TECHNISCHE UNIVERSITÄT CHEMNITZ

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CSC/09-01

Chemnitz Scientific Computing Preprints

Chemnitz Scientific Computing Preprints – ISSN 1864-0087

Impressum:

Chemnitz Scientific Computing Preprints — ISSN 1864-0087

(1995–2005: Preprintreihe des Chemnitzer SFB393)

Herausgeber:
Professuren für
Numerische und Angewandte Mathematik
an der Fakultät für Mathematik
der Technischen Universität Chemnitz

Postanschrift:
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http://www.tu-chemnitz.de/mathematik/csc/

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CSC/09-01 ISSN 1864-0087 **January 2009**

Contents

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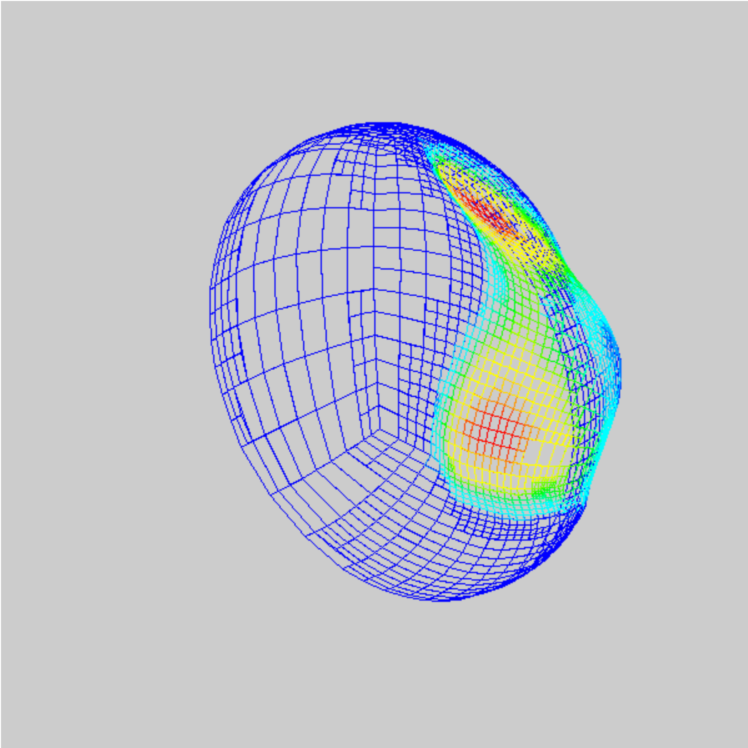

Figure 13: Some views

1 Introduction

1.1 Abstract

In this paper the obstacle description with Radial Basis Functions for contact problems in three dimensional elasticity will be done. A short introduction of the idea of Radial Basis Functions will be followed by the usage of Radial Basis Functions for approximation of isosurfaces. Then this isosurfaces are used for the obstacle-description in three dimensional elasticity contact problems. In the last part some computational examples will be shown.

1.2 Problem description

Let $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ be an elastic body with boundary $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$. The elastic body is clamped at his dirichlet boundary Γ_D and has a surface load at the neumann boundary Γ_N. Volume forces inside the body can also occur. In the displacement direction of the body exists a rigid obstacle. (see figure 1)

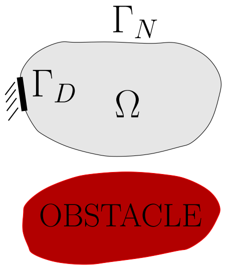

Figure 1: The contact problem

The basic problem is now to find a valid displacement field $u(x)$ for all $x \in \Omega$ such that no part of the body penetrates the obsctale. Details of the mathematical modelling and numerical solution of this problem with finite element method and subspace projection can be found in [6]. One of the important parts of the algorithm consists of the obstacle handling with projection methods which needs a fast penetration test for displaced nodes and evaluations of the outer normal of the obstacle.

The performance of this parts of the algorithm depends on the mathematical description of the obstacle, best performance can be achieved with obstacle definition as isosurface, i.e. as zero set of a function $F : \mathbb{R}^d \to \mathbb{R}$. The disadvantage of this description is, that only simple geometries can be defined in closed analytical form.

The aim of this paper is to introduce a kind of obstacle description as isosurface with the help of Radial Basis Functions which allows the description of complicated surfaces too.

2 Radial Basis Functions

The basic problem for the usage of Radial Basis Functions is to solve a interpolation problem of N arbitrary sampling points $x_1, x_2 \cdots x_N \in \mathbb{R}^d$ and given values $y_1, y_2 \cdots y_N \in \mathbb{R}$ to find a function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ which fulfills the interpolation condition

$$\tilde{f}(x_i) = y_i \qquad i = 1 \cdots N. \tag{1}$$

To find an ansatz for this function $\tilde{f}$ at first some basis functions will be introduced.

A function $\phi_i : \mathbb{R}^d \to \mathbb{R}$ is a radial basis function if there exists a function $\xi : \mathbb{R} \to \mathbb{R}$ satisfying $\phi_i(x) = \xi(||x - x_i||)$ for a fixed point $x_i \in \mathbb{R}^d$.

Commonly used types of RBF ansatz functions $\xi(r)$ with $r = ||x - x_i||$ are

$$\begin{aligned}
\text{Gaussian:} \quad \xi(r) &= \exp(-\beta r^2) \quad \text{for} \quad \beta > 0 \\
\text{Multiquadric:} \quad \xi(r) &= \sqrt{r^2 + 1} \\
\text{Polyharmonic spline:} \quad \xi(r) &= \begin{cases} r^k & \text{if k is odd} \\ r^k \log(r) & \text{if k is even} \end{cases} \\
\text{Thin plate spline:} \quad \xi(r) &= r^2 \log(r)
\end{aligned}$$

Other possibilities, especially for a large number of samplepoints are Radial Basis Functions with compact support, for example the so called "Wendland functions" [1]. With such basis functions the resulting interpolation system becomes sparse. For simple contact geometries the number of samplepoints is moderate and so this kind of basis is not used in this work.

Now let in every sampling point $x_i, i = 1 \cdots N$ such a radial basis function ϕ_i be defined. The ususal ansatz for an interpolation function is

$$\tilde{f}(x) = \sum_{i=1}^{N} \beta_i \phi_i(x) \tag{2}$$

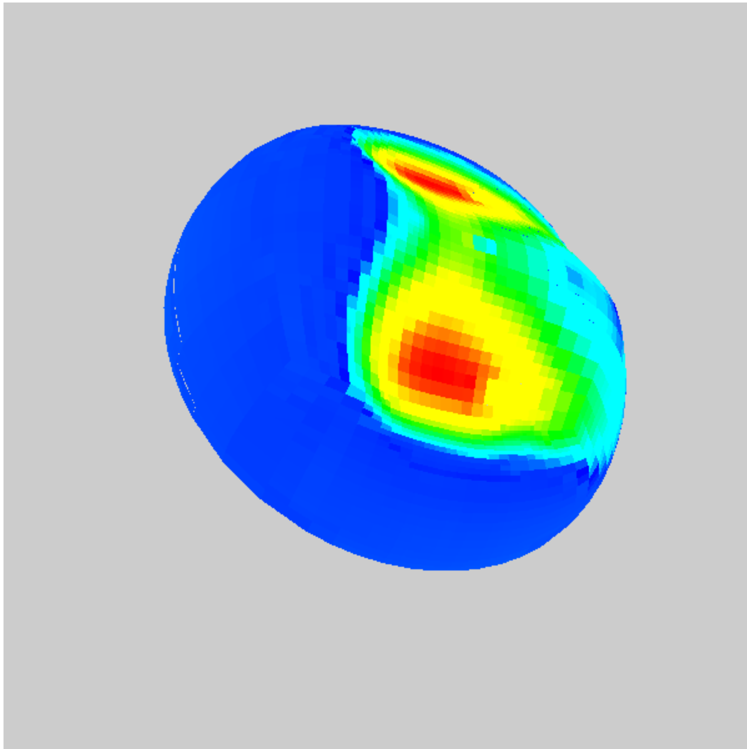

Figure 12: Some views

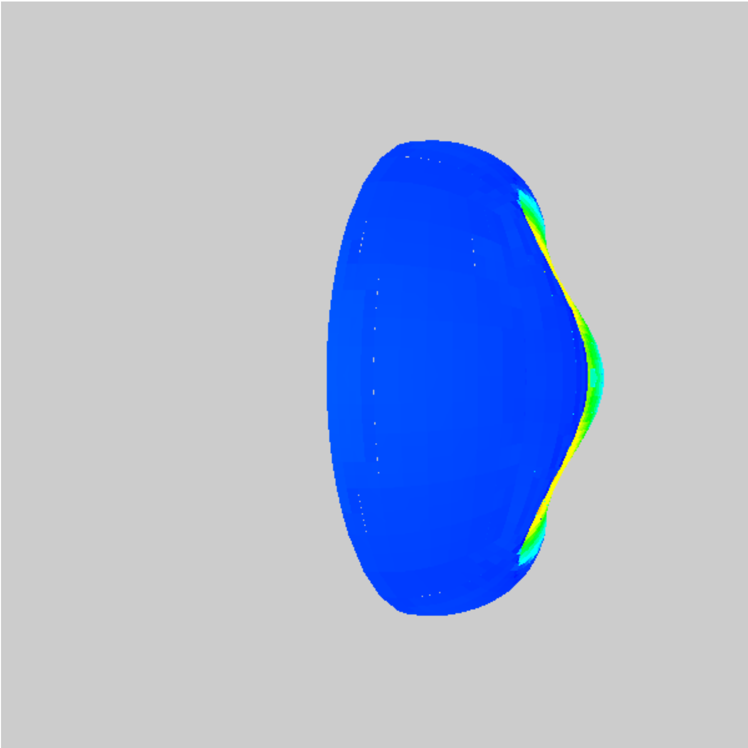

Figure 11: Some views

Together with the interpolation-condition (1) this leads to linear equations

$$\tilde{f}(x_i) := \sum_{j=1}^{N} \beta_j \phi_j(x_i) = y_i \quad i = 1 \cdots N$$

which can be written as a linear system

$$\Phi \beta^{RBF} = y$$

with the system matrix $\Phi := [\phi_j(x_i)]_{i,j=1}^{N}$ the vector $\beta^{RBF} = [\beta_1, \beta_2 \cdots \beta_N]^T$ of unknown coefficients and the right hand side $y = [y_1, y_2 \cdots y_N]^T$. Obviously the system matrix Φ is symmetric, one can show ([2]) that for a big variety of radial basis functions it is positive definite too.

The disadvantage of the radial basis function ansatz alone is that a high number of sample points is needed to get a sufficient exact approximation of a constant or a linear function. One possibility to cope with this problem is to add a polynomial part.

$$\tilde{f}(x) = \sum_{i=1}^{N} \beta_i^{RBF} \phi_i + p(x) \tag{3}$$

where the d-variate polynomial $p \in \pi_m(\mathbb{R}^d)$ of degree at most m is defined as

$$p(x) = \sum_{j=1}^{M} \beta_j^{poly} p_j(x),$$

with $M = dim(\pi_m(\mathbb{R}^d))$ and basis polynomials $p_j \quad j = 1 \cdots M$. Consequently one receive $(N + M)$ unknown coefficients, the interpolation system (1) consists of N equations, therefore a side condition

$$\sum_{j=1}^{N} \beta_j^{RBF} p(x_j) = 0 \quad \text{for all} \quad p \in \pi_m(\mathbb{R}^d) \tag{4}$$

is imposed. This leads to a linear system of dimension $(N + M)$

$$\begin{bmatrix} \Phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \beta^{RBF} \\ \beta^{POLY} \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} \tag{5}$$

A simple but good choice for $\pi_m(\mathbb{R}^d)$ are linear polynomials, i.e.

$$\begin{aligned} p_1(x) &= x_1 \\ p_2(x) &= x_2 \\ &\cdots \\ p_d(x) &= x_d \\ p_M(x) &= 1 \end{aligned}$$

for $x \in \mathbb{R}^d$, $M = d+1$ and x_i as the i-th component of x. This leads to a matrix P

$$P = \begin{bmatrix} x_1^T & 1 \\ x_2^T & 1 \\ & \cdots \\ x_N^T & 1 \end{bmatrix} .$$

The solution $\beta = \begin{bmatrix} \beta^{RBF} \\ \beta^{POLY} \end{bmatrix}$ of (5) holds the coefficients for the whole interpolation function

$$\tilde{f}(x) = \sum_{i=1}^{N} \beta_i \phi_i(x) + \sum_{i=1}^{M} \beta_{(N+i)} p_i(x) \quad . \tag{6}$$

3 Surface Approximation

In computational geometry one of the most useful descriptions of surfaces is the description by an implicit function. With the help of a function F with $F : \mathbb{R}^3 \rightarrow \mathbb{R}$ a surface is defined by the zero level of this function, i.e. we set as surface Σ

$$\Sigma := \{x \in \mathbb{R}^d : F(x) = 0\} \quad . \tag{7}$$

For example with

$$F : \mathbb{R}^3 \rightarrow \mathbb{R} \qquad F(x) := x_1^2 + x_2^2 + x_3^2 - R^2$$

the zero level of this F is the sphere with radius R and center $x = 0$.

Unfortuneality it is not possible to describe complicated surfaces like tool contours in deforming tools with simple analytical functional terms.

So the idea of the approximation of such surfaces comes up. The detailed description and examples of surfaces from scanned point clouds and their reconstruction can be found for example in [4] and [3].

For the explanation of the reconstruction idea see the 2 dimensional analogon of a curve reconstruction in the $\mathbb{R}^2$ in figure 2.

We choose N samplepoints $x_1 \cdots x_N$ and fix sampelvalues $y_1 \cdots y_N$ with

$x_1, x_4 \cdots x_{N-2}$	on the surface
$x_2, x_5 \cdots x_{N-1}$	inside the surface
$x_3, x_6 \cdots x_N$	outside the surface

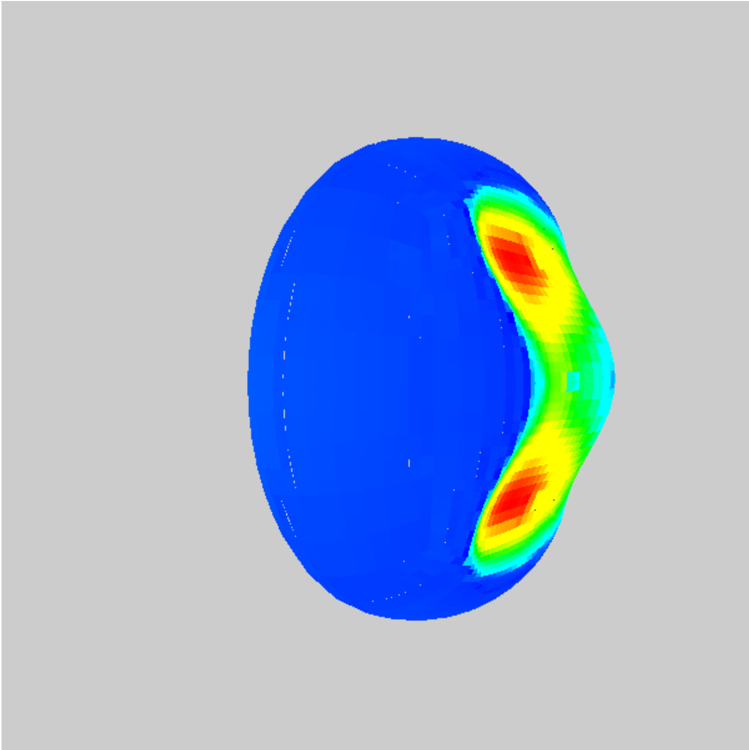

Figure 10: Some views

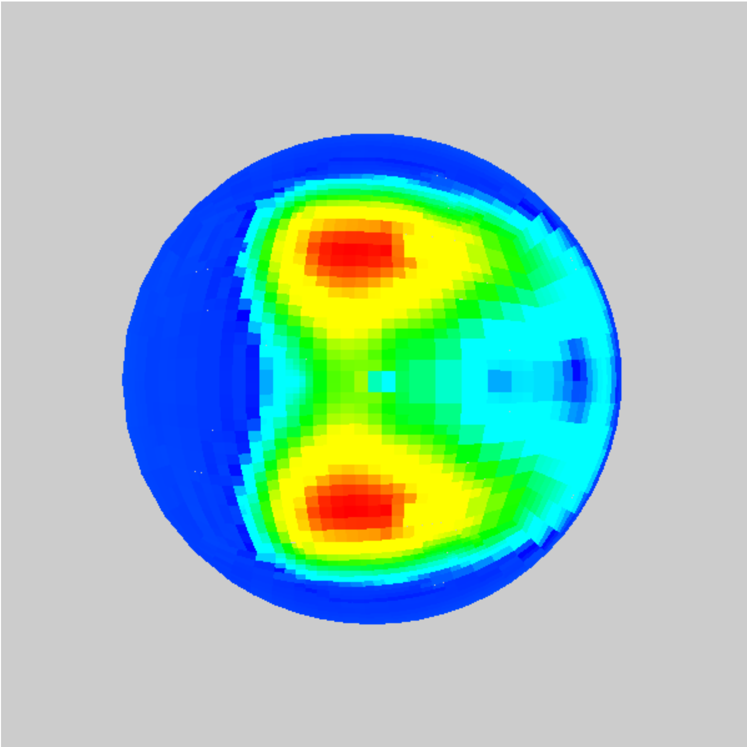

Figure 9: Some views

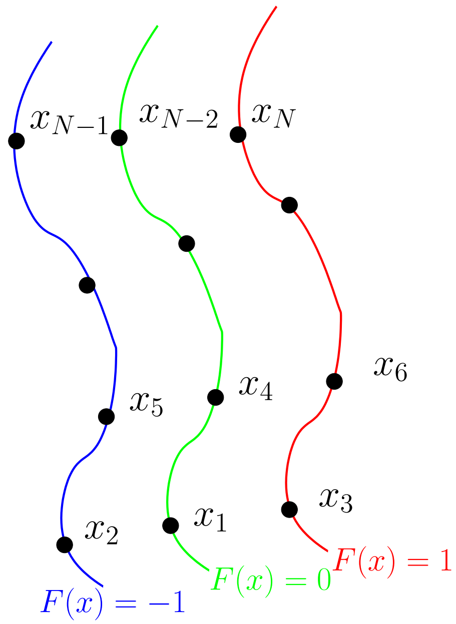

Figure 2: 2-d curve approximation

and set the corresponding values

$$\begin{array}{lll} y_i = 0 & \text{for} & i = 1, 4, \cdots N-2 \\ y_i = -1 & \text{for} & i = 2, 5, \cdots N-1 \\ y_i = 1 & \text{for} & i = 3, 6, \cdots N \end{array}$$

Then the radial basis interpolation system 5 is set up, solved and the surface defining function F is choosen as the RBF-interpolation from (6) as

$$F(x) = \sum_{i=1}^{N} \beta_i \phi_i(x) + \sum_{i=1}^{M} \beta_{(N+i)} p_i(x). \tag{8}$$

Depending on the smoothness of the surface it is also possible to use most of the sampling points on the zero level set on only a small number of samples with values not equal zero, but at least one sample must hold a value not equal zero to avoid the trivial zero solution $\beta = 0$ which leads to $F(x) = 0 \quad \forall x$.

4 Obstacle Description

A detailed description of the obstacle handling can be found in [5] and [6]. Especially a obstacle description by implicit functions F with $F : \mathbb{R}^3 \to \mathbb{R}$ with

$$F(x) \begin{cases} = 0 & x \text{ on the obstacle boundary} \\ < 0 & x \text{ inside the obstacle} \\ > 0 & x \text{ outside the obstacle.} \end{cases}$$

is done and implemented in the 2 and 3-dimensional adaptive finite element programs for elasticity problems.

To add a new kind of obstacle description with Radial Basis Functions the existing code was extented by the evaluation of the RBF-interpolation with precomputed coefficients β. So the penetration test can be done in the same manner by evaluation of $F(x+u)$. In case of penetration the outer normal of the surface is needed to define the subspace projector, it holds

$$\vec{n} = \nabla F$$

Therefore a gradient evaluation of (8) is needed. Depending on the choice of the Radial Basis Functions ϕ_i this can be done by analytical differentiation of the basis functions or by a numerical gradient approximation with the help of the difference quotient

$$\frac{\partial F}{\partial x_i} \approx \frac{F(x+\varepsilon e_i) - F(x-\varepsilon e_i)}{2\varepsilon} \tag{9}$$

with the unit vector $e_i = [0, 0, \cdots 0, 1, 0, \cdots 0]^T \in R^d$ and a small $\varepsilon > 0$.

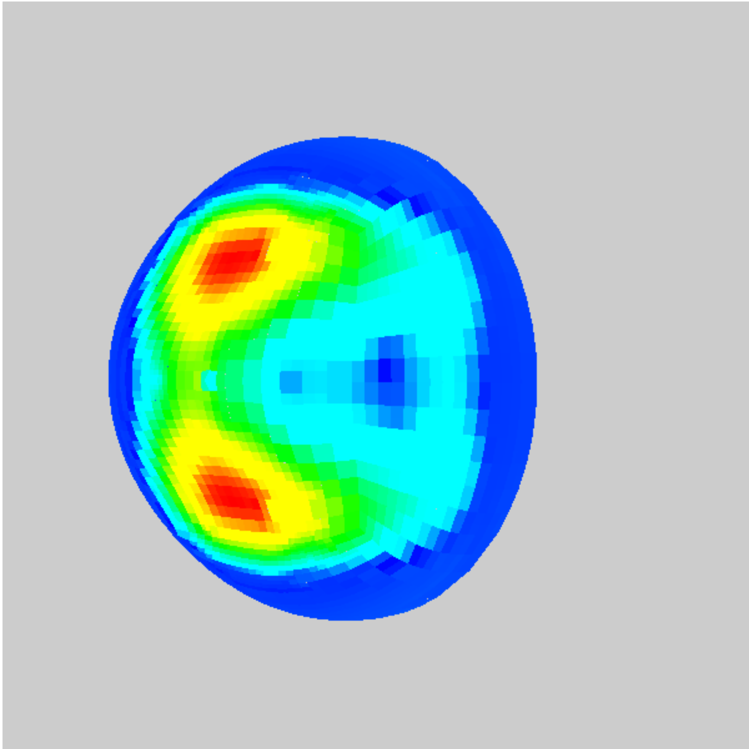

Figure 8: Some views

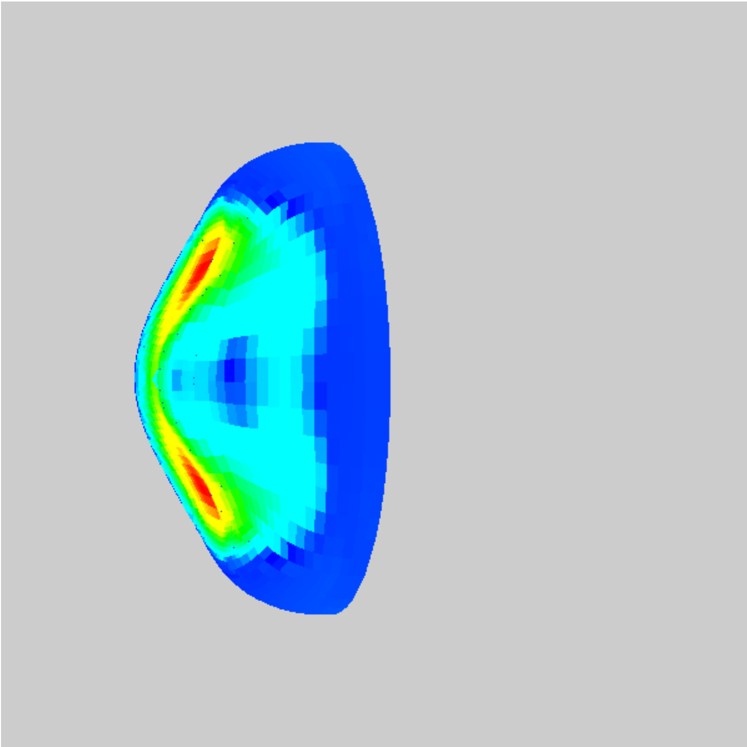

Figure 7: Some views

5 Computational Example

5.1 Obstacle Approximation

In the last section an numerical example will be shown. The surface points are sampled from the isosurface

$$\Sigma := \{x \in \mathbb{R}^3 : F(x) = 0\}$$

with

$$F(x) := x_3 + 0.2((0.7x_1)^3 + 2(cos(\frac{\pi}{2}x_2) - 1) + 3.7 \quad . \tag{10}$$

The obstacle is sampled over a regular x_1, x_2 grid with 9 steps in x_1 and 19 steps in x_2 direction. Every gridpoint creates 3 samplepoints, one for the zero niveau (the red marked points in figure 3) one for a positive niveau above and one negative niveau under the surface. This leads to 513 samplepoints, together with a linear polynomial approach the dimension of the interpolation system (5) becomes 517. After solving this system the resulting coefficients are used in the radial basis approximation 6 as approximation to 10.

In figure 4 the deformable elastic body is shown in his start position, it will be moved in negative z direction by a dirichlet boundary condition on the top of the half ball. Figure 5 shows the deformed body in contact with the obstacle. The color of the boundary surfaces of the body is choosen by the contact pressure value, zones without contact have a zero contact pressure.

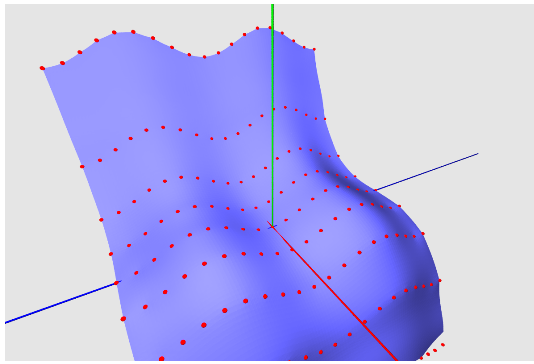

Figure 3: The Obstacle

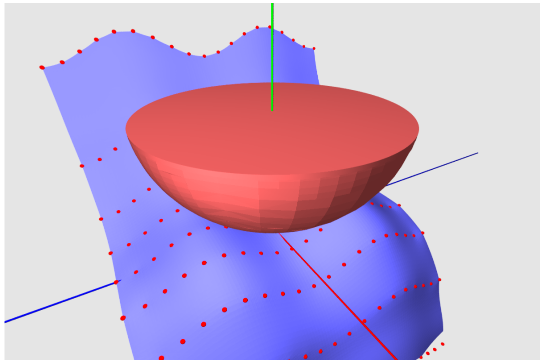

Figure 4: The elastic body in start position

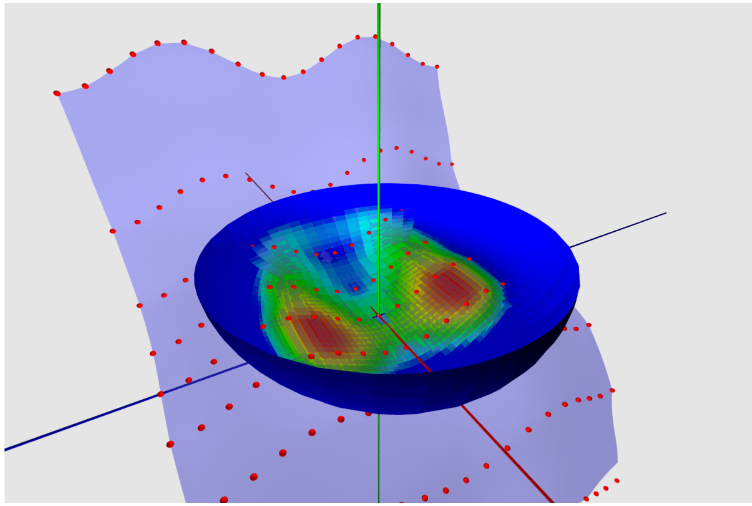

Figure 5: The deformed body with colored contact pressure

5.2 Some 3-d views of the deformed body

In this section some views of the boundary surfaces of the deformed body in different viewing angels are shown. The color of the body matchs with the contact pressure. Blue color stands for zero pressure (not in contact) and red parts are zones with maximal contact pressure.

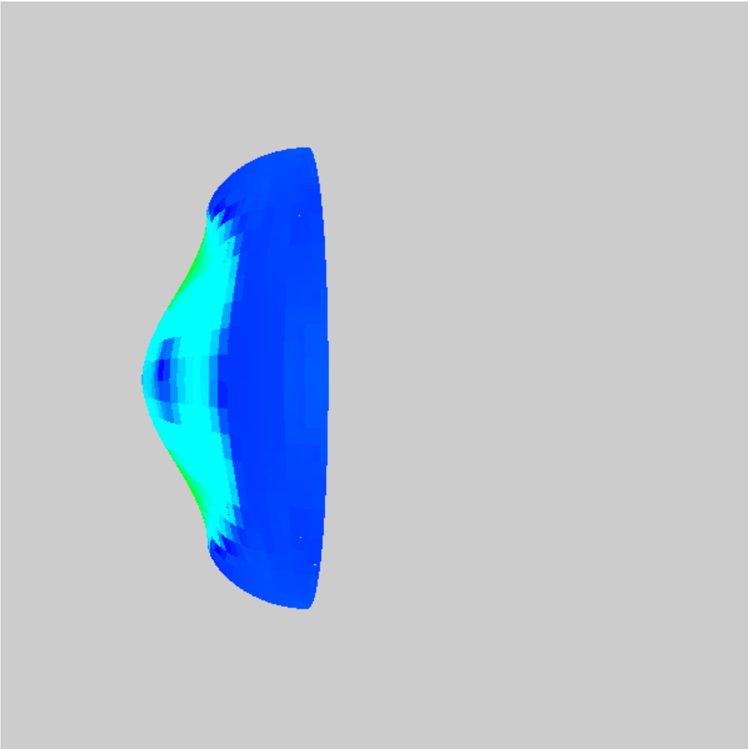

Figure 6: Some views