On complex-valued deautoconvolution of compactly supported functions with sparse Fourier representation *

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Abstract

Convergence rates results for the Tikhonov regularization of nonlinear ill-posed operator equations are missing, even for a Hilbert space setting, if a range type source condition fails and if moreover nonlinearity conditions of tangential cone type cannot be shown. This situation applies for a deautoconvolution problem in complex-valued L^2-spaces over finite real intervals, occurring in a slightly generalized version in laser optics. For this problem we show that the lack of applicable convergence rates results can be overcome under the assumption that the solution of the operator equation has a sparse Fourier representation. Precisely, we derive a variational source condition for that case, which implies a convergence rate immediately. The surprising observation is that a sparsity assumption imposed on the solution leads to success, although the used norm square is not known to be a sparsity promoting penalty in the Tikhonov functional.

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1 Introduction

In the seminal paper [8], Engl, Kunisch and Neubauer achieved a breakthrough for obtaining *convergence rates* for Tikhonov-regularized solutions to ill-posed operator equations

$$F(x) = y \tag{1.1}$$

with *nonlinear* forward operators $F : \mathcal{D}(F) \subseteq X \to Y$ mapping between a convex subset $\mathcal{D}(F)$ of a Hilbert space X and a Hilbert space Y, where $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote norms and inner products, respectively, for both spaces X and Y. Ill-posed equations (1.1) are mathematical manifestations of nonlinear inverse problems with numerous applications in natural sciences, engineering and finance.

Given an exact right-hand side $y \in F(\mathcal{D}(F))$ let $x^\dagger \in \mathcal{D}(F)$ be a corresponding solution to (1.1). Typically, only a noisy measurement of y is available. By $y^\delta \in Y$ we denote such a measurement and we assume that

$$\|y^\delta - y\| \le \delta \tag{1.2}$$

for some *noise level* $\delta > 0$.

The Tikhonov regularization method consists in approximating the exact solution $x^\dagger$ by minimizers x_α^δ of the Tikhonov functional

$$T_\alpha^\delta(x) = \|F(x) - y^\delta\|^2 + \alpha\,\|x - \bar{x}\|^2, \quad x \in \mathcal{D}(F). \tag{1.3}$$

Here, $\bar{x} \in X$ is a reference element playing the role of an initial guess for $x^\dagger$ and $\alpha > 0$ is the regularization parameter controlling the trade-off between the data fidelity term $\|F(x) - y^\delta\|^2$ and the penalty term $\|x - \bar{x}\|^2$. In [8] estimates

$$\|x_\alpha^\delta - x^\dagger\| = \mathcal{O}\big(\sqrt{\delta}\big) \qquad \text{as} \quad \delta \to 0 \tag{1.4}$$

for the solution error in terms of the noise level δ were obtained, where the regularization parameter α has to be chosen in the right way depending on

δ. Such *convergence rates* results are of high interest for understanding the behavior of regularization methods and also for comparing the performance and efficiency of different methods.

In the present paper we deal with a concrete ill-posed nonlinear operator F arising in optical measurement setups for characterizing ultra-short laser pulses. The operator is of autoconvolution-type and it is known that the convergence rate results of [8] and also more recent results are not applicable. Nevertheless, we derive an estimate (1.4) using a tailor-made proof. Starting with the assumption that the exact solution $x^\dagger$ has a sparse Fourier representation we obtain a *variational source condition*, also called variational inequality (cf. [16]) or variational smoothness assumption (cf. [10]), which is known to imply the desired convergence rate immediately. In this context, it seems to be a surprising observation that the assumption of a sparse solution yields a convergence rate for the Tikhonov regularization of the deautoconvolution problem, although the used penalty term is not sparsity promoting. For approaches and results concerning the regularization with sparsity constraints we refer, for example, to the papers [19, 20, 22] and to corresponding chapters and paragraphs in the monographs [18, 23, 24].

The structure of the paper is as follows: In Section 2 we introduce and discuss the considered autoconvolution operator F. Then in Section 3 we present an overview of existing convergence rates results for nonlinear operators and comment on their non-applicability to our specific operator. Section 4 derives a variational source condition for a slightly different operator and, based on that result, in Section 5 we present and prove our main convergence rate result for the autoconvolution operator F.

2 Autoconvolution operators

Having in mind its application in laser optics, see e.g. [12], we are interested in the autoconvolution operator $F : L^2(0,1) \to L^2(0,2)$ defined by

$$[F(x)](s) := \int\limits_{\max(s-1,0)}^{\min(s,1)} x(s-t)\, x(t)\, \mathrm{d}t, \quad s \in (0,2). \tag{2.1}$$

Here $L^2(0,1)$ and $L^2(0,2)$ denote the Hilbert spaces of square integrable Lebesgue measurable complex-valued functions on $(0,1)$ and $(0,2)$, respectively. If one interprets a function $x \in L^2(0,1)$ as a function defined on the whole real line $\mathbb{R}$ with support contained in $(0,1)$, then the operator F can be written as

$$[F(x)](s) = \int_{\mathbb{R}} x(s-t)\, x(t)\, \mathrm{d}t, \quad s \in \mathbb{R}, \tag{2.2}$$

where the support of $F(x)$ (as a function on $\mathbb{R}$) is contained in $(0,2)$. This is the usual autoconvolution operator on the real line, but restricted to functions with support in $(0,1)$. The operator F, enriched with a device dependent kernel function, plays an important role in the SD-SPIDER method for characterizing ultra-short laser pulses (see [1, 3, 11]). Ill-posedness of F has been shown in [9, 13] and [5]. Moreover, an algorithm for finding *global* minimizers of the Tikhonov functional (1.3) has been presented in [21], see also [1].

As a tool for proving our convergence rates result for Tikhonov's regularization method applied to this concrete operator F we also need another type of convolution operator. By $* : L^2(0,1) \times L^2(0,1) \to L^2(0,1)$ we denote the symmetric bilinear operator defined by

$$[x * \tilde{x}](s) := \int_0^s x(s-t)\, \tilde{x}(t)\, \mathrm{d}t + \int_s^1 x(s+1-t)\, \tilde{x}(t)\, \mathrm{d}t, \quad s \in (0,1). \tag{2.3}$$

Interpreting the functions in $L^2(0,1)$ as 1-periodic functions on $\mathbb{R}$, the operator $*$ attains the form

$$[x * \tilde{x}](s) := \int_0^1 x(s-t)\, \tilde{x}(t)\, \mathrm{d}t, \quad s \in (0,1), \tag{2.4}$$

which is the usual convolution of periodic functions. The following lemma shows that F and $*$ are closely related.

Lemma 2.1. *Let $A : L^2(0,2) \to L^2(0,1)$ be the linear operator defined by*

$$[Az](s) := z(s) + z(s+1), \quad s \in (0,1). \tag{2.5}$$

Then A is bounded with $\|A\| \leq \sqrt{2}$ and we have

$$AF(x) = x * x$$

for all $x \in L^2(0,1)$.

Proof. The boundedness follows from

$$\|Az\|^2 = \int_0^1 |z(t) + z(t+1)|^2 \mathrm{d}t \le \int_0^1 \left(2\,|z(t)|^2 + 2\,|z(t+1)|^2\right) \mathrm{d}t = 2\,\|z\|^2$$

and the equality of $A \circ F$ and $*$ immediately follows from (2.1) and (2.3). $\square$

For the sake of completeness we recall the convolution theorem for periodic functions explicitly: Denote by $(e^{(k)})_{k\in\mathbb{Z}}$ the canonical Fourier basis of $L^2(0,1)$, that is,

$$e^{(k)}(t) = \exp(2\,\pi\,k\,t\,\mathrm{i}), \quad t \in (0,1), \tag{2.6}$$

and by $x_k := \langle x, e^{(k)}\rangle$ and $\tilde{x}_k := \langle \tilde{x}, e^{(k)}\rangle$ the Fourier coefficients of x and $\tilde{x}$, respectively. Then

$$x * \tilde{x} = \sum_{k\in\mathbb{Z}} x_k\,\tilde{x}_k\,e^{(k)}. \tag{2.7}$$

In addition to the original deautoconvolution problem (1.1) with operator $F : L^2(0,1) \to L^2(0,2)$ from (2.1) we will also consider a modified deautoconvolution problem with forward operator $A \circ F : L^2(0,1) \to L^2(0,1)$ in Section 4, where the operator $A : L^2(0,2) \to L^2(0,1)$ has been introduced in Lemma 2.1 by formula (2.5). Lemma 2.1 and the well-known convolution theorem will be the relevant ingredients for our main convergence rate proof in Section 5.

3 Existing convergence rates results are not applicable

The convergence rates result of [8] (see also [7, Theorem 10.4]) is based on four assumptions: The operator F has to be *Fréchet differentiable* on its domain $\mathcal{D}(F)$. Denoting the Fréchet derivative at $x \in \mathcal{D}(F)$ by $F'(x) : X \to Y$ *Lipschitz continuity* of this derivative implying

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \le \frac{L}{2}\,\|\tilde{x} - x\|^2 \qquad \text{for all} \quad \tilde{x}, x \in \mathcal{D}(F) \tag{3.1}$$

is required. Moreover, the exact solution has to satisfy a *source condition*

$$x^\dagger - \bar{x} = F'(x^\dagger)^*\,w \tag{3.2}$$

with some $w \in Y$, and the source element w has to satisfy the *smallness condition*

$$L\,\|w\| < 1. \tag{3.3}$$

If all these assumptions are fulfilled and the regularization parameter is chosen as $\alpha = \alpha(\delta) \sim \delta$, then the convergence rate (1.4) can be proven.

In case of our autoconvolution operator $F : L^2(0,1) \to L^2(0,2)$ with $\mathcal{D}(F) = L^2(0,1)$ and Fréchet derivative

$$[F'(x)h](s) = 2 \int\limits_{\max(s-1,0)}^{\min(s,1)} x(s-t)\,h(t)\,\mathrm{d}t, \quad s \in (0,2),$$

for $x, h \in L^2(0,1)$ the Lipschitz condition (3.1) holds with $L = 2$, because we have for all $\tilde{x}, x \in L^2(0,1)$ that

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| = \|F(\tilde{x} - x)\| \le \|\tilde{x} - x\|^2.$$

If $\bar{x} = 0$ is chosen as reference element in the Tikhonov functional (1.3), which is the standard situation if no additional a priori information can be provided, then the source condition (3.2) together with the smallness condition (3.3) can only hold in the trivial case $x^\dagger = 0$ as the following Lemma 3.1 will show. Also for $\bar{x} \neq 0$, (3.2) is hardly satisfied in combination with (3.3) as was discussed in [5, Proposition 2.6].

Lemma 3.1. *For $F : L^2(0,1) \to L^2(0,2)$ from (2.1) and $\bar{x} = 0$ in* (1.3) *the source condition (3.2) attaining the form*

$$x^\dagger(t) = 2 \int\limits_0^1 x^\dagger(s)\,w(s+t)\,ds, \qquad 0 \le t \le 1, \quad w \in L^2(0,2), \tag{3.4}$$

can only hold together with the smallness condition (3.3) attaining the form

$$2\,\|w\| < 1 \tag{3.5}$$

if $x^\dagger = 0$.

Proof. The equation (3.4), the structure of which can simply be verified, can be considered as a fixed point equation $x^\dagger = Tx^\dagger$, where the linear operator T mapping in $L^2(0,1)$ is contractive whenever (3.5) holds. Hence by Banach's fixed point theorem $x^\dagger = 0$ is the uniquely determined solution of the fixed point equation. The contractivity of T becomes evident, because we have

$$\int_0^1 \int_0^1 |w(s+t)|^2 ds\, dt = \int_0^1 \left(\int_t^{t+1} |w(\tau)|^2 d\tau \right) dt \le \|w\|^2$$

and by the Cauchy Schwarz inequality

$$\|Tx^\dagger\| \le \left(\int_0^1 \int_0^1 |w(s+t)|^2 ds\, dt \right)^{1/2} \|x^\dagger\| \le 2\, \|w\|\, \|x^\dagger\|.$$

□

In the past 25 years the classical convergence rates result of [8] has been extended and generalized in many different directions. On the one hand we can replace (3.1) by a local nonlinearity condition of *tangential cone* type. That is, we assume that

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \le \sigma(\|F(x) - F(x^\dagger)\|) \qquad (3.6)$$

holds for all $x \in \mathcal{D}(F) \cap B_r(x^\dagger)$, where $B_r(x^\dagger)$ is a ball around $x^\dagger$ with sufficiently small radius $r > 0$ and $\sigma : [0, \infty) \to [0, \infty)$ is an *index function*, i.e., a strictly increasing and continuous function with $\sigma(0){=}0$. Then the source condition (3.2) yields convergence rates for Tikhonov regularization *without any smallness condition*. The obtained rate function is not the square root of the noise level but involves the index function σ from (3.6). Such rates results can also be extended to Tikhonov regularization in Banach spaces (cf., e.g., [4, 24]).

On the other hand, if a nonlinear operator F satisfies (3.6) for some index function σ, but the source condition (3.2) fails, then the *method of approximate source conditions* helps to compensate this deficit. Here one considers the *distance functions*

$$d(R) = \min\{\|x^\dagger - \bar{x} - F'(x^\dagger)^*\, w\| : w \in Y, \|w\| \le R\},$$

defined for $r \ge 0$, and obtains convergence rates depending on σ in (3.6) and on the decay of $d(R)$ for $R \to \infty$. In this context (3.2) serves as *benchmark source condition* and $d(R)$ measures its violation. Also these results can be extended to Banach space settings (cf., e.g., [15, 4]).

In case of our autoconvolution operator F a tangential cone type condition (3.6) has not been verified up to now. Thus, extensions of the classical convergence rates result which rely on such conditions cannot be applied to this operator.

Recently, based on the initial paper [16], it was shown that in particular for nonlinear ill-posed problems in Hilbert and Banach spaces *variational source conditions* can play a crucial role for obtaining convergence rates. This modern tool combines the expression of solution smoothness with respect to the forward operator F, previously expressed by source conditions like (3.2), and the structure of nonlinearity of F in a vicinity of the solution $x^\dagger$, previously expressed by conditions like (3.1) or (3.6). For the classical Hilbert space situation of Tikhonov regularization with a penalty functional of norm square type (cf. (1.3)) and error measure $\|x_\alpha^\delta - x^\dagger\|^2$ such variational source conditions attain the form

$$\beta\,\|x-x^\dagger\|^2 \le \|x-\bar{x}\|^2 - \|x^\dagger-\bar{x}\|^2 + \varphi(\|F(x)-F(x^\dagger)\|) \qquad \text{for all} \quad x \in \mathcal{M}. \tag{3.7}$$

Here $\beta \in (0,1]$ is some constant, φ is a concave index function, and $\mathcal{M} \subseteq X$ is a set which has to contain all Tikhonov minimizers x_α^δ for all sufficiently small δ, where α is assumed to be chosen somehow for each δ. Based on such a variational source condition one obtains the convergence rate

$$\|x_\alpha^\delta - x^\dagger\|^2 = \mathcal{O}(\varphi(\delta)) \qquad \text{as} \quad \delta \to 0$$

(cf., e.g., [17] and for more general error measures [10] and [14]).

Variational source conditions (3.7) can be obtained from the Lipschitz condition (3.1) in combination with the source condition (3.2) and the smallness condition (3.3) and also from the tangential cone type condition (3.6) in combination with the source condition (3.2). But as mentioned above both sets of conditions are not available for the autoconvolution operator F under consideration.

Nevertheless, in Section 5 we will derive a variational source condition for F from (2.1) if $x^\dagger \in L^2(0,1)$ has a sparse Fourier representation. For the sake of completeness let us mention at this point that in the past ten years diverse convergence rates results for Tikhonov regularization of nonlinear ill-posed problems under sparsity constraints have been published, also exploiting the approach of variational source conditions (see [23, §3.3]). However, corresponding derivations of suitable variational source conditions rely on tangential cone type nonlinearity conditions and therefore again do not apply to our deautoconvolution problem. Thus, the only chance to obtain a variational source condition for our F is a tailor-made proof.

4 Rates for a modified deautoconvolution problem

Before we verify convergence rates result for the operator $F : L^2(0,1) \to L^2(0,2)$ defined by (2.1), we study the simpler situation of deconvolving periodic functions. That is, we derive a variational source condition for the operator $A \circ F$ with A from Lemma 2.1, which then yields a convergence rate. Based on the variational source condition for $A \circ F$ we will derive a variational source condition for F in Section 5.

We only consider the case $\bar{x} = 0$ in the Tikhonov function (1.3) and want to obtain a variational source condition of the form

$$E(x, x^\dagger) \leq \|x\|^2 - \|x^\dagger\|^2 + c\,\|AF(x) - AF(x^\dagger)\| \qquad \text{for all} \quad x \in X \quad (4.1)$$

for some $c > 0$. Due to the non-injectivity of F as a consequence of $F(x^\dagger) = F(-x^\dagger)$ the norm square $\|x - x^\dagger\|^2$ is not suitable as error measure $E(x, x^\dagger)$, because with $x = -x^\dagger$ the inequality (4.1) would imply $\|2\,x^\dagger\|^2 \leq 0$. However, a reasonable choice is

$$E(x, x^\dagger) := (\mathrm{dist}(x, S))^2 \qquad (4.2)$$

with the distance

$$\mathrm{dist}(x, S) := \inf_{\tilde{x}^\dagger \in S} \|x - \tilde{x}^\dagger\|$$

between x and S, where

$$S := \{\tilde{x}^\dagger \in X : AF(\tilde{x}^\dagger) = AF(x^\dagger)\}.$$

Owing to the convolution theorem (2.7) we immediately see that

$$S = \{\tilde{x}^\dagger \in X : \tilde{x}_k^\dagger = x_k^\dagger \text{ or } \tilde{x}_k^\dagger = -x_k^\dagger \text{ for all } k \in \mathbb{Z}\}, \qquad (4.3)$$

where the $\tilde{x}_k^\dagger$ and the $x_k^\dagger$ are the Fourier coefficients of $\tilde{x}^\dagger$ and $x^\dagger$, respectively. From this observation we can derive a more handy expression for $\mathrm{dist}(x, S)$. First, we note that

$$(\mathrm{dist}(x, S))^2 = \sum_{k \in \mathbb{Z}} \min\bigl\{|x_k - x_k^\dagger|^2, |x_k + x_k^\dagger|^2\bigr\},$$

again with x_k and $x_k^\dagger$ denoting the Fourier coefficients of x and $x^\dagger$, respectively. Defining a sequence $(\xi_k(x))_{k \in \mathbb{Z}}$ by

$$\xi_k(x) := \begin{cases} 1, & \text{if } \mathrm{Re}(\overline{x_k}\, x_k^\dagger) \geq 0, \\ -1, & \text{else} \end{cases} \qquad (4.4)$$

a simple calculation shows that then

$$\min\big\{|x_k - x_k^\dagger|^2, |x_k + x_k^\dagger|^2\big\} = |x_k - \xi_k(x)\, x_k^\dagger|^2.$$

Thus,

$$(\mathrm{dist}(x,S))^2 = \sum_{k\in\mathbb{Z}} |x_k - \xi_k(x)\, x_k^\dagger|^2. \tag{4.5}$$

Now we are ready to prove a variational source condition (4.1) for $A \circ F$ for the error measure (4.2).

Proposition 4.1. *Let $x^\dagger$ have a sparse Fourier representation, that is, only N Fourier coefficients with respect to the basis (2.6) do not vanish. Then the variational source condition* (4.1) *with error measure* (4.2) *and constant $c = 2\sqrt{N}$ is satisfied.*

Proof. Let $x \in L^2(0,1)$ and denote by $(x_k)_{k\in\mathbb{Z}}$ and $(x_k^\dagger)_{k\in\mathbb{Z}}$ the Fourier coefficients of x and $x^\dagger$, respectively. Define $\tilde{x}^\dagger \in S$ by

$$\tilde{x}_k^\dagger := \xi_k(x) x_k^\dagger, \quad k \in \mathbb{Z},$$

with $\xi_k(x)$ as in (4.4). The convolution theorem (2.7) then yields

$$\begin{aligned}\|AF(x) - AF(x^\dagger)\|^2 &= \|AF(x) - AF(\tilde{x}^\dagger)\|^2 = \|(x - \tilde{x}^\dagger) * (x + \tilde{x}^\dagger)\|^2 \\ &= \sum_{k\in\mathbb{Z}} |x_k - \xi_k(x)\, x_k^\dagger|^2\, |x_k + \xi_k(x)\, x_k^\dagger|^2.\end{aligned}$$

A simple calculation shows that

$$|x_k + \xi_k(x)\, x_k^\dagger| \geq |x_k^\dagger|$$

for all $k \in \mathbb{Z}$. Denoting by

$$I := \{k \in \mathbb{Z} : x_k^\dagger \neq 0\}$$

the support of $x^\dagger$ with cardinality N we thus obtain

$$\|AF(x) - AF(x^\dagger)\|^2 \geq \sum_{k\in I} |x_k - \xi_k(x)\, x_k^\dagger|^2\, |x_k^\dagger|^2 = \sum_{k\in I} \big|\big(x_k - \xi_k(x)\, x_k^\dagger\big) x_k^\dagger\big|^2$$

and by applying the Cauchy Schwarz inequality and the triangle inequality the estimate

$$\begin{aligned}\|AF(x) - AF(x^\dagger)\|^2 &\geq \frac{1}{N}\left(\sum_{k\in I}\left|\left(x_k - \xi_k(x)\, x_k^\dagger\right)x_k^\dagger\right|\right)^2 \\ &\geq \frac{1}{N}\left|\sum_{k\in I}\overline{\left(x_k - \xi_k(x)\, x_k^\dagger\right)}\xi_k(x)x_k^\dagger\right|^2 \\ &= \frac{1}{N}\,|\langle \tilde{x}^\dagger, x - \tilde{x}^\dagger\rangle|^2.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\mathrm{dist}(x,S))^2 - \|x\|^2 + \|x^\dagger\|^2 &= \|x - \tilde{x}^\dagger\|^2 - \|x\|^2 + \|\tilde{x}^\dagger\|^2 \\ &= -2\,\mathrm{Re}\,\langle \tilde{x}^\dagger, x - \tilde{x}^\dagger\rangle \\ &\leq 2\,|\langle \tilde{x}^\dagger, x - \tilde{x}^\dagger\rangle|,\end{aligned}$$

completing the proof. □

5 Rates for the original deautoconvolution problem

Now we come back to our original autoconvolution operator F from (2.1) and use the variational source condition derived in Proposition 4.1 to obtain a variational source condition

$$E(x, x^\dagger) \leq \|x\|^2 - \|x^\dagger\|^2 + c\,\|F(x) - F(x^\dagger)\| \qquad \text{for all} \quad x \in \mathcal{M} \tag{5.1}$$

for F, where $\mathcal{M}$ will be an appropriate subset of $L^2(0,1)$.

At first we have to decide which error measure is to be used. As shown in [12, §4] the operator equation (1.1) has exactly two solutions: $x^\dagger$ and $-x^\dagger$. Thus, a reasonable error measure seems to be

$$E(x, x^\dagger) := (\mathrm{dist}(x, \{x^\dagger, -x^\dagger\}))^2 = \min\{\|x - x^\dagger\|^2, \|x + x^\dagger\|^2\}. \tag{5.2}$$

Proposition 5.1. *Let $x^\dagger$ have a sparse Fourier representation, that is, only N Fourier coefficients with respect to the basis (2.6) do not vanish. Then there are balls $B_r(x^\dagger)$ and $B_r(-x^\dagger)$ with radius*

$$r := \min_{k\in\mathbb{Z}\colon x_k^\dagger \neq 0} |x_k^\dagger|$$

around $x^\dagger$ and $-x^\dagger$, respectively, such that the variational source condition (5.1) *for the error measure* (5.2) *holds with*

$$\mathcal{M} = B_r(x^\dagger) \cup B_r(-x^\dagger) \qquad \text{and} \qquad c = 2\sqrt{2N}.$$

Proof. By Proposition 4.1 we have

$$\operatorname{dist}(x, S)^2 \le \|x\|^2 - \|x^\dagger\|^2 + 2\sqrt{N}\,\|AF(x) - AF(x^\dagger)\|$$

for all $x \in L^2(0,1)$. The set S has been described in (4.3). Now Lemma 2.1 immediately shows

$$\|AF(x) - AF(x^\dagger)\| \le \sqrt{2}\,\|F(x) - F(x^\dagger)\|.$$

To complete the proof it remains to verify the inequality

$$\operatorname{dist}(x, \{x^\dagger, -x^\dagger\}) \le \operatorname{dist}(x, S) \tag{5.3}$$

for $x \in \mathcal{M}$.

The desired inequality (5.3) is obviously satisfied if $\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \ge 0$ for all $k \in \mathbb{Z}$ and also if $\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \le 0$ for all $k \in \mathbb{Z}$ (cf. (4.5)), where the x_k and the $x_k^\dagger$ again denote the Fourier coefficients of x and $x^\dagger$, respectively. $\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \ge 0$ is a consequence of $\|x - x^\dagger\| \le r$, because this last inequality implies $|x_k - x_k^\dagger| \le |x_k^\dagger|$ for all k with $x_k^\dagger \ne 0$. Thus, $|x_k|^2 - 2\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \le 0$, which yields $\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \ge 0$. Analogously one obtains $\operatorname{Re}(\overline{x_k}\, x_k^\dagger) \le 0$ if $\|x + x^\dagger\| \le r$. Consequently (5.3) holds on the two balls $B_r(x^\dagger)$ and $B_r(-x^\dagger)$. □

The following convergence rate result as a consequence of Proposition 5.1 is valid for an appropriate a priori parameter choice of the regularization parameter α as well as for the a posteriori choice called *sequential discrepancy principle* in [2], for which we also refer to the paper [17, §4.2.1].

Corollary 5.2. *Under the assumptions of Proposition 5.1 the Tikhonov minimizers x_α^δ satisfy the convergence rate*

$$\operatorname{dist}(x_\alpha^\delta, \{x^\dagger, -x^\dagger\}) = \mathcal{O}\big(\sqrt{\delta}\big) \qquad \text{as} \quad \delta \to 0,$$

if the regularization parameter α is chosen a priori as $\alpha = \alpha(\delta) \sim \delta$ or a posteriori as $\alpha = \alpha(\delta, y^\delta)$ according to the sequential discrepancy principle.

Proof. As a result of [8, Theorem 2.3] the Tikhonov minimizers accumulate at $x^\dagger$ and $-x^\dagger$ if $\delta \to 0$. Thus, if δ is small enough, all Tikhonov minimizers lie in the set $\mathcal{M}$ on which the variational source condition holds (see Proposition 5.1). The derivation of convergence rates from a variational source condition with general non-negative error measures E can be found in [10, Chapter 4] and for the sequential discrepancy principle in [17, Theorem 2] and [2, Proposition 9]. $\square$

For a discussion of further details to Proposition 5.1 we recall that the corresponding constant in the variational inequality (5.1) is $c = 2\sqrt{2\,N}$. Then for an a priori choice of the regularization parameter

$$\underline{c}\delta \le \alpha(\delta) \le \overline{c}\delta$$

with fixed constants $0 < \underline{c} \le \overline{c} < \infty$ one obtains (e.g. along the lines of [10, Proof of Theorem 4.11]) the error estimate

$$\operatorname{dist}(x_\alpha^\delta, \{x^\dagger, -x^\dagger\}) \le \left(\frac{2}{\underline{c}} + \frac{\overline{c}}{2}\,c^2\right)^{1/2} \sqrt{\delta} = \left(\frac{2}{\underline{c}} + 4\,\overline{c}\,N\right)^{1/2} \sqrt{\delta} \tag{5.4}$$

This also yields

$$\operatorname{dist}(x_\alpha^\delta, \{x^\dagger, -x^\dagger\}) \le \left(\frac{2}{\underline{c}} + 4\,\overline{c}\right)^{1/2} \sqrt{N}\,\sqrt{\delta}, \tag{5.5}$$

showing that the $\mathcal{O}$-constant in the corollary does not grow faster than $\sqrt{N}$ if the number N of non-zero Fourier coefficients increases. The estimates (5.4) and (5.5), however, are only valid for sufficiently small noise levels $0 < \delta \le \overline{\delta}$, where the upper bound $\overline{\delta}$ depends on $r = \min_{k\in\mathbb{Z}:\, x_k^\dagger \neq 0} |x_k^\dagger|$ such that $x_{\alpha(\delta)}^\delta \in B_r(x^\dagger) \cup B_r(-x^\dagger)$ for all $\delta \in (0, \overline{\delta}]$.

Remark 5.3. From the above discussion it follows that for every element $x^\dagger \in L^2(0,1)$ with a sparse Fourier coefficient sequence $(x_k^\dagger)_{k\in\mathbb{Z}}$ with respect to the basis (2.6) there is a radius $r > 0$ such that for F from (2.1) a variational inequality

$$\|x - x^\dagger\|^2 \le \|x\|^2 - \|x^\dagger\|^2 + c\,\|F(x) - F(x^\dagger)\| \qquad \text{for all} \quad x \in B_r(x^\dagger) \tag{5.6}$$

is valid. Then from [23, Proposition 3.38] we derive the existence of a source element $w \in L^2(0,2)$ such that the source condition (3.4) holds. However, as a consequence of Lemma 3.1 we always have $2\,\|w\| \ge 1$.

Remark 5.4. One can also obtain a variational source condition (and thus rates) if the Fourier representation of $x^\dagger$ is not sparse. In case of the mapping $A \circ F$ considered in Section 4 one obtains

$$\beta\,(\mathrm{dist}(x,S))^2 \le \|x\|^2 - \|x^\dagger\|^2 + \varphi\big(\|AF(x) - AF(x^\dagger)\|\big)$$

for all $x \in L^2(0,1)$ with some constant $\beta \in (0,1)$ and with the concave index function

$$\varphi(t) = \inf_{n\in\mathbb{N}} \left(\frac{1}{1-\beta} \sum_{|k|>n} |x_k^\dagger|^2 + 2\,t\,\sqrt{2n+1} \right), \quad t \ge 0, \tag{5.7}$$

which is essentially determined by the decay of the Fourier coefficients of $x^\dagger$. The proof of this result uses a similar technique as applied in [6, Theorem 5.2].

Along the lines of the present section one can obtain such a variational source condition also for the operator F, but $x^\dagger$ and $-x^\dagger$ are not longer interior points of the set $\mathcal{M}$ on which the variational source condition holds. Thus, one cannot be sure whether the Tikhonov minimizers belong to $\mathcal{M}$, which is an important prerequisite for obtaining convergence rates. But if the Tikhonov minimizers are in $\mathcal{M}$, then the corresponding convergence rate is

$$\mathrm{dist}(x_\alpha^\delta, \{x^\dagger, -x^\dagger\}) = \mathcal{O}\big(\sqrt{\varphi(\delta)}\big) \qquad \text{as} \quad \delta \to 0.$$

Remark 5.5. A reason for the seemingly unmotivated occurrence of a sparsity assumption far away from sparsity promoting penalties in Tikhonov regularization might be the fact that at least for some *quadratic* operators the minimizers of the Tikhonov functional (1.3) are sparse. This is also the case for the operator $A \circ F$ written in the Fourier basis. Then $A \circ F$ can be considered as the mapping $G : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ of diagonal operator type defined by

$$[G(x)]_k := x_k^2, \quad k \in \mathbb{Z},$$

where $\ell^2(\mathbb{Z})$ denotes the space of all square summable *complex-valued* sequences over the index set $\mathbb{Z}$. The minimizers of the corresponding Tikhonov functional

$$T_\alpha^\delta(x) = \|G(x) - y^\delta\|^2 + \alpha\|x\|^2 \tag{5.8}$$

are given by

$$\left|[x_\alpha^\delta]_k\right| = \begin{cases} \sqrt{\left|y_k^\delta\right| - \frac{\alpha}{2}}, & \text{if } \left|y_k^\delta\right| > \frac{\alpha}{2}, \\ 0, & \text{else} \end{cases}$$

and

$$\arg[x_\alpha^\delta]_k = \frac{1}{2}\arg y_k^\delta \quad \text{or} \quad \arg[x_\alpha^\delta]_k = \frac{1}{2}\arg y_k^\delta + \pi \quad \text{if} \quad \left|[x_\alpha^\delta]_k\right| \neq 0.$$

This can be shown by evaluating the necessary conditions on $\left[x_\alpha^\delta\right]_k$ written in polar coordinates for all $k \in \mathbb{Z}$. Obviously only finitely many components of x_α^δ are not zero.

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