1

Spectral theory for nonstationary random potentials

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1.1 Introduction: Leaving stationarity

One-particle Schrödinger operators with random potentials are used to model quantum aspects of disordered electronic systems like unordered alloys, amorphous solids or liquids.

Starting from periodic potentials to describe perfect crystals, one is interested in the spectral properties of these operators. Periodic Schrödinger operators usually have absolutely continuous spectrum, which is connected physically to good conductance properties. The spectrum consists of bands, that is intervals, where one finds (absolutely continuous) spectrum. Between the bands there are so called forbidden zones without spectrum.

Anderson, Mott and Twose [0, 0] discovered – based on physical reasoning – that disordered systems should have a tendency to "localized states" in certain regimes of the energy spectrum, which reflects bad conductance properties of the solid in this energy regime.

In recent years there has been considerable progress concerning mathematically rigorous results on this phenomenon of **localization**. We refer to the bibliography where we chose some classics, some recent articles as well as books on the subject. However, all these results provide only one part of the picture that is accepted since the ground breaking work [0, 0] by Anderson, Mott and Twose. The effect of metal insulator transition is supposed to depend upon the dimension and the general picture is as follows: Once translated into the language of spectral theory there is a transition from a **localized phase** that exhibits pure point spectrum (= only bound states = no transport) to a **delocalized phase** with absolutely continuous spectrum (= scattering states = transport). What has been proven so far is the occurence of the former phase, well known under the name of localization. The missing part, delocalization, has not been settled for genuine random models.

There is need for an immediate disclaimer or, put differently, for an explanation of what we mean by "genuine".

An instance where a metal insulator transition has been verified rigorously is supplied by the **almost Mathieu operator**, a model with modest disorder for which the parameter that triggers the transition is the strength of the coupling. As references let us mention [0, 0, 0, 0, 0, 0] where the reader can find more about the literature on this true evergreen. Quite recently it has attracted a lot of interest especially among harmonic analysts; see [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]

The underlying Hilbert space is $l^2(\mathbb{Z})$. Consider parameters $\alpha, \lambda, \theta \in \mathbb{R}$ and define the selfadjoint, bounded operator $h_{\alpha,\lambda,\theta}$ by

$$(h_{\alpha,\lambda,\theta}u)(n) = u(n+1) + u(n-1) + \lambda\cos(2\pi(\alpha n + \theta))u(n),$$

for $u = (u(n))_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$.

Note that this operator is a discrete Schrödinger operator with a potential term with the coupling constant λ in front and the discrete analog of the Laplacian. For irrational α the potential term is an almost periodic function on $\mathbb{Z}$.

Basically, there is a metal insulator transition at the critical value 2 for the coupling constant λ. Since these operators are very close to periodic ones one can fairly label them as poorly disordered. Moreover, the proof of delocalization boils down to the proof of localization for a "dual operator" that happens to have the same form. In this sense, the almost Mathieu operator is not a genuine random model.

A second instance, where a delocalized phase is proven to exist is the Bethe lattice. See Klein's article [0].

Quite recently, an order parameter has been introduced by Germinet and Klein to characterize the range of energies where a multiscale scenario provides a proof of a localized regime, [0]. In their work the important parameter is the energy.

However, as we already pointed out above, for genuine random models, there is no rigorous proof of the existence of a transition or even of the appearance of spectral components other than pure point, so far. This is a quite strange situation: the unperturbed problem exhibits extended states and purely a.c. spectrum but for the perturbed one can prove the opposite spectral behavior only.

In this survey we are dealing with models that are not stationary in the sense that the influence of the random potential is not uniform in space. The precise meaning of this admittedly vague description differs from case to case but will be clear for each of them.

1.2 Sparse Random Potentials

The term sparse potentials is mostly known for potentials that have been introduced in the 1970's by Pearson [0] to construct Schrödinger operators on

the line with singular continuous spectrum. To use similar geometries to obtain a metal insulator transition can be traced back to Molchanov, Molchanov and Vainberg [0, 0] and Krishna [0, 0], see also [0, 0, 0]. Let us introduce three models for operators with sparse random potentials in $L^2(\mathbb{R}^d)$ taken from [0].

Model I:

$$H(\omega) = -\Delta + V_\omega,$$

where

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^m} \xi_k(\omega) f(x-k),$$

$f \leq 0$ is a compactly supported single site potential and the ξ_k are independent Bernoulli variables with $p_k := \mathbb{P}\{\xi_k = 1\}$.

Model II:

$$H(\omega) = -\Delta + V_\omega,$$

where

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^m} q_k(\omega)\xi_k(\omega) f(x-k),$$

with f and ξ as above and independent identically distributed random variables q_k.

Model III:

$$H(\omega) = -\Delta + V_\omega,$$

where

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^m} a_k q_k(\omega) f(x-k),$$

with q_i as above and a deterministic sequence a_i decaying fast enough at infinity (see [0] for a discrete analog of this model).

For the first two models it was proven in [0] that $[0,\infty)$ belongs to the absolutely continuous spectrum as long as p_k decays fast enough. To understand this appearance of a metallic regime, we recall the following facts from scattering theory:

We write $-\Delta = H_0$ so that the operators we are interested in can be written as $H = H_0 + V$. By $\sigma_{ac}(H)$ we denote the absolutely continuous spectrum, related to delocalized states.

Theorem 1. *(Cooks criterion)*
If for some $T_0 > 0$ and all ϕ in a dense set

$$\int_{T_0}^{\infty} \|V e^{-itH_0}\phi\| dt < \infty \qquad (*)$$

then $\Omega_- := lim_{t\to\infty} e^{itH} e^{-itH_0}$ exists and, consequently, $[0,\infty) \subset \sigma_{ac}(H)$, i.e., there are scattering states for H and any nonnegative energy.

The typical application rests on the fact that $(*)$ is satisfied if

$$|V(x)| \leq C(1+|x|)^{-(1+\epsilon)}, \qquad (**)$$

a condition that obviously fails to hold for almost every V_ω provided the p_k are not summable. However, in [0] the following result has been proved:

Theorem 2. *Assume that*

$$W(x) := \left(\mathbb{E}(V_\omega(x)^2)\right)^{\frac{1}{2}} \leq C(1+|x|)^{-(1+\epsilon)}.$$

Then V_ω satisfies Cook's criterion for a.e. ω.

The proof is short. So we reproduce it here.

Proof.

$$\mathbb{E}\left(\int_{T_0}^{\infty} \|V_\omega e^{-itH_0}\phi\| dt\right)$$

$$= \int_{T_0}^{\infty} \mathbb{E}\left(\int V_\omega(x)^2 |e^{-itH_0}\phi(x)|^2 dx\right)^{\frac{1}{2}} dt$$

$$= \int_{T_0}^{\infty} \left(\mathbb{E}\int V_\omega(x)^2 |e^{-itH_0}\phi(x)|^2 dx\right)^{\frac{1}{2}} dt$$

$$\leq \int_{T_0}^{\infty} \left(\int \mathbb{E}(V_\omega(x)^2) |e^{-itH_0}\phi(x)|^2 dx]\right)^{\frac{1}{2}} dt$$

$$= \int_{T_0}^{\infty} \|W(x) e^{-itH_0}\phi\| dt$$

One can apply this result if the p_k decay fast enough to guarantee sufficient decay of $W(x)$. For **Model I** and **Model II** operators (see [0]) with $p_k \sim k^{-\alpha}$ and f with compact support this condition reads $2 < \alpha$.

On the other hand one wants to have that $\sum_k p_k = \infty$, since otherwise V_ω has compact support a.s. by the Borel-Cantelli Lemma. Thus one gets both essential spectrum below zero and absolutely continuous spectrum above zero if

$$2 < \alpha < d.$$

For fixed $d \geq 3$ and $p_k \sim k^{-\alpha}$ one can moreover control the essential spectrum below 0 as done in [0]: the negative essential spectrum consists of a sequence of energies that can at most accumulate at 0.

To control the essential spectrum below zero, it is useful to introduce the model operators $H_f := H_0 + f$ and $H_n := H_0 + f(x - x_n)$ for a sequence of points $\{x_m\}_{m\in\mathbb{N}}$.

Theorem 3. *(Klaus) Let $\mathcal{E}$ denote the set of energies*

$$\mathcal{E} = \{E < 0 \,|\, \textit{there exists a sequence } n_j, \textit{ and energies } E_{n_j} \in \sigma(H_{n_j}) \textit{ with } E_{n_j} \to E\}.$$

Then $\sigma_{ess}(H) = \mathcal{E} \cup [0, \infty)$.

Using the sets

$$A_k(\Lambda) := \{\omega \,|\, \exists \text{ at least } k \text{ distinct points } n_l \in \Lambda \text{ with } \xi_{n_l} = 1\},$$

together with the bound

$$\mathbb{P}[A_k(\Lambda)] \leq \left(\sum_{i \in \Lambda} p_i\right)^k \leq |\Lambda|^{k-1} \sum_{i \in \Lambda} p_n^k,$$

one gets by a Borel-Cantelli argument, that the event

$$A_k := \bigcup_{L=1}^{\infty} A_k(\Lambda_L)$$

has zero probability. Calling a finite subset F of $\mathbb{Z}^d$ essential for H_ω if

$$\mathbb{P}\left[F + n \subset \Xi \text{ for infinitely many } n\right] = 1$$

with $\Xi := \{n \,|\, \xi_n = 1\}$ and

$$H_F := H_0 + \sum_{i \in F} f(\cdot - i),$$

from Klaus' theorem one gets the following.

Theorem 4. *Let $\mathcal{E} = \{E_n(F) \in H_F \,|\, F \textit{ is essential for } H_\omega\}$. If $\sum p_n^k < \infty$ for some k, then*

$$\sigma_{ess}(H_\omega) = \mathcal{E} \cup [0, \infty) \quad \mathbb{P}\textit{-almost surely}$$

Therefore, there exists essential spectrum below zero, if $\sum p_i = \infty$, and it is pure point, if $\sum p_i^k < \infty$ for some k.

For **Model III** operators the distribution $\mathbb{P}_0$ of the i.i.d. random variables q_k has a strong influence on the spectral behaviour in the following sense: If $\mathbb{P}_0$ has a bounded support, every realization of **Model III** is decaying at infinity and there is no hope to find essential spectrum below 0. However, if $\mathbb{P}_0$ does not have compact support, the potential $V_\omega(x)$ may admit a sequence $x_i \to \infty$ such that $V_\omega(x_i) \to -\infty$, thus allowing for essential, in fact, dense pure point spectrum below zero.

For details and more on sparse potentials, especially for models for which the negative spectrum has a richer structure and contains intervals, we refer the reader to [0].

Remarks 5. In [0] absence of (absolutely) continuous spectrum outside the spectrum of the unperturbed operator for certain random sparse models is proved reminiscent of Model I above and Model II from [0] but considerably more general. This is based on the techniques from [0, 0, 0].

1.3 Sparse Random Potentials and the Integrated Density Of States

A useful object in studying random Schrödinger operators is the (integrated) density of states $N(H_\omega, E)$. For ergodic random Schrödinger operators it can be defined as the thermodynamic limit of the eigenvalue counting function up to energy E, that is

$$N(H_\omega, E) := \lim_{L\to\infty} \frac{1}{|\Lambda_L|} N(H^D_{\omega,L}, E)$$

and

$$N(H^D_{\omega,L}, E) := \#\{\lambda_i \leq E \,|\, \lambda_i \in \sigma(H^D_{\omega,L})\},$$

where $H^D_{\omega,L}$ denotes the operator H_ω restricted to a box Λ_L of side length L with Dirichlet boundary conditions. Under appropriate conditions on the random potential this limit exists and is non-random. Also this limit tends to be independent of the boundary conditions used in the definition, that is, using Neumann boundary conditions on the boxes Λ_L gives the same limit.

It is not immediately clear, how to define an analogous object for sparse random potentials, because on the one hand there is nothing like ergodicity around them, on the other hand one has to guess the right normalisation substituting the volume $|\Lambda_L|$.

At least for sparse random potentials with $\sum p_n = \infty$, but $\sum p_n^2 < \infty$ it is possible to define a modified integrated density of states for energies $E < 0$ as

$$N(H_\omega, E) := \lim_{L\to\infty} \frac{1}{P_L} N(H^D_{\omega,L}, E)$$

with

$$P_L := \sum_{i\in\Lambda_L} p_i.$$

For **Model I** operators this gives

Theorem 6. *Let V_ω be a sparse random potential as in **Model I** with $\sum p_n = \infty$ and $\sum p_n^2 < \infty$. Then the limits*

$$N(H_\omega, E) := \lim_{L\to\infty} \frac{1}{P_L} N(H^D_{\omega,L}, E)$$

and

$$\tilde{N}(H_\omega, E) := \lim_{L\to\infty} \frac{1}{P_L} N(H^N_{\omega,L}, E)$$

exist for $E < 0$ *and are non-random. Moreover*

$$\tilde{N}(H_\omega, E) = N(H_\omega, E) = N(H_0 + f, E)$$

at every point of continuity of $N(H_0 + f, E)$ *with* $E < 0$.

An analogues proof works for **Model II** operators and gives:

Theorem 7. *Let* V_ω *be a sparse random potential as in* ***Model II*** *with* $\sum p_n = \infty$ *and* $\sum p_n^2 < \infty$. *Then the limits*

$$N(H_\omega, E) := \lim_{L\to\infty} \frac{1}{P_L} N(H^D_{\omega,L}, E)$$

and

$$\tilde{N}(H_\omega, E) := \lim_{L\to\infty} \frac{1}{P_L} N(H^N_{\omega,L}, E)$$

exist for $E < 0$ *and are non-random. Moreover*

$$\tilde{N}(H_\omega, E) = N(H_\omega, E) = \mathbb{E}\left[N(H_0 + q_0(\omega)f, E)\right]$$

at every point of continuity of $\mathbb{E}\left[N(H_0 + q_0(\omega)f, E)\right]$ *with* $E < 0$.

Remarks 8. Since $\sum p_n^2 < \infty$, by a Borel-Cantelli argument the essential set for H_ω consists of only one point. So the theorem tells us, that the modified integrated density of states "sees" exactly the essential spectrum below zero which is located at the eigenvalues of the model operator $H_0 + f$.

Since these operators are not ergodic in any sense, one has to look for a substitute for the ergodic theorems used to proof the existence and non-randomness of the integrated density of states. In [0] a strong law of large numbers has been proven to cover also these cases of sparse random potentials.

For details on the integrated density of states for sparse random potentials see [0].

1.4 Random surface models

Consider the following self-adjoint random operator in $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$:

$$H(\omega) = -\Delta + V_\omega,$$

where

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^m} q_k(\omega) f(x - (k, 0)),$$

the q_k are i.i.d. random variables and $f \geq 0$ is a single site potential that satisfy certain technical assumptions. This leads to the following geometry characterizing random surface models. Sometimes the upper half plane is considered only.

There is a lot of literature, mostly on the discrete case, using a decomposition into a bulk and a surface term see [0, 0, 0, 0, 0, 0, 0, 0, 0, 0].

The moral of the story is the appearance of a metal insulator transition at the edges of the unperturbed operator. We now concentrate on the continuum case, where we only know of [0, 0] as references. The existence of an a.c. component is proven in [0]. In the following, we present the result from [0], giving strong dynamical localization. Similar but somewhat different results have been announced in [0]. As discussed there, an additional Dirichlet boundary condition "stabilizes" the spectrum so that the appearance of negative spectrum requires a certain strength of the random perturbation. Therefore, proving localization at negative energies is easier (compared to the case without Dirichlet boundary conditions) since one is automatically dealing with a "large coupling" regime.

It is not hard to see that

$$\sigma(H(\omega)) = [E_0, \infty) \text{ where } E_0 = \inf \sigma(-\Delta + q_{\min} \cdot f^{\mathrm{per}}),$$

and

$$f^{\mathrm{per}} = \sum_{k \in \mathbb{Z}^m} f(x - (k, 0))$$

denotes the periodic continuation of f along the surface. Near the bottom of the spectrum E_0 one expects localization, i.e. suppression of transport as is typical for insulators. For nonnegative energies one expects extended states. To stress the existence of a metallic phase let us cite Theorem 4.3 of [0]

Theorem 9. *Let $H(\omega)$ be as below. Then we have, for every $\omega \in \Omega$:* $[0, \infty) \subset \sigma_{\mathrm{ac}}(H(\omega))$.

The idea of the *Proof* is that a wave packet with velocity pointing away from the surface will escape the influence of the surface potential and is asymptotically free. The rigorous implementation of this idea uses Enss' technique from scattering theory.

The model. (1) $0 < m < d$ and points in $\mathbb{R}^d = \mathbb{R}^m \times \mathbb{R}^{d-m}$ are written as pairs, if convenient;

(2) The single site potential $f \geq 0$, $f \in L^p(\mathbb{R}^d)$ where $p \geq 2$ if $d \leq 3$ and $p > d/2$ if $d > 3$, and $f \geq \sigma > 0$ on some open set $U \neq \emptyset$ for some $\sigma > 0$.

(3) The q_k are i.i.d. random variables distributed with respect to a probability measure μ on $\mathbb{R}$, such that $\operatorname{supp} \mu = [q_{\min}, 0]$ with $q_{\min} < 0$.

We will sometimes need further assumptions on the single site distribution μ:

(4) μ is *Hölder continuous*, i.e. there are constants $C, \alpha > 0$ such that

$\mu[a, b] \leq C(b-a)^{\alpha}$ for $q_{\min} \leq a \leq b \leq 0$.

(5) *Disorder assumption*: there exist $C, \tau > 0$ such that

$\mu[q_{\min}, q_{\min} + \varepsilon] \leq C \cdot \varepsilon^{\tau}$ for $\varepsilon > 0$.

What follows is the main result of [0].

Theorem 10. *Let $H(\omega)$ be as above with $\tau > d/2$ and assume that $E_0 < 0$.*

(a) *There exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon]$ the spectrum of $H(\omega)$ is pure point for almost every $\omega \in \Omega$, with exponentially decaying eigenfunctions.*

(b) *Assume that $p < 2(2\tau - m)$. Then there exists an $\varepsilon > 0$ such that in $[E_0, E_0 + \varepsilon] = I$ we have strong dynamical localization in the sense that for every compact set $K \subset \mathbb{R}^d$:*

$$\mathbb{E}\{\sup_{t>0} \||X|^p e^{-itH(\omega)} P_I(H(\omega))\chi_K\|\} < \infty$$

A consequence is pure point spectrum in the interval $[E_0, E_0 + \varepsilon] = I$. Together with the previous result on extended states we get the picture from Figure 4 that still leaves open some important questions.

1.5 The density of surface states

For random surface models it is also possible to define an appropriate analogue of the density of states, the density of surface states (see [0, 0, 0]).

Since one has to get rid of the bulk spectrum to recover properties of the surface potential (see [0]), one defines the density of surface states as the following limit.

$$\nu^S[f] := \lim_{L\to\infty} \frac{1}{L^m} \mathrm{tr}\left\{\chi_L \left(f(H_\omega) - f(H_0)\right)\right\}$$

for sufficiently smooth functions f, where χ_L is the characteristic function of a cube Λ_L. In [0] it was proven, that one needs $f \in C_0^3(\mathbb{R})$ at most. [0, 0] showed, that one needs $f \in C_0^1(\mathbb{R})$, but they used sign definite surface potentials.

This definition doesn't make use of boundary conditions, because one might guess, that the introduction of boundary conditions may have an effect on the density of surface states since they are at least of the same order as the perturbation by the potential.

So it may be surprising that it is also possible to define the density of surface states as the limit

$$\nu^{S,D}[f] := \lim_{L\to\infty} \frac{1}{L^m} \mathrm{tr}\left\{f(H^D_{\omega,L}) - f(H^D_0)\right\}$$

with Dirichlet boundary conditions or as the limit

$$\nu^{S,N}[f] := \lim_{L\to\infty} \frac{1}{L^m} \mathrm{tr}\left\{f(H^N_{\omega,L}) - f(H^N_0)\right\}$$

with Neumann boundary conditions. In [0] it has been proved, that for a wider class of random surface potentials (that allows also for sign indefinite random potentials) all these definitions give the same limit. This limit is also non-random and gives a distribution of order 1 (in fact it is the distributional derivative of a signed measure).

To prove this we strongly used Feynman-Kac representations of the Laplace transforms of the finite volume measures and showed that the difference of these Laplace transforms converge to 0 as $L \to \infty$. Together with the regularity of the density of surface states this gives also their independence of boundary conditions.

Using this Laplace transforms it was also possible to calculate the asymptotic behaviour of the integrated density of surface states for random Gaussian surface potentials (for details see [0]).

1.6 Lifshitz tails & Localization

Below the energy $E = 0$, the bottom of the spectrum of the unperturbed (free) operator the density of surface states is easily seen to be a positive Borel measure. Hence, we may speak of its distribution function

$$N_S(E) = \nu^s\left((-\infty, E)\right) \quad for \quad E < 0.$$

Very recently, it was proved in [0] that $N_S(E)$ shows a very characteristic decay at E_0, the bottom of the spectrum. This behavior, known as Lifshitz behavior in the case of stationary random potentials, is given by:

$$N_S(E) \sim e^{-c(E-E_0)^{-m/2}} \qquad (*)$$

as $E \searrow E_0$. This result is then used in [0] to prove Anderson localization for surface models without assuming the disorder assumption (5). [0] modify the proof in [0] replacing (5) by estimates that follow from $(*)$.

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