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A convergence theorem for Dirichlet forms with applications to boundary value problems with varying domains

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0 Introduction

We study continuity of boundary problems with varying domains. To explain this in more detail, let us consider our standard example: Denote by H_{G_n} the Dirichlet Laplacian on the open set $G_n \subset \mathbb{R}^d$. The basic question which we adress is, whether we have convergence

$$H_{G_n} \longrightarrow H_G,$$

if the sets G_n converge to G in an appropriate sense. Two notions of convergence for the operators appear suitable: Generalized convergence in the strong resolvent sense (srs) and in the norm resolvent sense (nrs) (the "generalized" refers to the fact that the H_{G_n} act in different Hilbert spaces; we will frequently omit it). We shall introduce these concepts in some detail below but first we briefly describe the content of the following sections.

In Section 1 we are concerned with convergence in srs. The main result, Theorem 1.3, says that $H_{G_n} \xrightarrow{srs} H_G$ if $\overline{\lim}\, G_n$, $\underline{\lim}\, G_n$ and G are equivalent in an appropriate sense. This is, in fact, valid for measurable, not necessarily open sets, and for generators H of regular Dirichlet forms. The Dirichlet form setting turns out to allow a very convenient definition of H_G. Roughly speaking, the Dirichlet boundary condition can be described as an infinite potential. It is this viewpoint which allows the simple formulation of Theorem 1.3 . The main tool for its proof, monotone convergence of forms has already been used in this context by Rauch and Taylor [16], Simon [18] and Weidmann [24]. In Corollary 1.4 we show how their results can be derived from Theorem 1.3 .

Section 2 deals with nrs convergence which yields much stronger spectral theoretic consequences. Here, the abstract monotone convergence results are not sufficient, and a new aspect enters the picture: localization of the perturbation. To understand what this means, take the unperturbed Dirichlet form $\mathfrak{h}$ and consider a monotone sequence of positive potentials V_n. If the difference $V_n - V_m$ is zero outside some fixed compact set for all n, m, then $\mathfrak{h} + V_n$ converges in nrs. This follows from the main result of the present article, Theorem 2.1, which states more: The potentials can be replaced by certain signed measures, the set on which the differences live only needs to be of finite capacity, and we even have Hilbert–Schmidt convergence of the semigroup differences. Apart from a factorization argument from operator ideals, the proof of Theorem 2.1 relies on an inequality from probabilistic potential theory. Since this is the only instance where (via the Feynman–Kac formula) probability theory enters, we have chosen to present its proof in an appendix. The application of our perturbation theorem to the motivating question is given in Theorem 2.2. It says that $H_{G_n} \xrightarrow{nrs} H_G$ if the G_n satisfy the condition of Theorem 1.3 and, moreover, the symmetric differences $G_n \triangle G$ are contained in a fixed set of finite capacity.

We now introduce srs and nrs convergence, following Simon's paper [18]; see also [24, 25]. To this end fix a Hilbert space $\mathfrak{H}$, which corresponds to $L_2(X)$ in our application. To each closed form $\mathfrak{t} \geq \gamma$ with domain $D(\mathfrak{t})$ (not necessarily dense in $\mathfrak{H}$), we can associate a self adjoint operator T in the Hilbert space $\overline{D(\mathfrak{t})}^{\mathfrak{H}}$ according to [15], Chap. VI, Thm. 2.1. If $\phi : \mathbb{R} \to \mathbb{R}$ is bounded and measurable we define

$$\phi(\mathfrak{t}) := \phi(T) \oplus 0 \text{ on } \overline{D(\mathfrak{t})} \oplus D(\mathfrak{t})^{\perp}.$$

In particular, $e^{\mathfrak{t}}$ and $(\mathfrak{t} + E)^{-1}, E$ suitable, are defined. We say that $\mathfrak{t}_n$ converges to $\mathfrak{t}$ in srs, if $(\mathfrak{t} + E)^{-1} = s - \lim(\mathfrak{t}_n + E)^{-1}$ for a suitable E; this is denoted by $\mathfrak{t}_n \xrightarrow{srs} \mathfrak{t}$ or $T_n \xrightarrow{srs} T$, respectively. Similarly,

$$\mathfrak{t}_n \xrightarrow{nrs} \mathfrak{t} \iff \|(\mathfrak{t}_n + E)^{-1} - (\mathfrak{t} + E)^{-1}\| \to 0.$$

As in the densely defined case, one can easily prove the following

Proposition A *Let* $\mathfrak{t}, \mathfrak{t}_n (n \in \mathbb{N})$ *be closed forms, with spectral resolution* $E(t) = 1_{(-\infty,t]}(\mathfrak{t}), E_n(t) := 1_{(-\infty,t]}(\mathfrak{t}_n)$, *respectively. Then*
(a) $\mathfrak{t}_n \xrightarrow{srs} \mathfrak{t} \Longrightarrow E(\lambda) = s - \lim_n E_n(\lambda)\ (\lambda \in \rho(T))$.
(b) $\mathfrak{t}_n \xrightarrow{nrs} \mathfrak{t} \Longrightarrow E(\lambda) = \|\cdot\| - \lim_n E_n(\lambda)\ (\lambda \in \rho(T))$.

1 Strong convergence for perturbed domains

In this section we extend and unify results of Rauch and Taylor [16], Weidmann [24] and Simon [18] concerning strong continuity under perturbation of domains. The framework of Dirichlet forms appears to be well suited for that study. Thus. let X denote a locally compact, second countable space, which is endowed with a Radon measure m with full support and let us assume that

(I) $\mathfrak{h}$ is a closed , regular Dirichlet form in $L_2(X, m)$.

This means that $\mathfrak{h} : D \times D \to \mathbb{R}$ is a nonnegative symmetric bilinear form which induces an inner product $(\cdot|\cdot)_{\mathfrak{h}}$ on the linear space D by $(u|v)_{\mathfrak{h}} := \mathfrak{h}[u,v] + (u|v)$ with the following properties: $(D, (\cdot|\cdot)_{\mathfrak{h}})$ is complete (*"closed"*), $D \cap C_c(X)$ is dense both in $(D, (\cdot|\cdot)_{\mathfrak{h}})$ and in $(C_o(X), \|\cdot\|_\infty)$ (*"regular"*), and $u \in D$ implies that $u^+ \wedge 1 \in D, \|u^+ \wedge 1\|_{\mathfrak{h}} \leq \|u\|_{\mathfrak{h}}$ (*"Dirichlet form"*); for the details we refer to [13].

1.1 Example. Let $X \subset \mathbb{R}^d$ be open, m the Lebesgue measure, and $X \ni x \mapsto a(x)$ a locally integrable function with values in the symmetric matrices such that $\lambda \leq a(x) = (a_{ij}(x)) \leq \mu$ for some $\lambda, \mu > 0$. Then $D = W_0^{1,2}(X)$,

$$\mathfrak{h}[u,v] = \sum_{i,j} \int a_{ij}(x) \partial_i u(x) \partial_j v(x) dx$$

defines a regular Dirichlet form (see [5], Section 1.2). For $a(x) = 1$ this form is associated with the Dirichlet Laplacian on X.

The *capacity* induced by $\mathfrak{h}$ will play a key role in the subsequent discussion. It is defined as follows: for $A \subset X$,

$$\mathrm{cap}(A) = \inf\{\|f\|_{\mathfrak{h}}^2; A \subset U, U \text{ open}, f \geq 1_u\}.$$

Obviously, $\mathrm{cap}(A) \geq m(A)$ for all $A \subset X$. Since $\mathfrak{h}$ is regular, $\mathrm{cap}(A) < \infty$ for all relatively compact A.

Assume, for the moment, that $G \subset X$ is open. Then, in analogy with the Dirichlet Laplacian $-\Delta_G$ whose form domain is $W_0^{1,2}(G) = \overline{W^{1,2} \cap C_c(G)}$,

$$\mathfrak{h}_G := \overline{\mathfrak{h}|D \cap C_c(G)}$$

can be considered as the form, whose operator H_G is obtained from H by imposing Dirichlet boundary conditions at $B := X \setminus G$. If $m(B) > 0$, one has to view H_G as an operator in $L_2(G) = \overline{D(\mathfrak{h}_G)}$. To obtain a representation of $\mathfrak{h}_G$ which is better suited for our purpose and, at the same time, allows sets G which are merely measurable, we have to recall that every $u \in D$ admits a *quasi-continuous* version $\tilde{u}$ (for every $\varepsilon > 0$ there exists an open set U with $\mathrm{cap}(U) < \varepsilon$ outside of which $\tilde{u}$ is continuous), see [13], Thm. 3.1.3, p. 65. Such a $\tilde{u}$ is unique q.e., i.e. up to sets of capacity zero. It is not hard to check that

$$D_0(G) = \{u \in D; \tilde{u} = 0 \text{ q.e. on } X \setminus G\}$$

is a closed subspace of D for arbitrary $G \subset X$ and that

$$D_0(G) = \overline{D \cap C_c(G)}^{\mathfrak{h}} = D(\mathfrak{h}_G)$$

if G is open. Hence it is consistent to write $\mathfrak{h}_G$ and H_G for the form $\mathfrak{h}$ restricted to $D_0(G)$ and the associated self adjoint operator. (The analogous spaces $W_0^{1,2}(E)$ have been introduced and studied in [10, 11].) It is sometimes illustrative to think of H_G as given by $H + V$, where V is a potential which is infinite on $X \setminus G$.

Having in mind the Laplacian and thinking in terms of quantum mechanical models one might picture that such an infinite barrier forces wave functions to vanish outside G. Our next aim is to put this intuition into precise mathematical terms, introducing "measure perturbations". The right class of measures is

$$M_0 = \{\mu : \mathfrak{B} \to [0,\infty]; \mu\, \sigma\text{–additive}, B \in \mathfrak{B}, \mathrm{cap}(B) = 0 \Longrightarrow \mu(B) = 0\},$$

which has been studied in [4, 23] for the $-\Delta$ case, and in [19] in the generality considered here. In particular, it is known that

$$D(\mathfrak{h} + \mu) := \{u \in D; \tilde{u} \in L_2(\mu)\},\ (\mathfrak{h} + \mu)[u, v] := \mathfrak{h}[u, v] + \int \tilde{u}\tilde{v}\, d\mu$$

defines a closed form. Note that $Vdm \in M_0$ for every measurable $V \geq 0$ and that $\mathfrak{h} + Vdm$ is just the sum of the forms of H and the multiplication operator V. The corresponding operators are well–studied, at least for locally integrable V and $H = -\Delta$, and often this self adjoint operator, the so–called form sum, is meant implicitly, if one writes $-\Delta + V$.

For $B \subset \mathfrak{B}$,

$$\infty_B(M) := \infty \cdot \mathrm{cap}(B \cap M)$$

(with the convention $\infty \cdot 0 = 0$) defines a measure $\infty_B \in M_0$ which takes only the values 0 or ∞. It is clear that

$$\mathfrak{h}_G = \mathfrak{h} + \infty_{X \setminus G} \geq \mathfrak{h} + \infty \cdot 1_{X \setminus G}$$

for every $G \in \mathfrak{B}$. Under rather weak regularity conditions this last inequality is in fact an equality. This is the case for closed G and for G which satisfy the segment property; see the example below. Let us first introduce some notation: For $\mu, \nu \in M_0$ we write $\mu \prec \nu$ or $\mu \sim \nu$ if $\mathfrak{h} + \mu \leq \mathfrak{h} + \nu$ or $\mathfrak{h} + \mu = \mathfrak{h} + \nu$ in the sense of forms. We want to caution the reader that $\mu \prec \nu$ does not imply that $\mu \leq \nu$ as set functions. As a further abbreviation we sometimes use $A \sim B$ to indicate the equivalence $\infty_A \sim \infty_B$. From $\mathrm{cap}(A \triangle B) = 0$ it obviously follows that $A \sim B$. The converse is not true, as we see in

1.2 Example. Consider the classical Dirichlet form on $\mathbb{R}^d$, $A := \{x_1 > 0\}$, $B := \{x_1 \geq 0\}$. Then $A \sim B$ while $\mathrm{cap}(B \setminus A) \neq 0$. Moreover, $\infty_A \sim \infty_B \sim \infty 1_A$. This easily follows by translating elements which vanish on A q.e. along the x_1-direction and using the fact that $D_0(X \setminus B)$ is closed. This argument can be "localized" to prove the following simple fact:
Let $G \subset \mathbb{R}^d$ be open and let G satisfy the segment property at $R \subset \partial G$ by which we mean (cf. [1], p. 54) that for all $x \in R$ there exists a neighborhood U_x and a nonzero $y_x \in \mathbb{R}^d$ such that $U_x \cap \overline{G} + ty_x \subset G$ for every $t \in (0, 1)$. Then $X \setminus (G \cup R) \sim X \setminus G$.

Using the notions introduced above and the notation

$$\overline{\lim} B_n := \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} B_k\, , \underline{\lim} B_n := \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} B_k$$

for a sequence (B_n) in $\mathfrak{B}$, the main result of the present section reads as follows:

1.3 Theorem. *Assume* (I). *Let* $B_n \in \mathfrak{B}$ *for* $n \in \mathbb{N}$ *and assume that* $\overline{\lim} B_n \sim \underline{\lim} B_n$. *Then*

$$\mathfrak{h} + \infty_{B_n} \xrightarrow{srs} \mathfrak{h} + \infty_B$$

for every $B \sim \overline{\lim} B_n$.

Proof. Define $A_n := \bigcap_{k \geq n} B_k$, $C_n := \bigcup_{k \geq n} B_k$. Then $A_n \nearrow \underline{\lim} B_n =: A$, $C_n \searrow \overline{\lim} B_n =: C$. By monotone form convergence theorems (cf. [18], Thm. 3.2 and Thm. 4.1, [24], Satz 3.1), it follows that $\mathfrak{h} + \infty_{A_n} \xrightarrow{srs} \mathfrak{h} + \infty_A$ and $\mathfrak{h} + \infty_{C_n} \xrightarrow{srs} \mathfrak{h} + \infty_C$. As $\mathfrak{h} + \infty_{A_n} \leq \mathfrak{h} + \infty_{B_n} \leq \mathfrak{h} + \infty_{C_n}$ and $\mathfrak{h} + \infty_A = \mathfrak{h} + \infty_C$ the asserted convergence follows. □

Together with Proposition A, this Theorem implies strong convergence of the spectral projections of $\mathfrak{h} + \infty_{B_n}$ and $\mathfrak{h} + \infty_B$. From the discussion in [16] and [24] it follows that the eigenprojections $E_n(\lambda)$ converge even in norm for λ below the essential spectrum of $\mathfrak{h} + \infty_B$.

Coming back to "real Dirichlet boundary conditions" let us consider open sets G_n for $n \in \mathbb{N}$, and set $B_n := X \setminus G_n$. The above theorem tells us that the sequence H_{G_n} converges in strong resolvent sense if $\overline{\lim} B_n \sim \underline{\lim} B_n$. Its limit is of the form H_G with an open set G if and only if there is a closed set B which satisfies $B \sim \overline{\lim} B_n$. Rauch and Taylor [16] and Weidmann [24] give sufficient conditions for this to happen in the setting of Example 1.1. We now show how their results can be derived from Theorem 1.3. Here we use the notation $K \subset\subset G$, if K is a compact set in X which is contained in G.

1.4 Corollary. *Let* $\mathfrak{h}$ *be as in Example 1.1, and let* G, G_n *be open sets.*

(a) (Cf. [16], Lemma 1.1) *Assume*
  (i) $\forall K \subset\subset G \; \exists n_o \in \mathbb{N} : K \subset G_n \; (n > n_o)$,
  (ii) $\forall K \subset\subset X \setminus \overline{G} \; \exists n_o \in \mathbb{N} : K \subset X \setminus \overline{G_n} \; (n > n_o)$,
  (iii) $\infty_{X \setminus G} \sim \infty 1_{X \setminus \overline{G}}$.
  Then $\overline{\lim} X \setminus G_n \sim \underline{\lim} X \setminus G_n \sim X \setminus G$.

(b) (Cf. [24], Satz 4.8) *Assume*
  (i) $\forall K \subset\subset G : \mathrm{cap}(K \setminus G_n) \to 0$ *for* $n \to \infty$,
  (ii) $\forall K \subset\subset \mathbb{R}^d : \lambda(K \cap (G_n \setminus G)) \to 0$ *for* $n \to \infty$.
  (iii) *G admits a locally finite covering* (U_n) *with the following properties:* $U_1 \cap G_n \subset G$ $(n \in \mathbb{N})$, $\forall j > 1 \; \exists a_j \in \mathbb{R}^d$ *such that* $(U_j \cap \overline{G}) + ta_j \subset G$ *for* $t \in (0, 1)$.
  Then $H_{G_n} \xrightarrow{srs} H_G$.

Proof. Let $B := X \setminus G$, $B_n := X \setminus G_n$.
(a): By (i) we have $\underline{\lim} G_n \subset G$, so that

$$\overline{\lim} B_n \prec \infty_B$$

(iii) implies that $\infty_B \sim \infty_{B^o} \sim \infty 1_{B^o}$. Hence it follows from (ii) that

$$\infty_{B^o} \prec \underline{\lim}\, B_n^o \prec \underline{\lim}\, B_n .$$

(b): We use the "subsequence of a subsequence argument" already employed in [24]. Thus we have to check that every subsequence of (B_n) has a subsequence to which we can apply Theorem 1.3. To avoid triple indices, we denote the given subsequence by (B_n). Using (i), (ii) and a diagonal argument, we find a subsequence such that

$$\mathrm{cap}(K \setminus \underline{\lim}\, G_{n_k}) = 0 \; (K \subset\subset G),$$

$$\lambda(K \cap (\overline{\lim}\, G_{n_k} \setminus G)) = 0,$$

which can be restated in the following way (using the inner regularity of cap and the Lebesgue measure, λ):

$$\mathrm{cap}(\overline{\lim}\, B_{n_k} \setminus B) = 0, \; \lambda(B \setminus \underline{\lim}\, B_{n_k}) = 0.$$

This implies

$$\infty_{\overline{\lim}\, B_{n_k}} \prec \infty_B, \; \infty 1_B \prec \infty_{\underline{\lim}\, B_{n_k}} .$$

Denote $A := \underline{\lim}\, B_{n_k}$, $C := \overline{\lim}\, B_{n_k}$ and $A^i := A \cap U_i$, $B^i := B \cap U_i$, $C^i := C \cap U_i$ for $i \in \mathbb{N}$, where U_i is as in (iii). Then $\infty_{C^i} \prec \infty_{B^i}$, $\infty 1_{B^i} \prec \infty 1_{A^i}$ for $i \in \mathbb{N}$. Since $G_n \cap G_1 \subset G$ $(n \in \mathbb{N})$, we have that $B \cap U_1 \subset B_n \cap U_1$ for all $n \in \mathbb{N}$ which yields $\infty_{B^1} \prec \infty_{A^1}$. Therefore,

$$C^1 \sim B^1 \sim A^1 .$$

It remains to check $\infty_{B^i} \prec \infty 1_B$ for $i > 1$, because then, for suitable open U_o,

$$\begin{aligned} \infty_C &\sim \sum_{i=0}^{\infty} \infty_{C^i} \prec \infty_B \\ &\sim \infty_{B^0} + \infty_{B^1} + \sum_{i>1} \infty_{B^i} \\ &\prec \infty_{A^0} + \infty_{A^1} + \sum_{i>1} \infty_{1_B} \\ &\prec \infty_A , \end{aligned}$$

which gives the asserted equality. The inequality $\infty_{B^i} \prec \infty 1_B$ follows with the segment property (as in Example 1.2), since $\partial B^i \subset \partial B \cap U_i$. □

We note that our Theorem 1.3 clearly applies to the example from [24], which does not satisfy the conditions of [16, 24]. More generally it gives convergence for all monotone sequences generalizing an observation by Simon; cf [18], Example 1, p. 383. This illustrates one big advantage of Theorem 1.3: it does not require a priori regularity assumptions on the limit set.

In [24], the classical Dirichlet form is replaced by a form of the type $\mathfrak{h} + q$, where $q^+ \in L^1_{loc}(A^c)$ for some closed set A of Lebesgue measure zero and q^- satisfies a condition which implies form boundedness with bound zero with respect to $-\Delta$. It is easy to see that one may include much more singular perturbations. For instance, Theorem 1.3 remains valid if one replaces $\mathfrak{h}$ (in the general Dirichlet form setting) by $\mathfrak{h} + \mu$, where $\mu^+ \in M_o$ and μ^- is form bounded with respect to $\mathfrak{h}$ with bound less than 1.

2 Convergence of measure perturbations

In this section we deal with generalized norm resolvent convergence for perturbed Dirichlet forms. One major application is again to boundary value problems with varying domains. While Theorem 1.3 rested on an abstract convergence theorem for monotone sequences of forms (and the Dirichlet form setting was necessary for the definition of Dirichlet boundary conditions) the main tool in this section is a convergence theorem for perturbations of Dirichlet forms by measures, Theorem 2.1. The crucial assumption is that the perturbation is "localized in space" in the sense that it takes place on a set of finite capacity. The following assumption will be very important in the sequel

(II)	$\mathfrak{h}$ is a closed regular Dirichlet form such that $e^{-t\mathfrak{h}}$ induces a bounded linear operator from $L_1(X)$ to $L_\infty(X)$.

Let us say that a measure $\mu \in M_o$ is *supported* on a set $Y \subset X$ if $\mu \prec \infty_Y$, and call Σ a *quasi–support* of μ if Σ is a minimal (with respect to $\prec$) quasi–closed supporting set. This is in accordance with the definition of quasi–support given in [14], where the authors restrict themselves to smooth measures. The existence of a quasi–support is not a priori clear. Since this is not our main concern we defer the proof to Corollary 2.3, where we also show that Σ is characterized by $\infty \cdot \mu \sim \infty_\Sigma$.

In view of possible applications we allow perturbations of the form $\mu^+ - \mu^-$, where $\mu^+ \in M_o$ and μ^- satisfies a Kato condition. We briefly explain the latter but refer the reader to [2, 22] for details. Consider

$$\hat{S}_K := \{\mu \in M_o; \langle \mu, (H+E)^{-1} \cdot \rangle \in L_1(X; m)' = L_\infty(X, m)\},$$

$$c_E(\mu) := \|\langle \mu, (H+E)^{-1} \cdot \rangle\|_\infty,$$

$$c(\mu) := \inf_{E>0} c_E(\mu).$$

This class generalizes the class K_d of potentials (for the classical Dirichlet form on $\mathbb{R}^d$) in the sense that $V \geq 0$, $V \in K_d$ implies that $V\lambda \in \hat{S}_K$, $c(V\lambda) = 0$. From [22], Theorem 3.1 we know that $\mu \in \hat{S}_K$ implies that μ is $\mathfrak{h}$–bounded with bound $c(\mu)$. Thus, for $c(\mu) < 1$, the KLMN Theorem allows us to define $\mathfrak{h} - \mu$, and some crucial properties of the corresponding semigroups depend only on $c(\mu)$ (see [22] for an account of this type of results). A sequence (μ_n) in $M_o - \hat{S}_K$ is called *monotone*, if $\mathfrak{h} + \mu_n$ is a monotone (increasing or decreasing) sequence of forms.

2.1 Theorem. *Assume* (II). *Let* $\mu_n^+ \in M_o$, $\mu^- \in \hat{S}_K$ *with* $c(\mu^-) < 1/2$ *and let* $\mu_n = \mu_n^+ - \mu^-$. *Assume that* (μ_n) *is monotone and that* $\mu_n - \mu_1$ *is supported on a set* Σ_n *for* $n \in \mathbb{N}$ *such that* $\sup_{n\in\mathbb{N}} \operatorname{cap}(\Sigma_n) < \infty$. *Then, for all* $t > 0$,

$$\|e^{-t(\mathfrak{h}+\mu_n)} - e^{-t(\mathfrak{h}+\mu_m)}\|_{HS} \longrightarrow 0 \textit{ for } n,m \to \infty.$$

In particular, $\mathfrak{h} + \mu_n \xrightarrow{nrs} \mathfrak{h}_\infty$ *for suitable* $\mathfrak{h}_\infty$, *and* $\sigma_{ess}(H_n) = \sigma_{ess}(H_\infty)$ *for the associated self adjoint operators.*

Proof. From [22], Theorem 5.1 we infer that for every $t > 0$ there exists $c(t) \geq 0$ such that

$$\|e^{-t(\mathfrak{h}+\mu_n)} : L_2 \to L_\infty\| \leq c(t)$$

for all $n \in \mathbb{N}$. From Proposition A.3 in the appendix we know that $(e^{-t(\mathfrak{h}+\mu_n)} - e^{-t(\mathfrak{h}+\mu_1)})1 \in L_2$ with an estimate $\|\ldots\|_2 \leq C \cdot \operatorname{cap}(\Sigma_n)^{1/2}$ for all $n \in \mathbb{N}$. Since either $D_n(t) := e^{-t(\mathfrak{h}+\mu_n)} - e^{-t(\mathfrak{h}+\mu_1)}$ or $-D_n(t)$ acts positivity preserving, this implies

$$\|D_n(t) : L_\infty \to L_2\| \leq C \cdot \operatorname{cap}(\Sigma_n)^{1/2}.$$

Using [6], 11.2 and 11.6, we conclude that

$$D_n(t)e^{-t(\mathfrak{h}+\mu_n)} \in \mathfrak{H}\mathfrak{S},$$

$$\|\ldots\|_{HS} \leq C \cdot c(t) \cdot \operatorname{cap}(\Sigma_n)^{1/2}.$$

(This kind of factorization argument has also been used in [21], where it is explained in more detail.) The same arguments together with duality show that

$$\|D_n(t)e^{-t(\mathfrak{h}+\mu_1)}\|_{HS} \leq C \cdot c(t) \cdot \operatorname{cap}(\Sigma_n)^{1/2}.$$

Therefore, for arbitrary $t > 0$, $(D_n(t))$ is a bounded sequence in $\mathfrak{H}\mathfrak{S}$. As the μ_n are monotone, the kernels for $D_n(t)$ form a monotone sequence (cf. [22], Appendix B). Hence, $\|D_n(t) - D_m(t)\|_{HS} \to 0$ for $n,m \to \infty$. The assertion concerning the nrs convergence follows from the representation

$$(\mathfrak{h} + \mu_n + E)^{-1} - (\mathfrak{h} + \mu_m + E)^{-1} = \int_0^\infty e^{-Et}[e^{-t(\mathfrak{h}+\mu_n)} - e^{-t(\mathfrak{h}+\mu_m)}]dt,$$

(which holds for $E \geq 1$) with the help of Lebesgue's theorem. The stability of the essential spectrum can be shown with an appeal to Weyl's theorem as in [21], Corollary to Theorem 1. □

As a first application of this convergence result we state the following nrs–version of Theorem 1.3 :

2.2 Theorem. *Let B_n be Borel sets, for $n \in \mathbb{N}$. Assume that $\underline{\lim}\, B_n \sim \overline{\lim}\, B_n$ and that $B_n \triangle B_m \subset \Sigma$ $(n, m \in \mathbb{N})$ for some Σ with finite capacity. Then*

$$\mathfrak{h} + \infty_{B_n} \xrightarrow{nrs} \mathfrak{h} + \infty_B$$

for every $B \sim \overline{\lim}\, B_n$.

2.3 Remark. (1) It will be clear from the following proof that in the above theorem $\mathfrak{h}$ can be replaced by $\mathfrak{h} + \mu$, where $\mu^+ \in M_o$ and $\mu^- \in \hat{S}_K$ with $c(\mu^-) < 1/2$.

(2) In the situation of the theorem, Proposition A implies that the spectral projections depend continuously on the domain. This yields continuity of discrete eigenvalues and the associated eigenfunctions.

Proof. We use the notation A_n etc. from the proof of Theorem 1.3. We want to apply Theorem 2.1 to the sequence $\mu_n := \infty_{A_n}$. For $n \leq m$, $A_n \subset A_m$, and $A_m \setminus A_n$ supports $\mu_m - \mu_n$. Moreover, $A_m \setminus A_n \subset \Sigma$, so that the assumptions of Theorem 2.1 are fulfilled and we get

$$\mathfrak{h} + \infty_{A_n} \xrightarrow{nrs} \mathfrak{h} + \infty_A = \mathfrak{h} + \infty_B .$$

Similarly,

$$\mathfrak{h} + \infty_{C_n} \xrightarrow{nrs} \mathfrak{h} + \infty_C = \mathfrak{h} + \infty_B ,$$

from which the asserted convergence follows since

$$\mathfrak{h} + \infty_{A_n} \leq \mathfrak{h} + \infty_{B_n} \leq \mathfrak{h} + \infty_{C_n} \quad \square$$

We now proceed to a further application of Theorem 2.1 which deals with "large coupling limits". Before stating the result in full generality let us consider a special case which has been analyzed in detail by Baumgärtel and Demuth [3, 8]. Fix a closed set $B \subset X$ and consider $\mathfrak{h} + n\mathbf{1}_B$. If $n \to \infty$, this sequence converges in srs to $\mathfrak{h} + \infty\mathbf{1}_B$ (by monotone convergence results). As we already remarked above, rather weak regularity assumptions on B will ensure that $\mathfrak{h} + \infty\mathbf{1}_B = \mathfrak{h}_G$, for $G = X \setminus B$. If B is compact then the convergence will even take place with respect to nrs, whether or not the limit equals $\mathfrak{h}_G$, as we see from the following Corollary (we want to note, however, that in [3, 8] unbounded B are also included).

2.4 Corollary. *Let $\mu \in M_o$ with quasi–support Σ. If $\operatorname{cap}(\Sigma) < \infty$, then*

$$\mathfrak{h} + n \cdot \mu \xrightarrow{nrs} \mathfrak{h} + \infty_\Sigma \text{ for } n \to \infty.$$

Proof. By Theorem 2.1 it follows that the $\mathfrak{h} + n\mu$ converge in nrs. To identify its limit, it suffices to check that

$$\mathfrak{h} + n\mu \xrightarrow{srs} \mathfrak{h} + \infty_\Sigma.$$

By monotone convergence theorems for forms, the limit $\mathfrak{h}_\infty$ of the left hand side can easily be described as

$$D(\mathfrak{h}_\infty) = \{u \in D(\mathfrak{h} + \mu); \int \tilde{u}^2 d\mu = 0\}, \; \mathfrak{h}_\infty = \mathfrak{h}|D(\mathfrak{h}_\infty).$$

Consequently, we have to check that there exists a Σ such that the following holds for all $u \in D$, $u \geq 0$:

$$\tilde{u} = 0 \text{ q.e. on } X \setminus \Sigma \Longleftrightarrow \tilde{u} = 0 \quad \mu - \text{a.e.}$$

Clearly, $I := \{u \in D; \tilde{u} = 0 \ \mu\text{–a.e.}\}$ is a closed subspace of D which satisfies the ideal property. Hence Theorem 1 in [20] guarantees the existence of Σ. (Although one cannot use the idea of [14] directly to prove the existence of a quasi–support for non–smooth measures, a look at [14] and [20] shows that part of the arguments are similar.) In view of our definition of quasi–support we still have to show that Σ is a minimal supporting set. Assume that $S \subset X$, S supports μ, i.e. $\mu \prec \infty_S$ and $\infty_S \prec \infty_\Sigma$. Then we have $\infty_\Sigma = \infty\mu \leq \infty_S$. □

A An inequality from probabilistic potential theory

This section is devoted to proving Proposition A.3, which is essential for the proof of Theorem 2.1. Recall that assumption (I) guarantees the existence of a Markov process $(\Omega, (\mathbb{P}^x; x \in X), (X_t; t \geq 0))$ with state space $X \cup \{\infty\}$ such that

$$e^{-t\mathfrak{h}} f(x) = \mathbb{E}^x(f \circ X_t)$$

for all $f \in L_p$, $1 \leq p \leq \infty$. Perturbation of $\mathfrak{h}$ by a measure μ can be represented by an additive functional A in the sense that

$$e^{-t(\mathfrak{h}+\mu)} f(x) = \mathbb{E}^x(f.\circ X_t \cdot e^{-A(t)}).$$

This has been established in different generality in [2, 9, 13, 23]. We will use this identity, the celebrated *Feynman–Kac formula*, only for measures of the form Vm with $V \in L_\infty$, for which the additive functional A takes the form

$$A(t) = \int_0^\infty V \circ X_s ds,$$

see [13], p. 133 or [17], Theorem X.68, p.279, where actually only the case of the classical Dirichlet form is studied explicitely. As a first consequence we describe the additive functional corresponding to the measure ∞_U, where U is open. Of course, this is nothing new for the experts. Since we didn't find a reference for it in full generality (for the classical case, see [23], (4.5), Feller generators are treated in [9], Thm. 4.4) we have chosen to prove the following

A1 Lemma. *Let $U \subset X$ be open, and set*

$$T_U(t)(\omega) := \int_0^t 1_U \circ X_s(\omega) ds,$$

which is the total time up to t, which the particle ω spent in U. Let

$$\tau_U(\omega) := \inf\{s \geq 0; X_s \notin U\}$$

denote the first exit time of U. Then

$$e^{-t(\mathfrak{h}+\infty_U)}f(x) = \mathbb{E}^x[f \circ X_t \cdot 1_{\{T_U(t)=0\}}] \geq \mathbb{E}^x[f \circ X_t \cdot 1_{\{\tau_U < t\}}].$$

Thus, formally, we can calculate $e^{-t(\mathfrak{h}+\infty_U)}$ by setting $V = \infty \cdot 1_U$ in the Feynman–Kac formula for functions. This is actually the philosophy behind the following proof.

Proof. Let $V_n := n \cdot 1_U$. Then

$$\mathfrak{h} + V_n \xrightarrow{srs} \mathfrak{h} + \infty \cdot 1_U = \mathfrak{h} + \infty_U.$$

In fact, the convergence is just monotone form convergence, and the equality follows from [13], Lemma 3.1.4, p. 65. Using the Feynman–Kac formula for V_n and letting $n \to \infty$ we arrive at the asserted equality. The inequality is obvious. □

Fix a set B of finite capacity, and let τ_B be given as in the preceeding lemma. In what follows, we shall use that

$$e_B(x) := \mathbb{E}^x[e^{-\tau_B}]$$

defines a quasi–continuous $e_B \in D$, the so–called *1–equilibrium potential* of B. It satisfies $e_B \geq 1_B$ q.e. and $\|e_B\|_{\mathfrak{h}}^2 = \text{cap}(B)$, which means that it is the "minimizing element" in the definition of the capacity (see Section 1).

A2 Lemma. *Assume that* $\text{cap}(B) < \infty$. *Then* $\mathbb{P}^{\cdot}[\tau_B \leq t] \in L_1$,

$$\|\mathbb{P}^{\cdot}[\tau_B \leq t]\|_1 \leq e^t \text{cap}(B).$$

Proof. Since $\mathbb{P}^x[\tau_B \leq t] \leq e^t e_B(x)$ for e_B as above, it suffices to check that $e_B \in L_1$ with $\|e_B\|_1 \leq \text{cap}(B)$. To this end, let $g \in L_2$, $0 \leq g \leq 1$. Then

$$\begin{aligned} \langle e_B, g\rangle &= (\mathfrak{h}+1)[e_B, (H+1)^{-1}g] \\ &= \int_X (H+1)^{-1} g(x) d\nu_B(x) \\ &\leq \nu_B(X) = \text{cap}(B), \end{aligned}$$

where we used the *equilibrium measure* ν_B, cf. [13],]3.3, p. 75 and the fact that $0 \leq (H+1)^{-1}g \leq 1$. From the above inequality, the assertion concerning the norm of e_B follows. □

A3 Proposition. *Let* μ^+, $\nu^+ \in M_o$, $\mu^- = \nu^- \in \hat{S}_K$, $c(\mu^-) < 1/2$. *Moreover assume that* $\mu^+ \geq \nu^+$ *and that* $\mu^+ - \nu^+$ *is supported on a set* Σ *of finite capacity. Then*

$$\|(e^{-t(\mathfrak{h}+\nu)} - e^{-t(\mathfrak{h}+\mu)})1\|_2 \leq C \cdot \mathrm{cap}(\Sigma),$$

with a constant C *which only depends on* μ^-.

Proof. Since cap is outer regular, we may assume Σ to be open. By assumption, we have $\mu^+ \leq \nu^+ + \infty_\Sigma$ which implies

$$e^{-t(\mathfrak{h}+\mu)}f(x) \geq e^{-t(\mathfrak{h}-\mu^-+\nu^++\infty_\Sigma)}f(x)$$

for all $f \geq 0$ by [22], Corollary B.3. Hence it suffices to prove the estimate for $\mu = \nu + \infty_\Sigma$. Using [22], Theorem 3.5 we may restrict ourselves to the case $\nu = Vm$, with $V \in L_\infty$, $c_E(V^-) < 1/2$ for suitable $E > 0$ (see the definitions preceding Theorem 2.1). For such V we calculate

$$(e^{-t(\mathfrak{h}+V)} - e^{-t(\mathfrak{h}+V+\infty_\Sigma)})1(x)$$

$$= \mathbb{E}^x[e^{-\int_0^t V \circ X_s ds}(1 - 1_{\{T_\Sigma(t)=0\}})]$$

(here we used Lemma A1)

$$\leq \mathbb{E}^x[e^{-\int_0^t V^- \circ X_s ds} \cdot 1_{\{\tau_\Sigma \leq t\}}]$$

(again by A1)

$$\leq (\mathbb{E}^x[e^{-\int_0^t 2V^- \circ X_s ds}])^{1/2} \cdot (\mathbb{P}^x[\tau_\Sigma \leq t])^{1/2},$$

where we used the Cauchy–Schwarz inequality in $L_2(\mathbb{P}^x)$ for the last step. Since the second factor defines an L_2–function by Lemma A2 we only need to show that $\mathbb{E}^x[e^{-\int_0^t 2V^- \circ X_s ds}]$ is bounded with a bound only depending on $2c_E(V^-) = c_E(2V^-)$. This in turn follows from [22], Theorem 3.3 and the observation

$$\mathbb{E}^x[e^{-\int_0^t 2V^- \circ X_s ds}] = e^{-t(\mathfrak{h}-2V^-)}1(x) \leq \|e^{-t(\mathfrak{h}-2V^-)} : L_1 \to L_1\| \quad \square$$

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