 < \nu < \frac{1}{2}, \tag{1.12}$$

and *logarithmic source conditions* (cf. [17])

$$x^\dagger = \overline{x} + f(F'(x^\dagger)^* F'(x^\dagger)) \, w, \quad w \in X, \quad f(t) := (-\log t)^{-\mu}, \;\; \mu > 0. \tag{1.13}$$

As first option the nonlinearity condition

$$F'(x) = R(x, x^\dagger) F'(x^\dagger), \;\; \|R(x, x^\dagger) - I\|_{Y \to Y} \le C_R \, \|x - x^\dagger\|^\kappa, \; 0 < \kappa \le 1, \tag{1.14}$$

for some constant $0 < C_R < \infty$ and all $x \in \overline{B}_r(x^\dagger) \subset \mathcal{D}(F)$ is recommended. Then the mean value theorem in integral form yields (cf. [16, p.28])

$$\|F(x)-F(x^\dagger)-F'(x^\dagger)(x-x^\dagger)\| = \| \int_0^1 [F'(x^\dagger+t(x-x^\dagger))-F'(x^\dagger)](x-x^\dagger)dt\|$$

$$\leq \| \int_0^1 [R(x^\dagger + t(x-x^\dagger), x^\dagger) - I]\, F'(x^\dagger)(x-x^\dagger)\, dt\|$$

$$\leq C_R \left(\int_0^1 t^\kappa \, dt \right) \|F'(x^\dagger)(x-x^\dagger)\| \, \|x-x^\dagger\|^\kappa$$

and hence

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x-x^\dagger)\| \leq \frac{C_R}{1+\kappa} \|F'(x^\dagger)(x-x^\dagger)\| \, \|x-x^\dagger\|^\kappa. \tag{1.15}$$

Now the inequality (1.15) implies on the one hand that

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x-x^\dagger)\| \leq \widetilde{C} \, \|F'(x^\dagger)(x-x^\dagger)\| \tag{1.16}$$

holds for some constant $0 < \widetilde{C} < \infty$ and all $x \in \overline{B}_r(x^\dagger)$. On the other hand, by using the triangle inequality, from (1.15) we even derive the tangential cone condition (1.2) in the case of sufficiently small $r > 0$, which is then also a consequence of (1.14).

As second option the nonlinearity condition

$$F'(x) = F'(x^\dagger)R(x, x^\dagger), \quad \|R(x, x^\dagger) - I\|_{X \to X} \leq C_R \, \|x-x^\dagger\|^\kappa, \; 0 < \kappa \leq 1, \tag{1.17}$$

for some constant $0 < C_R < \infty$ and all $x \in \overline{B}_r(x^\dagger) \subset \mathcal{D}(F)$ has been suggested, which is very different from the tangential cone condition but can be verified for inverse problems with boundary measurements (cf., e.g., [5]). For Hölder and logarithmic rates under (1.17) we refer to [26, Theorem 2.1] and should mention in this context that for the proof of those convergence rates a condition of form (1.17) must be valid with a uniform constant C_R for all x and $x^\dagger$ lying in a small ball.

On the other hand, when the benchmark source condition (1.7) fails or the source element $v \in Y$ in (1.7) violates the smallness condition (1.8) and if moreover neither a condition (1.3) with any concave index function φ nor the condition (1.17) are satisfied, but only a nonlinearity condition (1.4) holds, then to our knowledge the literature provides *no convergence rate result*. Hence, this situation of low solution smoothness in combination with a poor structure of nonlinearity describes an unexplored area with respect to convergence rates for the Tikhonov regularization. In Section 2 we will show that this situation may arise for the real-valued autoconvolution problem on the unit interval.

This variety of autoconvolution problems, occurring for example in the deconvolution of appearance potential spectra, leads to operator equations (1.1) which are locally ill-posed everywhere. As the numerical case studies in [10] show, the strength of ill-posedness is somewhat reduced if for a support of solutions $x^\dagger$ on $[0, 1]$ the full data of $F(x^\dagger)$

are observed on $[0,2]$. The question of ill-posedness must be reset in the case of the complex-valued autoconvolution equations motivated by an application from laser optics (cf. [13]). We will show in Section 3 that both locally well-posed and ill-posed situations occur for such complex-valued problems with full data in dependence of the domain $\mathcal{D}(F)$ under consideration.

2. Autoconvolution for real functions on the unit interval

In this section, we consider the autoconvolution operator F on the space $X = Y = L^2(0,1)$ of quadratically integrable real functions over the unit interval $[0,1]$. Then (1.1) attains the form

$$[F(x)](s) := \int_0^s x(s-t)x(t)dt = y(s), \quad 0 \le s \le 1, \tag{2.1}$$

with $F : L^2(0,1) \to L^2(0,1)$ and $\mathcal{D}(F) = L^2(0,1)$. This operator equation of quadratic type occurs in physics of spectra, in optics and in stochastics, often as part of a more complex task (see, e.g., [2, 6, 35]). A series of studies on deautoconvolution and regularization have been published for the setting (2.1), see for example [7, 24, 25, 32]. Some first basic mathematical analysis of the autoconvolution equation can already be found in the paper [14]. Moreover, a regularization approach for general quadratic operator equations was suggested in the recent paper [12], corresponding numerical case studies have been presented in [4].

Example 2.1. The simple example of a sequence belonging to $\overline{B}_r(x^\dagger) \subset L^2(0,1)$,

$$x_n(t) = \begin{cases} x^\dagger(t) & \text{if } 0 \le t \le 1 - \frac{1}{n} \\ x^\dagger(t) + r\sqrt{n} & \text{if } 1 - \frac{1}{n} < t \le 1 \end{cases} \qquad (n = 2, 3, \ldots),$$

with $\|x_n - x^\dagger\| = r$, but

$$\|F(x_n) - F(x^\dagger)\| \le 2r \int_0^{1/n} |x^\dagger(t)|dt \le \frac{2r}{\sqrt{n}} \|x^\dagger\| \to 0 \quad \text{as} \quad n \to \infty,$$

shows that the equation (2.1) is *locally ill-posed* at every point $x^\dagger \in L^2(0,1)$.

This ill-posedness occurs although the corresponding nonlinear operator F is *not compact* (cf. [14, Prop. 4]). However, its linearization is compact, since

$$[F'(x)h](s) = 2\int_0^s x(s-t)h(t)dt, \quad 0 \le s \le 1, \quad h \in L^2(0,1), \tag{2.2}$$

characterizes the *Fréchet derivative* $F'(x) : L^2(0,1) \to L^2(0,1)$ of F in all points $x \in L^2(0,1)$ and any linear convolution operator mapping $x \in L^2(0,1) \mapsto a * x \in L^2(0,1)$ with $a \in L^2(0,1)$ is compact. Based on Titchmarsh's theorem (cf. [14, Lemma 3]) it can be shown that $F'(x^\dagger)$ is just an *injective* operator if

$$\sup\{\, \bar{t} \in [0,1] : \; x^\dagger(t) = 0 \;\text{ a.e. on }\; [0,\bar{t}]\} \;=\; 0. \tag{2.3}$$

If a solution $x^\dagger$ to (2.1) satisfies the condition (2.3), then $x^\dagger$ and $-x^\dagger$ are the only solutions of this equation, i.e. the solution is twofold. On the other hand, it was also shown in [14, Theorem 2] that F is *weakly continuous*, hence *weakly sequentially closed.* Moreover, $F'(x)$ is Lipschitz continuous and satisfies the condition

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| = \|F(x - x^\dagger)\|^2 \le \|x - x^\dagger\|^2, \tag{2.4}$$

for all $x, x^\dagger \in L^2(0,1)$, such that the nonlinearity condition (1.4) is fulfilled with $K = 1$ and for arbitrarily large balls $\overline{B}_r(x^\dagger)$. A further assertion on nonlinearity is formulated in the following proposition.

Proposition 2.2. *For the autoconvolution operator F mapping in $L^2(0,1)$ and any element $x^\dagger \in L^2(0,1)$ there is no index function η in combination with a radius $r > 0$ such that*

$$\|F(x) - F(x^\dagger)\| \le \hat{C}\, \eta(\|F'(x^\dagger)(x - x^\dagger)\|) \tag{2.5}$$

for some constant $0 < \hat{C} < \infty$ and all $x \in \overline{B}_r(x^\dagger)$.

Proof. To construct a contradiction it is enough to find a sequence $\{x_n\}_{n=1}^\infty \subset \overline{B}_r(x^\dagger)$ such that $\|F'(x^\dagger)(x_n - x^\dagger)\| \to 0$ as $n \to \infty$, but $\lim\limits_{n\to\infty} \|F(x_n) - F(x^\dagger)\| > 0$. Along the lines of Example 4 from [14] we can consider the sequence of functions $x_n = x^\dagger + \Delta_n \in \overline{B}_r(x^\dagger)$ with $\Delta_n(t) = \sqrt{2}r \sin(\pi n t)$ and $\|\Delta_n\| = r > 0$. Taking into account the weak convergence $x_n - x^\dagger \rightharpoonup 0$ in $L^2(0,1)$ we have $\|F'(x^\dagger)(x_n - x^\dagger)\| \to 0$ and for any index function η also $\eta(\|F'(x^\dagger)(x_n - x^\dagger)\|) \to 0$ as $n \to \infty$, because $F'(x^\dagger)$ is a compact operator. However, F is not compact and $\lim\limits_{n\to\infty} \|F(x_n) - F(x^\dagger)\| = \lim\limits_{n\to\infty} \|(2x^\dagger + \Delta_n) * \Delta_n\| = \lim\limits_{n\to\infty} \|\Delta_n * \Delta_n\| = \frac{r^2}{\sqrt{6}} > 0$. This proves the proposition. Note that we have used in this context the limit $\lim\limits_{n\to\infty} \|x^\dagger * \Delta_n\| = 0$, which is again a consequence of the compactness of linear convolution operators. □

Now the following corollary of Proposition 2.2 is valid.

Corollary 2.3. *For the autoconvolution operator from Proposition 2.2 a condition (1.16) and consequently a nonlinearity condition (1.14) cannot hold. Moreover also the tangential cone condition (1.2) cannot hold with a small constant $0 < C < 1$.*

Proof. From (1.16) we would obtain by using the triangle inequality

$$\|F(x) - F(x^\dagger)\| \le \|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| + \|F'(x^\dagger)(x - x^\dagger)\|$$

$\leq (\widetilde{C}+1)\,\|F'(x^\dagger)(x-x^\dagger)\|$ and hence (2.5) with $\eta(t)=t$, which contradicts Proposition 2.2. By taking into account that (1.16) is a consequence of the nonlinearity condition (1.14) we see that also (1.14) cannot hold. Moreover, a tangential cone condition (1.2) would yield $\|F(x)-F(x^\dagger)\| \leq \|F(x)-F(x^\dagger)-F'(x^\dagger)(x-x^\dagger)\| + \|F'(x^\dagger)(x-x^\dagger)\| \leq C\,\|F(x)-F(x^\dagger)\| + \|F'(x^\dagger)(x-x^\dagger)\|$, and in particular with $0<C<1$

$$\|F(x)-F(x^\dagger)\| \leq \frac{1}{1-C}\,\|F'(x^\dagger)(x-x^\dagger)\|,$$

which is also incompatible with Proposition 2.2. $\square$

With the following proposition we will show that also the nonlinearity condition (1.17) cannot hold.

Proposition 2.4. *For the autoconvolution operator F mapping in $L^2(0,1)$ a nonlinearity condition (1.17) cannot hold.*

Proof. For $x^\dagger=0$ the assertion is obviously true since $F'(x^\dagger)$ is the zero-operator in this case, but there are non-zero operators $F'(x)$ for elements x in any ball $\overline{B}_r(0)$. Hence we can restrict our proof to the case that $x^\dagger \neq 0$. Now let us assume that condition (1.17) is satisfied. From (1.17) we have that, for all $x \in \overline{B}_r(x^\dagger)$, $R(x,x^\dagger): X \to X$ denotes bounded linear operators with a uniform norm bound and

$$\|R(x,x^\dagger)^* - I\|_{X\to X} = \|R(x,x^\dagger)-I\|_{X\to X} \leq C_R\, r^\kappa$$

for all those operators and their adjoints. Let us define, for all $s\in[0,1]$, the functions

$$x_s(t) := \begin{cases} x(s-t) & \text{for } 0\le t\le s \\ 0 & \text{else} \end{cases}, \quad x_s^\dagger(t) := \begin{cases} x^\dagger(s-t) & \text{for } 0\le t\le s \\ 0 & \text{else} \end{cases}.$$

Then we have for arbitrary $v\in L^2(0,1)$

$$[F'(x)v](s) = 2\int_0^s x(s-t)v(t)dt = 2\int_0^1 x_s(t)v(t)dt$$

and

$$[F'(x^\dagger)R(x,x^\dagger)v](s) = 2\int_0^s x^\dagger(s-t)R(x,x^\dagger)v(t)dt$$

$$= 2\int_0^1 x_s^\dagger(t)R(x,x^\dagger)v(t)dt = 2\int_0^1 R(x,x^\dagger)^* x_s^\dagger(t)v(t)dt.$$

Hence,

$$\int_0^1 x_s(t)v(t)dt = \int_0^1 R(x,x^\dagger)^* x_s^\dagger(t)v(t)dt \qquad \text{for all} \quad v\in L^2(0,1),$$

which yields for all $0 \le s \le 1$ the equality

$$R(x,x^\dagger)^*\, x_s^\dagger \;=\; x_s\,. \tag{2.6}$$

To construct a contradiction we consider $x := x^{\{n\}}$ $(n = 1, 2, ...)$ with $x^{\{n\}}(t) := x^\dagger(t) + \sqrt{2}r\sin(\pi n t)$. From the last equality we get

$$R(x^{\{n\}},x^\dagger)^*(x_1^\dagger - x_{1-\frac{1}{n}}^\dagger) = R(x^{\{n\}},x^\dagger)^* x_1^\dagger - R(x^{\{n\}},x^\dagger)^* x_{1-\frac{1}{n}}^\dagger = x_1^{\{n\}} - x_{1-\frac{1}{n}}^{\{n\}}.$$

For the norms of $x_1^\dagger - x_{1-\frac{1}{n}}^\dagger$ and $x_1^{\{n\}} - x_{1-\frac{1}{n}}^{\{n\}}$ we obtain

$$\begin{aligned}
||x_1^\dagger - x_{1-\frac{1}{n}}^\dagger||^2 &= \int_0^{1-\frac{1}{n}} (x^\dagger(1-t) - x^\dagger(1-\tfrac{1}{n}-t))^2 \mathrm{d}t + \int_{1-\frac{1}{n}}^1 x^\dagger(1-t)^2 \mathrm{d}t \\
&= \int_0^{1-\frac{1}{n}} (x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t))^2 \mathrm{d}t + \int_0^{\frac{1}{n}} x^\dagger(t)^2 \mathrm{d}t \\
&\to 0 \;\text{ as }\; n \to \infty
\end{aligned}$$

and

$$\begin{aligned}
||x_1^{\{n\}} - x_{1-\frac{1}{n}}^{\{n\}}||^2 &= \int_0^{1-\frac{1}{n}} (x^{\{n\}}(1-t) - x^{\{n\}}(1-\tfrac{1}{n}-t))^2 \mathrm{d}t + \int_{1-\frac{1}{n}}^1 x^{\{n\}}(1-t)^2 \mathrm{d}t \\
&= \int_0^{1-\frac{1}{n}} (x^\dagger(t+\tfrac{1}{n}) + \sqrt{2}r\sin(\pi n(t+\tfrac{1}{n})) - x^\dagger(t) - \sqrt{2}r\sin(\pi n t))^2 \mathrm{d}t \\
&\quad + \int_0^{\frac{1}{n}} (x^\dagger(t) + \sqrt{2}r\sin(\pi n t))^2 \mathrm{d}t \\
&\ge \int_0^{1-\frac{1}{n}} 2\Big(\frac{\sqrt{2}}{2}(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)) - 2r\sin(\pi n t)\Big)^2 \mathrm{d}t.
\end{aligned}$$

Now the simple inequality $2(a+b)^2 \ge b^2 - 2a^2$ with

$$a := \frac{\sqrt{2}}{2}(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)) \quad \text{and} \quad b := -2r\sin(\pi n t) \quad \text{yields}$$

$$\begin{aligned}
&\int_0^{1-\frac{1}{n}} 2\Big(\frac{\sqrt{2}}{2}(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)) - 2r\sin(\pi n t)\Big)^2 \mathrm{d}t \\
\ge &\int_0^{1-\frac{1}{n}} \Big(4\big(\tau\sin(\pi n t)\big)^2 - \big(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)\big)^2\Big) \mathrm{d}t \\
\ge &\int_0^{1-\frac{1}{n}} 4r^2\sin^2(\pi n t)\mathrm{d}t - \int_0^{1-\frac{1}{n}} \big(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)\big)^2 \mathrm{d}t \\
= &2r^2(1-\frac{1}{n}) - \int_0^{1-\frac{1}{n}} \big(x^\dagger(t+\tfrac{1}{n}) - x^\dagger(t)\big)^2 \mathrm{d}t \\
&\to 2r^2 \;\text{ as }\; n \to \infty.
\end{aligned}$$

By (1.17) we obtain

$$\begin{aligned}\|x_1^{\{n\}} - x_{1-\frac{1}{n}}^{\{n\}}\| &\le \|x_1^\dagger - x_{1-\frac{1}{n}}^\dagger\| \, \|R(x^{\{n\}}, x^\dagger)^*\|_{X\to X} \\ &\le \|x_1^\dagger - x_{1-\frac{1}{n}}^\dagger\| \left(\|R(x^{\{n\}}, x^\dagger) - I\|_{X\to X} + \|I\|_{X\to X}\right). \\ &\le \|x_1^\dagger - x_{1-\frac{1}{n}}^\dagger\| \, (C_R r^\kappa + 1)\end{aligned}$$

Taking the limit $n \to \infty$ this turns to $2r^2 \le 0$, which is a contradiction. Thus the proof is complete. □

Unfortunately, by now we cannot prove for any element $x^\dagger \neq 0$ that the autoconvolution operator (2.1) in $L^2(0,1)$ satisfies the tangential cone condition (1.2) or its attenuation (1.3) for some concave index function φ. Taking into account the triangle inequality we can reformulate the corresponding open problem in the following form:

Open problem 2.5. *For which elements $x^\dagger \neq 0$ do we have a concave index function φ in combination with a radius $r > 0$ for the autoconvolution operator F mapping in $L^2(0,1)$ such that*

$$\|F'(x^\dagger)(x - x^\dagger)\| \le \bar{C}\, \varphi(\|F(x) - F(x^\dagger)\|) \tag{2.7}$$

holds for some constant $0 < \bar{C} < \infty$ and all $x \in \overline{B}_r(x^\dagger)$.

For solutions $x^\dagger$, which violate the condition (2.7) for all concave index functions φ, to our best knowledge convergence rates for the Tikhonov regularization (1.5) applied to equation (2.1) can currently be established if and only if

$$x^\dagger(t) = \overline{x}(t) + \int_t^1 x^\dagger(s-t)\, v(s) ds, \quad 0 \le t \le 1,\ v \in L^2(0,1), \ \ \|v\| < 1, \tag{2.8}$$

holds, which expresses here the benchmark source condition (1.7) together with the smallness condition (1.8). Then we have for $\alpha(\delta) \sim \delta$ the rate (1.9) from [8, Theorem 10.4]. Necessary conditions to accomplish (2.8) concerning the interplay of $x^\dagger$ and $\overline{x}$ are formulated in the subsequent proposition.

Proposition 2.6. *Apart from the trivial case $\overline{x} = x^\dagger$, the condition (2.8) can only hold if $x^\dagger \neq 0$ and if the reference element $\overline{x} \in L^2(0,1)$ is chosen such that*

$$\frac{\|x^\dagger - \overline{x}\|}{\|x^\dagger\|} < 1 \tag{2.9}$$

and $x^\dagger - \overline{x}$ is a continuous function on $[0,1]$ with $\overline{x}(1) = x^\dagger(1)$. Hence, for the appropriate choice of $\overline{x}$ the value $x^\dagger(1)$ must be known. Furthermore, for the choice $\overline{x} = 0$ there is no $x^\dagger \neq 0$ which satisfies (2.8).

Proof. For $\overline{x} = x^\dagger$, (2.8) is always satisfied with $v = 0$. By using the norm-conserving linear transformation $L: v \mapsto \widetilde{v}$ in $L^2(0,1)$ defined as $[Lv](t) = \widetilde{v}(t) := v(1-t),\ 0 \le t \le 1$, we can rewrite the equation in (2.8) as

$$x^\dagger(1-t) - \overline{x}(1-t) = \int\limits_0^t \widetilde{v}(t-s)\, x^\dagger(s) ds, \quad 0 \le t \le 1,$$

or short in convolution form as $Lx^\dagger = L\overline{x} + [Lv] * x^\dagger$. However, the transformation $x \mapsto Tx := L(L\overline{x} + [Lv] * x)$ is a contractive, affine linear mapping for fixed $\|v\| < 1$ and by Banach's fixed point theorem there is a uniquely determined solution $x^\dagger \in L^2(0,1)$ satisfying the equation in (2.8). For $\overline{x} = 0$ we have $x^\dagger = 0$ as uniquely determined solution to that equation for all such source elements v. Now we can estimate $\|x^\dagger - \overline{x}\| \le \|x^\dagger\|\|v\| < \|x^\dagger\|$, for all nonzero solutions $x^\dagger$, which yields the necessary condition (2.9). Moreover, $x^\dagger - \overline{x}$ is a continuous function as the result of the convolution of the two functions $\widetilde{v}$ and $x^\dagger$ from $L^2(0,1)$, and thus we have $\overline{x}(1) = x^\dagger(1)$ as another necessary condition imposed on $x^\dagger$ to satisfy (2.8). □

Remark 2.7. If $x^\dagger \neq 0$ solves the operator equation (2.1) then there is also a second different solution $-x^\dagger$. If $\langle x^\dagger, \overline{x} \rangle = 0$, then the reference element $\overline{x} \in L^2(0,1)$ has the same norm distance to both solutions and the $\overline{x}$-minimum-norm solution is not unique. Hence, the benchmark source condition (2.8) can apply for at most one of the solutions $x^\dagger$ or $-x^\dagger$. Otherwise by (1.9) the regularized solutions x_α^δ would converge with $\alpha = \alpha(\delta) \sim \delta$ simultaneously to both solutions $x^\dagger$ and $-x^\dagger$. However, from (2.9) we have for $x^\dagger \neq 0$

$$\|x^\dagger - \overline{x}\|^2 = \|x^\dagger\|^2 - 2\,\langle x^\dagger, \overline{x} \rangle + \|\overline{x}\|^2 < \|x^\dagger\|^2$$

and thus

$$\|\overline{x}\|^2 < 2\,\langle x^\dagger, \overline{x} \rangle, \quad x^\dagger \neq 0. \tag{2.10}$$

Now the necessary condition (2.10 for obtaining (2.8) shows that under $\langle x^\dagger, \overline{x} \rangle = 0,\ x^\dagger \neq 0$, the benchmark source condition cannot hold at all.

3. Local well-posedness and ill-posedness occurring in the complex-valued autoconvolution equation

The complex-valued and full data analog to equation (2.1) was motivated by problems of ultrashort laser pulse characterization arising in the context of the self-diffraction SPIDER method, and we refer to [13, 28] for physical details. Taking into account $L^2_{\mathbb{C}}$-spaces of quadratically integrable complex-valued functions over finite real intervals we set $X = L^2_{\mathbb{C}}(0,1),\ Y = L^2_{\mathbb{C}}(0,1)$, and consider the operator equation (3.1)

$$F(x) = y, \qquad [F(x)](s) := \begin{cases} \int\limits_0^s x(s-t)x(t)dt & \text{if} \quad 0 \le s \le 1 \\ \int\limits_{s-1}^1 x(s-t)x(t)dt & \text{if} \quad 1 < s \le 2 \end{cases}$$

with $F : L^2_{\mathbb{C}}(0,1) \to L^2_{\mathbb{C}}(0,2)$ and $\mathcal{D}(F) = L^2_{\mathbb{C}}(0,1)$. Every function $x \in L^2_{\mathbb{C}}(0,1)$ can be represented as $x(t) = A(t)\, e^{i\phi(t)},\ 0 \le t \le 1$, with the nonnegative *amplitude (modulus)* function $A = |x|$ and the *phase* function $\phi:\ [0,1] \to \mathbb{R}$.

Example 3.1. Since the Example 2.1 fails in the full data case, i.e. if $y(s)$ is observed for all $0 \le s \le 2$, for showing the ill-posedness a new sequence construction became necessary. So it was outlined in [10, Prop. 2.3] that the sequence,

$$x_n(t) = x^\dagger(t) + \Psi_{\frac{1}{2}-\frac{1}{n}}(t),\ \ 0 \le t \le 1, \qquad \Psi_\beta(t) := \frac{r\sqrt{1-2\beta}}{t^\beta},\ \ \beta > 0,$$

with $\|x_n - x^\dagger\| = r$ and $\|F(x_n) - F(x^\dagger)\| \to 0$ as $n \to \infty$ is appropriate for the full data case. This sequence even applies to $x^\dagger \in L^2_{\mathbb{C}}(0,1)$ and shows local ill-posedness everywhere also for the complex-valued case $F : L^2_{\mathbb{C}}(0,1) \to L^2_{\mathbb{C}}(0,2)$ (cf. [13, Example 3.1]).

In Example 3.1, the perturbation function $\Psi_\beta(t)$ is real-valued and has a weak pole at $t = +0$. Then, for fixed n, the amplitude function $|x_n|$ is not in $L^\infty(0,1)$ and hence x_n need not belong to $L^\infty_{\mathbb{C}}(0,1)$. The next example, however, shows that local ill-posedness everywhere can also be shown by means of sequences $\{x_n\}_{n=1}^\infty \subset L^\infty_{\mathbb{C}}(0,1)$.

Example 3.2. For the complex-valued case the linear operator $F'(x) : L^2_{\mathbb{C}}(0,1) \to L^2_{\mathbb{C}}(0,2)$ defined as

$$[F'(x)h]\,(s) = 2\,[x * h]\,(s), \quad 0 \le s \le 2, \quad h \in L^2_{\mathbb{C}}(0,1),$$

is compact and represents the Fréchet derivative of F at all points $x \in L^2_{\mathbb{C}}(0,1)$. Hence for weakly convergent sequences $z_n \rightharpoonup 0$ in $L^2_{\mathbb{C}}(0,1)$, i.e. if $\langle z_n, z\rangle \to 0$ as $n \to \infty$ holds for all $z \in L^2_{\mathbb{C}}(0,1)$, we have the norm convergence $\lim\limits_{n\to\infty} \|z_n * h\| = 0$ for all $h \in L^2_{\mathbb{C}}(0,1)$. Consequently, for $x_n = x^\dagger + z_n$ we have $\|F(x_n) - F(x^\dagger)\| = \|z_n * (z_n + 2x^\dagger)\| \to 0$ if and only if $\|z_n * z_n\| \to 0$ as $n \to \infty$. When we set $z_n(t) = r\, e^{i\,n^2 t^2},\ 0 \le t \le 1$, for fixed and arbitrary $r > 0$ then we have $\|z_n\| = r$ and $z_n \rightharpoonup 0$ in $L^2_{\mathbb{C}}(0,1)$ and $\|z_n * z_n\| \to 0$ as $n \to \infty$ in the norm of $L^2_{\mathbb{C}}(0,2)$. Hence, the problem is locally ill-posed at any point $x^\dagger \in L^2_{\mathbb{C}}(0,1)$.

It was formulated in [13] as an open question whether the deautocovolution process remains always instable if *only phase perturbations* occur. This question is motivated by the laser pulse problem, where a complex-valued measuring tool based kernel function $k(s,t)$ is added to the integral equation (3.1), but the amplitude function $A = |x^\dagger|$ as part of the solution $x^\dagger(t) = A(t)\,e^{i\phi^\dagger(t)},\ 0 \le t \le 1$, can be measured and only the phase function $\phi^\dagger$ is to be determined from observed $y \in L^2_{\mathbb{C}}(0,2)$. The following proposition gives a negative answer to this question for the simplified case (3.1) with trivial kernel $k(s,t) \equiv 1$.

Proposition 3.3. *For solutions $x^\dagger(t) = A(t)\,e^{i\phi^\dagger(t)}$ to the complex-valued autoconvolution equation (3.1) with a fixed amplitude function $A \in L^\infty(0,1)$, which is not almost everywhere on $[0,1]$ the zero function, we restrict the domain of the operator $F : \mathcal{D}(F) \subset L^2_{\mathbb{C}}(0,1) \to L^2_{\mathbb{C}}(0,2)$ to*

$$\mathcal{D}(F) := \{x(t) = A(t)\,e^{i\phi(t)},\ 0 \le t \le 1,\ \phi : [0,1] \to \mathbb{R}\} \subset L^2_{\mathbb{C}}(0,1).$$

Then there exist phase functions $\phi^\dagger$ such that (3.1) is locally well-posed at $x^\dagger$.

Proof. We set $K_1 := \|A\|_{L^1(0,1)} > 0,\ K_\infty := \|A\|_{L^\infty(0,1)}$ and will show local well-posedness at points $x^\dagger(t) = A(t)\,e^{i\phi^\dagger(t)}$ with $\phi^\dagger(t) \equiv \omega$, $0 \le t \le 1$, and an arbitrary real constant ω. Then we have for all $x \in \overline{B}_{K_1}(x^\dagger) \cap \mathcal{D}(F)$ the local Hölder condition

$$\|x - x^\dagger\| \le 2^{3/4}\sqrt{\frac{K_\infty}{K_1}}\,\sqrt{\|F(x) - F(x^\dagger)\|} \tag{3.2}$$

with Hölder exponent 1/2, which yields the local well-posedness at the point $x^\dagger$. Namely, using the Hölder inequality we have for all $x(t) = A(t)\,e^{i\phi(t)}$ the estimate

$$\begin{aligned}
||F(x)-F(x^\dagger)|| &= \sqrt{\int_0^2 \left| \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(e^{2i\omega} - e^{i\phi(s-t)}e^{i\phi(t)})dt \right|^2 ds} \\
&\ge \frac{1}{\sqrt{2}} \int_0^2 \left| \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(e^{2i\omega} - e^{i\phi(s-t)}e^{i\phi(t)})dt \right| ds
\end{aligned}$$

and further by setting $\zeta(t) := \phi(t) - \omega$

$$\begin{aligned}
&\int_0^2 \left| \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(e^{2i\omega} - e^{i\phi(s-t)}e^{i\phi(t)})dt \right| ds \\
&= \int_0^2 \left| \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(1 - e^{i\zeta(s-t)}e^{i\zeta(t)})dt \right| ds
\end{aligned}$$

$$\geq \int_0^2 \left| \mathrm{Re} \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(1-e^{i(\zeta(s-t)+\zeta(t))})dt \right| ds$$

$$= \int_0^2 \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(1-\cos(\zeta(s-t)+\zeta(t))dtds.$$

By changing the order of integration and exploiting addition theorems we moreover obtain

$$\int_0^2 \int_{\max(0,s-1)}^{\min(1,s)} A(s-t)A(t)(1-\cos(\zeta(s-t)+\zeta(t))dtds$$

$$= \int_0^1 \int_t^{t+1} A(s-t)A(t)(1-\cos(\zeta(s-t))\cos(\zeta(t))+\sin(\zeta(s-t))\sin(\zeta(t)))dsdt$$

$$= \int_0^1 A(t) \int_t^{1+t} A(s-t)dsdt - \int_0^1 A(t)\cos(\zeta(t)) \int_t^{t+1} A(s-t)\cos(\zeta(s-t))dsdt$$

$$+ \int_0^1 A(t)\sin(\zeta(t)) \int_t^{t+1} A(s-t)\sin(\zeta(s-t))dsdt$$

$$= \int_0^1 A(t) \int_0^1 A(s)dsdt - \int_0^1 A(t)\cos(\zeta(t)) \int_0^1 A(s)\cos(\zeta(s))dsdt$$

$$+ \int_0^1 A(t)\sin(\zeta(t)) \int_0^1 A(s)\sin(\zeta(s))dsdt$$

$$= \left(\int_0^1 A(s)ds\right)^2 - \left(\int_0^1 A(s)\cos(\zeta(s))ds\right)^2 + \left(\int_0^1 A(s)\sin(\zeta(s))ds\right)^2$$

$$\geq \left(\int_0^1 A(s)(1+\cos(\zeta(s)))ds\right)\left(\int_0^1 A(s)(1-\cos(\zeta(s)))ds\right).$$

On the other hand, we have

$$
\begin{aligned}
||x - x^\dagger||^2 =& \int_0^1 |A(t)e^{i\omega} - A(t)e^{i\phi(t)}|^2 dt \\
=& \int_0^1 A(t)^2 |1 - e^{i\zeta(t)}|^2 dt \\
=& \int_0^1 A(t)^2((1 - \cos(\zeta(t)))^2 + \sin^2(\zeta(t)))dt \\
=& \int_0^1 A(t)^2(1 - 2\cos(\zeta(t)) + \cos^2(\zeta(t)) + \sin^2(\zeta(t)))dt \\
=& 2\int_0^1 A(t)^2(1 - \cos(\zeta(t)))dt \\
\leq& 2K_\infty \int_0^1 A(t)(1 - \cos(\zeta(t)))dt.
\end{aligned}
$$

Owing to $1 - \cos(\zeta(t)) \leq 2$ we have also the estimate

$$
\begin{aligned}
\int_0^1 A(s)(1 + \cos(\zeta(s)))ds =& 2\int_0^1 A(s)ds - \int_0^1 A(s)(1 - \cos(\zeta(s)))ds \\
\geq& 2K_1 - \sqrt{\int_0^1 A(s)^2(1 - \cos(\zeta(s)))^2 ds} \\
\geq& 2K_1 - \sqrt{2\int_0^1 A(s)^2(1 - \cos(\zeta(s)))ds} \\
=& 2K_1 - ||x - x^\dagger||.
\end{aligned}
$$

This yields for $x \in \overline{B}_{K_1}(x^\dagger)$

$$
\int_0^1 A(s)(1 + \cos(\zeta(s)))ds \geq K_1
$$

and hence by combining the above estimates (3.2), which proves the proposition.

$\square$

Finally, we note that well-posedness situations for the real autoconvolution operator $F : \mathcal{D}(F) \subset L^2(0,1) \to L^2(0,2)$ and specific compact domains $\mathcal{D}(F)$ were already outlined in [10] by exploiting Fourier transforms. In these cases, one can even verify the modulus of continuity of the inverse operator F^{-1}.

4. Conclusions and open questions

It is certainly future work to obtain progress with respect to the Open Problem 2.5. For the general theory it will be of interest to derive (low order) convergence rates for ill-posed nonlinear operator equations (1.1) when the solution $x^\dagger$ is *too nonsmooth* to satisfy the benchmark source condition (1.7) and if moreover the nonlinearity structure of the operator F around $x^\dagger$ is *too poor* to fulfil a condition (1.3) for some index function φ or a condition of type (1.17). As another open question from Section 2 we can ask for *smoothness classes* $\mathcal{M}$ of solutions $x^\dagger$ of the real-valued autoconvolution equation in $L^2(0,1)$ such that for all $x^\dagger \in \mathcal{M}$ a convergence rate

$$\|x^\delta_{\alpha(\delta,y^\delta)} - x^\dagger\| = \mathcal{O}\left(\theta(\delta)\right) \qquad \text{as} \quad \delta \to 0, \tag{4.1}$$

with some fixed concave index function θ is obtained for the Tikhonov regularization (1.5). For ill-posed operator equations $Ax = y$ with bounded linear operators $A : X \to Y$ possessing a nonclosed range in Y such smoothness classes $\mathcal{M}$ yielding (4.1) are usually dense subsets $\mathcal{M} = \{x \in X : x = Gv,\ v \in X\}$ of the Hilbert space X characterized by ranges of some bounded linear operators $G : X \to X$ with unbounded Moore-Penrose inverse $G^\dagger$ where G and A are connected by some *link condition* (cf. [20]). For nonlinear problems the smoothness classes have a much more complicated structure. For the real-valued autoconvolution problem (2.1) the benchmark source condition (1.7) leading to (4.1) with $\theta(t) = \sqrt{t}$ is quite illustrative. Here, the smoothness class $\mathcal{M}$ collects all elements $x^\dagger$ solving the fixed point equation $x^\dagger(t) = \overline{x}(t) + \int_t^1 x^\dagger(s-t)v(s)ds,\ 0 \le t \le 1$, where the source elements v pass through the open unit ball in $L^2(0,1)$. This fixed point equation is uniquely solvable for all such v, because the mapping $x \to \overline{x} + \frac{1}{2}F'(x)^*v$ is contractive and Banach's fixed point theorem applies.

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