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ADMISSIBLE AND REGULAR POTENTIALS FOR SCHRÖDINGER FORMS

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0. INTRODUCTION

For a nonnegative potential $V \in L_{1,\mathrm{loc}}(\mathbf{R}^\nu)$ one can define a selfadjoint realization of the formal Schrödinger operator $-\Delta + V$ using quadratic forms. To be more precise, one uses the "form sum" $-\Delta \dot{+} V$, i.e. the sum $h + t_V$, where h and t_V are the closed forms which correspond to the semibounded operators $-\Delta$ and V, respectively. This is possible since $h + t_V$ is densely defined. (Note that $C_c^\infty(\mathbf{R}^\nu) \subset D(h) \cap D(t_V)$.)

An example in [6] shows that $h + t_V$ may be densely defined without the requirement $V \in L_{1,\mathrm{loc}}(\mathbf{R}^\nu)$. It is therefore legitimate to ask for a characterization of those potentials for which $D(h) \cap D(t_V)$ is dense. This question leads us to the notion of "admissible potentials" which was introduced by J. Voigt [7]. Voigt also defined the somewhat smaller class of regular potentials. A regular potential has the important property that $-\Delta \dot{+} V$ is a restriction of the maximal operator $(-\Delta + V)_{\max}$, where $D((-\Delta + V)_{\max}) := \{f \in L_2(\mathbf{R}^\nu) : -\Delta f,\ Vf \in L_{1,\mathrm{loc}}(\mathbf{R}^\nu), -\Delta f + Vf =: (-\Delta + V)_{\max} f \in L_2(\mathbf{R}^\nu)\}$.

Therefore $-\Delta \dot{+} V$ is a "reasonable" selfadjoint "realization" of the formal Schrödinger operator $-\Delta + V$. Indeed, this is a special case of a result of Voigt ([7], Theorem 7.4). We shall prove an extension of the latter in Theorem 3.4.

Our main results are Theorems 3.2 and 3.3 where we give equivalent conditions for a potential V in order to be $U_0(\cdot)$-admissible or $U_0(\cdot)$-regular, respectively, where $(U_0(t);\, t \geqslant 0)$ is the semigroup associated with $-\Delta \dot{+} V_1$ on $L_2(\Omega)$ ($\Omega \subset \mathbf{R}^\nu$ open) for rather general V_1. The results state that, roughly speaking, V is admissible or regular, respectively, if it is locally integrable outside of "small sets", where the smallness is measured in terms of the capacity.

In the first two sections we have chosen the abstract setting of regular Dirichlet forms although we are mainly interested in Schrödinger forms.

The main reason is that we only need to truncate and mollify functions in the form domain, and the concept of a regular Dirichlet form enables these manipulations. Therefore regular Dirichlet forms are, in our opinion, the adequate setting for a great variety of statements concerning Schrödinger forms. Note, for example, that Corollary 2.5 gives a generalization of a result of Cycon ([1], Theorem 2).

The paper is organized as follows: In Section 1 we characterize $U(\cdot)$-admissibility and $U(\cdot)$-regularity of potentials where $(U(t)\,;\,t \geq 0)$ is the semigroup associated with a regular Dirichlet form t. It will turn out in Theorems 1.5 and 1.6, that a nonnegative potential V is regular and admissible, respectively, if it satisfies an integrability condition outside of open sets with arbitrarily small t-capacity.

Section 2 is devoted to the study of perturbations of Dirichlet forms with negative potentials. The form h of the first paragraph is, in fact, a regular Dirichlet form. But we are also interested in Schrödinger operators $-\Delta + V_1$ with V_1 not necessarily semibounded below. Hence, the associated forms will not be Dirichlet forms, in general. In Theorem 2.2 we will prove that for a certain class of negative potentials V_0 regularity and admissibility will not be affected if one replaces t by $t_0 =$ "$t - t_{-V_0}$". (Note that the last symbol is, in general, not defined.) The main tool of this section is Proposition 2.4, which might be of some interest of its own, even in the case of Schrödinger forms.

In Section 3, finally, we are able to state the main Theorems 3.2 and 3.3 as simple consequences of the results of Sections 1 and 2.

Concluding the introduction I want to express my thanks to J. Voigt. It is a great pleasure to acknowledge his help and encouragement.

1. REGULAR AND ADMISSIBLE POTENTIALS FOR REGULAR DIRICHLET FORMS

In order to fix the notation we want to recall some basic definitions and properties. (For details we refer to [2] and [7].)

In what follows let X be a σ-compact Hausdorff space (i.e. X is locally compact and $X = \bigcup_{n \in \mathbb{N}} X_n$, where X_n is open and relatively compact) with a Borel measure μ.

We consider a symmetric form $t \geq c$ with dense demain D on the complex Hilbert space $(L_2(X, \mu), (\cdot\,|\,\cdot))$. We assume t to be closed, i.e., $(D; (\cdot\,|\,\cdot)_t)$ is a Hilbert space, where $(\cdot\,|\,\cdot)_t = t[\cdot\,,\cdot] + (1-c)\cdot(\cdot\,|\,\cdot)$ and $\|\cdot\|_t$ denotes the corresponding norm. (Note that for $t \geq c'$ the corresponding norms are equivalent.) Since t is closed there exists an associated nonnegative selfadjoint operator T. Hence, $-T$ generates a C_0-semigroup $(U(t)\,;\,t \geq 0)$ on $L_2(X, \mu)$, the associated semigroup. If $V: X \to [0;\infty]$ is measurable, then

$$D(t_V) := \{f \in L_2(X, \mu) : V^{1/2} f \in L_2(X, \mu)\} =$$
$$= \{f \in L_2(X, \mu) : V|f|^2 \in L_1(X, \mu)\}, \quad t_V[f, g] := (V^{1/2} f \mid V^{1/2} g)$$

defines a closed nonnegative form, which is densely defined if and only if V is finite μ-a.e. Letting $V^{(n)}(x) := (V \wedge n)(x)$ (we denote by $f \wedge g$ ($f \vee g$) the pointwise minimum (maximum) of two functions f, g) it follows from perturbation theory of C_0-semigroups that $-T - V^{(n)}$ is the generator of a C_0-semigroup. If

$$U_V(t) := \text{s-}\lim_{n\to\infty} \exp t(-T - V^{(n)})$$

defines a C_0-semigroup $(U_V(t) ; t \geqslant 0)$, we call V *$U(\cdot)$-admissible*. If V is $U(\cdot)$-admissible, then so is αV for $0 < \alpha \leqslant 1$. If, in addition,

$$\text{s-}\lim_{\alpha\to 0} U_{\alpha V}(t) = U(t) \quad \text{for } t \geqslant 0,$$

we say that V is *$U(\cdot)$-regular*.

Note that $U(\cdot)$-admissibility depends on the semigroup $(U(t) ; t \geqslant 0)$. The characterization in Proposition 1.1 is more convenient in our context, but we wanted to give the original definition introduced in [7], where a slightly different situation is considered. In [7] the semigroups are defined on $L_p(X, \mu)$ and therefore are assumed to be positivity preserving. In the Hilbert space case this requirement may be dropped since the limit $U_V(t)$ exists for $t \geqslant 0$ by monotone convergence of forms (cf. [5], Theorem 3.1).

To be more precise,

$$t + t_{V^{(n)}} \to t + t_V$$

in strong resolvent sense, which proves part (a) of

1.1. Proposition (cf. [7], Proposition 5.8). *Let t be a closed form, T and $(U(t) ; t \geqslant 0)$ be the associated selfadjoint operator and C_0-semigroup respectively and $V: X \to [0; \infty]$ be measurable.*

(a) *V is $U(\cdot)$-admissible if and only if $t + t_V$ is densely defined. In this case $-(T + V)$ is the generator of $(U_V(t) ; t \geqslant 0)$.*

(b) *V is $U(\cdot)$-regular if and only if $D \cap D(t_V)$ is dense in $(D, (\cdot \mid \cdot)_t)$.*

For the proof we refer to [7], proof of Proposition 5.8.

t is called *Dirichlet form*, if

$$g \in D \Rightarrow |g| \in D \quad \text{and} \quad t[|g|] \leqslant t[g],$$

and

$$g \in D \text{ real valued} \Rightarrow g \wedge 1 \in D \quad \text{and} \quad t[g \wedge 1] \leqslant t[g],$$

where we use the notation $t[f, f] =: t[f]$.

We want to remark that the first property implies (cf. [4], Theorem XIII 50) that $U(t)$ is positivity preserving and in particular reality preserving.

This yields

$$f \in D \Rightarrow \operatorname{Re} f \in D \quad \text{and} \quad t[\operatorname{Re} f] \leqslant t[f].$$

A Dirichlet form t is said to be *regular*, if $D \cap C_c(X)$ is dense in $(D, \|\cdot\|_t)$ and in $(C_c(X), \|\cdot\|_\infty)$ (cf. [2], p. 5 and p. 6 where real Hilbert spaces are considered).

1.2. REMARK. Let t be a regular Dirichlet form.

(a) For a closed set A, $D \cap C_c(X \setminus A)$ is dense in $(C_c(X \setminus A), \|\cdot\|_\infty)$ and in $L_2(X \setminus A, \mu)$.

(b) For $f, g \in D \cap L_\infty(X, \mu)$ we have $f \cdot g \in D$ and

$$\|f \cdot g\|_t \leqslant \|f\|_t \cdot \|g\|_\infty + \|f\|_\infty \cdot \|g\|_t.$$

For the proof we refer to [2], Lemma 1.4.2 (iii) and [2], Theorem 1.4.2 (ii).

Throughout the rest of this paper let us assume:

(A) t denotes a regular Dirichlet form with domain D. T and $(U(t)\,;\, t \geqslant 0)$ are the associated selfadjoint operator and C_0-semigroup, respectively, and $V: X \to [0:\infty]$ is a measurable function.

To get some idea what $U(\cdot)$-admissible and $U(\cdot)$-regular potentials look like let us state

1.3. REMARK. (a) If $V \in L_{1,\mathrm{loc}}(X \setminus A)$, where A is a closed set and $\mu(A) = 0$, then V is $U(\cdot)$-admissible.

(b) If $V \in L_{1,\mathrm{loc}}(X)$, then V is $U(\cdot)$-regular.

The proof is an immediate consequence of Proposition 1.1 and Remark 1.2 (a).

An example in [6] shows, however, that V may be $U(\cdot)$-regular without being locally integrable. In order to characterize $U(\cdot)$-admissibility and $U(\cdot)$-regularity we introduce the t-capacity:

For an open set $U \subset X$ let

$$c_t(U) := \inf\{\|f\|_t^2 : f \in D,\, f \geqslant \chi_U\}, \quad (\inf \emptyset = \infty),$$

be the *t-capacity of U*.

For arbitrary $A \subset X$, we set

$$c_t(A) := \inf\{c_t(U) : U \text{ open},\ U \supset A\}$$

(cf. [2], p. 61).

For the reader's convenience we recall some basic properties concerning capacities.

1.4. LEMMA. (a) *For any open* U *such that* $c_t(U) < \infty$ *there exists a unique* $e_U \in D$ *satisfying* $\chi_U \leqslant e_U \leqslant 1$ *and* $c_t(U) = \|e_U\|_t^2$, *the so-called "1-equilibrium potential" of* U.

(b) *For any* $f \in D$ *there exists an* $\tilde{f}$ *such that* $f = \tilde{f}$ μ-a.e. *and for each* $\varepsilon > 0$ *there exists an open* U *satisfying* $c_t(U) < \varepsilon$ *and* $\tilde{f} \mid (X \setminus U)$ *is continuous.*

(c) *For any sequence* (U_n) *of open sets we have*

$$c_t\Big(\bigcup_{n\in\mathbf{N}} U_n\Big) \leqslant \sum_{n\in\mathbf{N}} c_t(U_n).$$

(d) *For* $f \in D$ *and* $\tilde{f}$ *as in* (b)

$$c_t([\tilde{f} \geqslant \lambda]) \leqslant \frac{1}{\lambda^2} \|f\|_t^2$$

holds.

For the proof we refer to [2], Lemma 3.1.1, Lemma 3.1.2, Theorem 3.1.3 and Lemma 3.1.5.

1.5. THEOREM. *Assume* (A). *Then the following statements are equivalent.*

(i) V *is* $U(\cdot)$*-regular or, equivalently,* $D \cap D(t_V)$ *is dense in* $(D, \|\cdot\|_t)$.

(ii) *For each* $\varepsilon > 0$ *there exists an open set* $U \subset X$ *such that* $c_t(U) < \varepsilon$ *and* $\chi_{X\setminus U} V \in L_{1,\mathrm{loc}}(X, \mu)$.

(iii) *For each* $\varepsilon > 0$ *there exists an* $f \in D$ *such that* $0 \leqslant f \leqslant 1$, $\|f\|_t < \varepsilon$ *and* $(1 - f)V \in L_{1,\mathrm{loc}}(X, \mu)$.

Proof. (i) $\Rightarrow$ (ii) Let $X = \bigcup_{k\in\mathbf{N}} X_k$, $\varepsilon > 0$. Using 1.4(c) it suffices to prove (ii) with X replaced by X_k. Since t is regular, there is a $\varphi \in D \cap C_c(X)$ such that $\varphi \geqslant \chi_{X_k}$. Let $0 \leqslant \psi \leqslant \varphi$, $\psi \in D \cap D(t_V)$ such that $\|\varphi - \psi\|_t \leqslant \varepsilon$, and choose a representative $\tilde{\psi}$ and an open set W according to Lemma 1.4(b), i.e. $c_t(W) < \varepsilon$ and $\tilde{\psi} \mid (X\setminus W)$ is continuous. Therefore, $U := X_k \cap ([\tilde{\psi} < 1/2] \cup W)$ is open. (Recall that $[\tilde{\psi} < 1/2]$ is relatively open in $X\setminus W$.)

$$c_t(U) \leqslant c_t(X_k \cap [\tilde{\psi} < 1/2]) + c_t(W) \leqslant$$

$$\leqslant c_t(X_k \cap [\varphi - \tilde{\psi} \geqslant 1/2]) + c_t(W) \leqslant$$

$$\leqslant 4\|\varphi - \psi\|_t^2 + c_t(W) \leqslant 4\varepsilon^2 + \varepsilon.$$

By construction, we have $0 \leqslant \chi_{X\setminus U} V \leqslant 4 \cdot \psi^2 V \in L_1$ since $\psi \in D(t_V)$.

(ii) $\Rightarrow$ (iii) Let $f = e_U$ with U being as in (ii). Then $\chi_U \leqslant f \leqslant 1$ which implies $(1 - f)V \leqslant \chi_{X\setminus U} V \in L_{1,\mathrm{loc}}$ and $\|f\|_t = \sqrt{\varepsilon}$.

(iii) $\Rightarrow$ (i) Using (iii) we find a sequence $(f_n) \subset D$ such that $\|f_n\|_t \to 0$ and $(1 - f_n)V \in L_{1,\mathrm{loc}}$. Let $\varphi \in D \cap C_c(X)$, $\|\varphi\|_\infty \leqslant 1$.

In order to show $\varphi \in \overline{D \cap D(t_V)}^{\|\cdot\|_t}$ it suffices to find a sequence $(\varphi_n) \subset D \cap D(t_V)$ such that $\varphi_n \to \varphi$ weakly in $(D, (\cdot|\cdot)_t)$. Let $\varphi_n := \varphi(1 - f_n) \in D \cap D(t_V)$. Then $\|\varphi_n\|_t \leqslant \|\varphi\|_t + \|f_n\|_t$. Since $\varphi_n \to \varphi$ in $L_2(X, \mu)$ we have

$$\forall g \in D(T) : (\varphi - \varphi_n \,|\, g)_t = (\varphi - \varphi_n \,|\, Tg) \to 0,$$

which implies $\varphi_n \to \varphi$ weakly in $(D, (\cdot|\cdot)_t)$. ▨

We want to remark that, in our opinion, the implication (i) ⇒ (ii) of the above theorem is more surprising than the other implications, and that the proof of (iii) ⇒ (i) is very reminiscent of the proof of the theorem in [6]. Roughly speaking, Theorem 1.4 states that V is $U(\cdot)$-regular if and only if it is locally integrable t-q.e. (t-quasi everywhere, i.e. outside of sets with t-capacity zero).

1.6. THEOREM. *Assume* (A). *Then the following statements are equivalent:*

(i) *V is $U(\cdot)$-admissible or, equivalently, $D \cap D(t_V)$ is dense in $L_2(X, \mu)$.*

(ii) *For each $\varepsilon > 0$ there exist a closed set $A \subset X$ and an open set $U \subset X$ such that $\mu(A) = 0$, $c_t(U) < \varepsilon$ and $\chi_{X \setminus U} V \in L_{1,\text{loc}}(X \setminus A)$.*

(iii) *For each $\varepsilon > 0$ there exist a closed set $A \subset X$ and $f \in D$ such that $\mu(A) = 0$, $0 \leqslant f \leqslant 1$, $\|f\|_t < \varepsilon$ and $(1 - f)V \in L_{1,\text{loc}}(X \setminus A)$.*

Proof. (i) ⇒ (ii) It suffices to prove (ii) with X replaced by X_k. In fact, let $\varepsilon > 0$ and choose $U^{(k)}, A^{(k)} \subset X_k$ such that $\mu(A^{(k)}) = 0$, $c_t(U^{(k)}) < \varepsilon \cdot 2^{-k}$ and $\chi_{X \setminus U^{(k)}} V \in L_{1,\text{loc}}(X \setminus A^{(k)})$.

$$U := \bigcup_{k \in \mathbf{N}} U^{(k)} \text{ is open and } c_t(U) \leqslant \sum_{k \in \mathbf{N}} c_t(U^{(k)}) < \varepsilon$$

$$A := X \setminus \Big(\bigcup_{k \in \mathbf{N}} X_k \setminus A^{(k)}\Big) \text{ is closed and } \mu(A) = 0.$$

Moreover, if $K \subset X \setminus A$, K compact then $K \subset \bigcup_{i=1}^{N} X_i \setminus A^{(i)}$. Therefore $\chi_k \cdot \chi_{X \setminus U} V \in L_1(X)$. Hence $\chi_{X \setminus U} V \in L_{1,\text{loc}}(X \setminus A)$. In order to prove the assertion for X_k pick a sequence $(\varphi_n) \subset D \cap D(t_V)$ $0 \leqslant \varphi_n \leqslant 1$ such that $\varphi_n \to \chi_{X_k}$ in $L_2(X, \mu)$. By Lemma 1.4(b) we may choose an open set U with $c_t(U) < \varepsilon$ and $\tilde{\varphi}_n$ such that $\tilde{\varphi}_n(X \setminus U)$ is continuous. Let $A_n := [\varphi_n \leqslant 1/2] \cap X_k \setminus U$ which is closed, since $[\tilde{\varphi}_n \leqslant 1/2]$ is relatively closed in $X \setminus U$. $A := \bigcap_{n \in \mathbf{R}} A_n$ is closed and $\mu(A) = 0$, since $\varphi_n \to 1$. Since $\chi_{X \setminus U} \cdot V \cdot \chi_{X \setminus A_i} \leqslant \chi_{X \setminus U} 4\varphi_i^2 V \in L_1$, we have that $\chi_{X \setminus U} \cdot V \in L_{1,\text{loc}}(X \setminus A)$.

(ii) ⇒ (iii) Choose A and U according to (ii) for $\varepsilon > 0$. Then $f = e_U$ and A satisfy $\|f\|_t \leqslant \sqrt{\varepsilon}$ and $(1 - f)V \in L_{1,\mathrm{loc}}(X \setminus A)$.

(iii) ⇒ (i) Let $\varphi \in D \cap C_c(X)$ such that $0 \leqslant \varphi \leqslant 1$. It suffices to prove the existence of a sequence $(\varphi_n) \subset D \cap D(t_V)$ such that $\varphi_n \to \varphi$ in $L_2(X, \mu)$. By (iii) there exist sequences $(f_n) \subset D$ and (A_n) of closed sets such that $(1 - f_n)V \in L_{1,\mathrm{loc}}(X \setminus A_n)$.

For $n \in \mathbf{N}$ let $\psi_n \in D \cap C_c(X \setminus A_n)$, $0 \leqslant \psi_n \leqslant \varphi$ such that $\psi_n \to \varphi$ in $L_2(X, \mu)$ (such a sequence exists by Remark 1.2 and since $\mu(A_n) = 0$). Define

$$\varphi_n := \psi_n \wedge (1 - f_n) \in D \cap D(t_V)$$

(by construction) and $\varphi_n \to \varphi$ in $L_2(X, \mu)$ by the dominated convergence theorem. ▨

2. PERTURBATION OF DIRICHLET FORMS BY NEGATIVE POTENTIALS

In this section we shall treat the perturbation of the given regular Dirichlet form t by a negative measurable $V_0 : X \to [-\infty, 0]$. We denote $V_0^{(n)} := V_0 \wedge (-n)$. Following [7], Definition 2.2(b) we say that V_0 is *$U(\cdot)$-admissible*, if

$$\text{s-lim} \exp t(-T - V_0^{(n)}) =: U_0(t)$$

exists for $t \geqslant 0$ and $(U_0(t)\,;\, t \geqslant 0)$ defines a C_0-semigroup, whose generator will be denoted by $-T_0$.

2.1. Proposition (cf. [7], Proposition 5.7). *Let V_0: $X \to [-\infty, 0]$ be measurable.*

(a) *Then are equivalent:*

(i) *V_0 is $U(\cdot)$-admissible and $T_0 \geqslant c$.*

(ii) *$t + t_{V_0^{(n)}} \geqslant c$ for $n \in \mathbf{N}$.*

(iii) *$t_{-V_0} \leqslant t - c$ in the sense of forms*, i.e., *$D(t_{-V_0}) \supset D$ and the inequality holds pointwise.*

In this case $t + t_{V_0^{(n)}} \to t_0$ in strong resolvent sense, where t_0 is the form corresponding to T_0.

Moreover, D is a dense subspace of $(D(t_0), (\cdot\,|\,\cdot)_{t_0})$, and

$$t_0[f] \leqslant t[f] + \int V_0(x)\,|f(x)|^2\,\mathrm{d}\mu(x) \quad (f \in D).$$

If, furthermore, the right hand side of the above inequality defines a closable form, then its closure equals t_0.

Proof. For the proof of the equivalence we refer to the proof of [7], Proposition 5.7. Since t_0 is the regular part of $t - t_{-V_0} \mid D$ (cf. [5], Theorem 3.2) the other assertions follow. ▧

For the rest of this section we fix

Let $V_0 : X \to [-\infty, 0]$ be measurable and $U(\cdot)$-admissible and assume furthermore that

(B)
$$t_0[f, g] = t[f, g] + \int V_0(x) f(x) \bar{g}(x) \, d\mu(x)$$

holds for $f, g \in D$. Denote by T_0 and $(U_0(t) ; t \geqslant 0)$ the selfadjoint operator and semigroup corresponding to t_0, respectively.

Since t is regular it follows from Proposition 2.1(iii) that $V_0 \in L_{1,\mathrm{loc}}(X, \mu)$.

We are now able to state the main theorem of this section.

2.2. THEOREM. *Assume* (A) *and* (B).

(a) *V is $U(\cdot)$-admissible $\Leftrightarrow$ V is $U_0(\cdot)$-admissible.*

(b) *V is $U(\cdot)$-regular $\Leftrightarrow$ V is $U_0(\cdot)$-regular.*

For the proof of the theorem we need some preliminary results.

2.3. LEMMA. *Assume* (A) *and* (B). *Then $f \in D(t_0)$ implies* $\mathrm{Re}\, f, |f| \in D(t_0)$ *and* $t_0[\mathrm{Re} f] \leqslant t_0[f]$, $t_0[|f|] \leqslant t_0[f]$.

Proof. By assumption, t is a Dirichlet form, so that [4], Theorem XIII.50 implies that $U(t)$ is positivity preserving for $t \geqslant 0$. Hence $\exp t(-T - V_0^{(n)})$ is positivity preserving, which extends to $U_0(t)$ for $t \geqslant 0$ by strong convergence. Using [4], Theorem XIII.50, again we get the assertion. ▧

The next Proposition is crucial for the proof of Theorem 2.2.

2.4. PROPOSITION. *Assume* (A) *and* (B) *and let* $L := \{f \in D(t_0) \cap L_\infty(X, \mu) :$ $:\mathrm{supp} f$ *compact*$\}$. *Then $L \subset D$ and L is a dense subspace of* $(D(t_0), \|\cdot\|_{t_0})$. *More precisely, for each $f \in D(t_0)$ there exists a sequence $(\varphi_n) \subset L$ such that $|\varphi_n| \leqslant |f|$ and $\varphi_n \to f$ in $(D(t_0), \|\cdot\|_{t_0})$.*

Proof. Let $\varphi \in L$. Using Lemma 2.3 we assume without restriction φ to be real valued and nonnegative. Let $\psi \in D \cap C_c(X)$ such that $\psi \geqslant \varphi$ (which exists by the regularity of t). Since D is dense in $(D(t_0), \|\cdot\|_{t_0})$ there exists a sequence $(\psi_n) \subset D$ such that $\psi_n \to \varphi$. Again using Lemma 2.3 and the regularity of t we conclude that $\varphi_n := (\psi_n \vee 0) \wedge \psi \in D$ for each $n \in \mathbf{N}$ and, moreover, $(\|\varphi_n\|_{t_0})$ is bounded. Hence, $\varphi_n \to \varphi$ weakly in $(D(t_0), (\cdot \mid \cdot)_{t_0})$, and $(\varphi_n) \subset \{f \in D : |f| \leqslant \psi\}$.

Since the last set is convex, there exists a sequence $(g_n) \subset \{f \in D : |f| \leqslant \psi\}$ which converges to φ in $(D(t_0), (\cdot|\cdot)_{t_0})$.

$$t[\varphi_n - \varphi_m] = t_0[\varphi_n - \varphi_m] - \int V_0(x)\, |\varphi_n(x) - \varphi_m(x)|^2 \, d\mu(x).$$

Since

$$V_0(x)|\varphi_n(x) - \varphi_m(x)|^2 \leqslant 2V_0(x)|\psi(x)|^2 \quad (n, m \in \mathbf{N})$$

and $V_0|\psi|^2 \in L_1(X, \mu)$ we may pass over to a subsequence (φ_n) such that the right hand side of the last inequality tends to zero as $n, m \to \infty$. Using that t is closed it follows that $f \in D$.

As D is dense in $D(t_0)$, so is $D \cap C_c(X)$. For real valued, nonnegative $f \in D(t_0)$ (which we may assume using Lemma 2.3) there exists $(\varphi_n) \subset D \cap C_c(X)$ such that $\varphi_n \to f$ in $D(t_0)$. Using Lemma 2.3 again we find that $f_n := f \wedge \varphi_n \in L$ and $(t_0[f_n])$ is bounded. Mimicking the above given arguments for the convex set $\{h \in L : |h| \leqslant f\}$ the assertion follows. ▨

We think that Proposition 2.4, which provides the heart of the proof of Theorem 2.2 might be of interest of its own. In fact, the following corollary is a generalization of a result of Cycon. One sees immediately how one can deduce Theorem 2 in [1] from Corollary 2.5 by using mollifiers. Even in the case of Schrödinger forms, this provides a more general result; in [1] the negative part of the potential must satisfy (C′), which is more restrictive than (C) (see Section 3, take V as in (C″), V not locally square integrable). What seems to be more interesting is, in our opinion, the different method of proof.

2.5. COROLLARY. *Assume* (A) *and* (B) *and let* $L := \{f \in D(t_0) \cap L_\infty(X, \mu):$ $:\operatorname{supp} f$ *compact*$\}$. *Then* $L \cap D(t_V)$ *is dense in* $D(t_0 + t_V)$. *More precisely, for each* $f \in D(t_0) \cap D(t_V)$ *there exists a sequence* $(\varphi_n) \subset L$ *such that* $|\varphi_n| \leqslant |f|$ *and* $\varphi_n \to f$ *in* $(D(t_0 + t_V), \|\cdot\|_{t_0 + t_V})$.

Proof of Theorem 2.2. (a) "⇒" is clear, since $D(t_0) \supset D$. "⇐" follows from Proposition 2.4 and Corollary 2.5.

(b) "⇒" Let $f \in D(t_0)$. By Proposition 2.4 we may assume without restriction $f \in L$. By assumption there exists $(f_n) \subset D \cap D(t_V)$ such that $f_n \to f$ in $(D, \|\cdot\|_t)$. Without restriction $|f_n| \leqslant |f|$.

$$t_0[f - f_n] = t[f - f_n] + \int V_0(x)\, |f_n(x) - f(x)|^2 \, d\mu(x).$$

Using the dominated convergence theorem (for a pointwise converging subsequence of (f_n), if necessary), it follows that

$$f_n \to f \quad \text{in} \quad (D(t_0), \|\cdot\|_{t_0}).$$

"$\Leftarrow$" By assumption $D(t_0) \cap D(t_V)$ is dense in $D(t_0)$. Using 2.5, this implies that $L \cap D(t_V)$ is dense in $D(t_0)$ and, moreover, that for each $f \in L$ there exists a sequence $(f_n) \subset L \cap D(t_V)$ such that $|f_n| \leqslant |f|$ and $f_n \to f$ in $D(t_0)$. Using the dominated convergence theorem again, we claim that $f_n \to f$ in $(D, \|\cdot\|_t)$ which concludes the proof since L is dense in D. ∎

3. APPLICATION TO SCHRÖDINGER FORMS

In this section we are going to state our main results Theorem 3.2 and Theorem 3.3. In what follows we let $X := \Omega$ an open subset of $\mathbf{R}^\nu$, equipped with the ν-dimensional Lebesgue measure. Moreover, we will generally assume

(C) Let $D(h) = \mathring{W}^{1,2}(\Omega)$, $h[f,g] := \sum_1^\nu (\partial_i f \mid \partial_i g)$, *and* $V_1 \in L_{1,\text{loc}}(\Omega)$ be real valued, such that (B) is satisfied with $V_0 := -(V_1)^-$ and $t := h + t_{(V_1)^+}$, i.e. $(h + t_{(V_1)^+} - t_{(V_1)^-})$ is semibounded and closable. We denote $h_1 := h + t_{(V_1)^+}$ and by h_0 the closure of $h_1 - t_{(V_1)^-} \mid D(h_1)$.

Obviously h and h_1 are regular Dirichlet forms. The associated operators and semigroups are denoted by $-\Delta$, H_1, H_0 and $U(\cdot)$, $U_1(\cdot)$, $U_0(\cdot)$, respectively.

3.1. Remark. Assume

(C′) $V_1 \in L_{2,\text{loc}}(\Omega)$ real valued, such that $-\Delta + V_1 \mid C_c^\infty(\Omega)$ is semibounded

or

(C″) $V_1 \in L_{1,\text{loc}}(\Omega)$ real valued, such that $(V_1)^-$ is $-\Delta$-form small.

Then V_1 satisfies (C).

Next, we introduce the classical (1.2)-capacity (cf. [3] for an alternative definition). For an open subset $U \subset \mathbf{R}^\nu$ let

$$c_{1,2}(U) := \inf\{\|f\|_{1,2}^2 = \sum_{i=1} \|\partial_i f\|^2 + \|f\|^2 : f \in W^{1,2}(\mathbf{R}^\nu),\ f \geqslant \chi_U\}.$$

For arbitrary A let

$$c_{1,2}(U) := \inf\{c_{1,2}(U) : U \supset A,\ U \text{ open}\}.$$

Since $c_{1,2}$ is the capacity of a regular Dirichlet form (h as above for the special case $\Omega = \mathbf{R}^\nu$) the assertions of Lemma 1.4 hold for the (1.2)-capacity. The main theorems of this section read as follow:

3.2. THEOREM. *Assume* (C) *and let* $V:\Omega \to [0:\infty]$ *be measurable. Then the following statements are equivalent*

(i) V *is* $U_0(\cdot)$*-admissible.*

(ii) V *is* $U(\cdot)$*-admissible.*

(iii) V *is* $U_1(\cdot)$*-admissible.*

(iv) *For each* $\varepsilon > 0$ *there exists a closed set* $A \subset \mathbf{R}^\nu$ *and an open set* $U \subset \mathbf{R}^\nu$ *such that* $\lambda(A) = 0$, $c_{1,2}(U) < \varepsilon$ *and* $\chi_{X\setminus U} V \in L_{1,\mathrm{loc}}(\Omega\setminus A)$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is an immediate consequence of Theorem 2.2. (Note that we may replace t by h_1 and t_0 by h since $-(V_1)^+ := V_0$ satisfies (B).) (ii) $\Rightarrow$ (iv), since $c_{1,2}(U) \leqslant c_h(U)$ for any open set $U \subset \Omega$. (iv) $\Rightarrow$ (ii) is proved by mimicking the proof of (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) of Theorem 1.6. ▨

With analogous arguments one finds

3.3. THEOREM. *Assume* (C) *and let* $V : \Omega \to [0;\infty]$ *be measurable. Then the following statements are equivalent:*

(i) V *is* $U_0(\cdot)$*-regular.*

(ii) V *is* $U(\cdot)$*-regular.*

(iii) V *is* $U_1(\cdot)$*-regular.*

(iv) *For each* $\varepsilon > 0$ *there exists an open set* $U \subset \mathbf{R}^\nu$ *such that* $c_{1,2}(U) < \varepsilon$ *and* $\chi_{X\setminus U}\cdot V \in L_{1,\mathrm{loc}}(\Omega)$.

As indicated in the introduction we shall now prove that for a $U(\cdot)$-regular V, $H_0 \dotplus V$ is a restriction of the "maximal operator" $H_{\max} := \{f \in L_2(\Omega) : \Delta f, V_1 f, Vf \in L_{1,\mathrm{loc}}(\Omega), \Delta f + V_1 f + Vf =: H_{\max} f \in L_2(\Omega)\}$, under suitable conditions on V_1.

3.4. THEOREM. *Assume* (C) *and let* V *be* $U(\cdot)$*-regular.*

(a) *For* $f \in D(H_0 \dotplus V)$ *we have* $Vf \in L_{1,\mathrm{loc}}$ *and* $H_0 f \in L_{1,\mathrm{loc}}$ *in the distributional sense*, i.e. *there exists a* $g \in L_{1,\mathrm{loc}}(\Omega)$ *such that* $\langle f, H_0\varphi\rangle = \langle g, \varphi\rangle$ $(\varphi \in C_c^\infty(\Omega))$. *Moreover*

$$(H_0 \dotplus V)f = H_0 f + Vf.$$

(b) *If moreover* V_1 *is as in* (C′) *or* (C″), *then*

$$H_0 \dotplus V \subset H_{\max}.$$

Proof. Let $f \in D(H_0 \dotplus V)$, $\varphi \in C_c^\infty(\Omega) \subset D(h_0)$. Since V is regular we find a sequence $(f_n) \subset D(h_0) \cap D(t_V)$ such that $f_n \to \varphi$ in $D(h_0)$. Using 2.5 we may assume $|f_n| \leqslant |\varphi|$. Hence

$$((H_0 \dotplus V)f\,|\,f_n) = h_0[f, f_n] + t_V[f, f_n],$$

which implies

$$\lim_{n\to\infty}\int_\Omega V(x)f(x)\bar{f}_n(x)\,dx = \lim_{n\to\infty}\{((H_0 \dotplus V)f\,|\,f_n) - h_0[f,f_n]\} =$$

$$= ((H_0 \dotplus V)f\,|\,\varphi) - h_0[f,\varphi]$$

exists. Since φ was arbitrary, it follows that $Vf \in L_{1,\mathrm{loc}}(\Omega)$ as well as

$$H_0 f = (H_0 \dotplus V)f - Vf \in L_{1,\mathrm{loc}}(\Omega)$$

in the distributional sense.

If moreover V_1 is as in (C′) or (C″), then $V_1 f \in L_{1,\mathrm{loc}}(\Omega)$, so that we may conclude

$$H_0 f = -\Delta f + V_1 f$$

in the distributional sense, which proves (b). ▧

We want to remark that Theorem 3.4 and its proof are very reminiscent of [7], Theorem 7.4 and the proof given there. The assumptions in 7.4 concerning the negative part of the potential are more restrictive. Take, for example, $V(x) = -|x|^{-2}$ in $\Omega = \mathbf{R}^5$. Then V satisfies (C′) but $V \notin \hat{K}_5$.

It is, in our opinion, an interesting question, whether (b) of Theorem 3.4 holds without the additional requirement.

Concluding the final section we would like to point out another problem: The reader will have noticed so far, that the assumption "$h_1 - t_{V_1^-}$ is closable" was crucial for this section. It might be as well that $h_1 - t_{V_1^-}$ is closable whenever it is semibounded, but we were not able to prove that.

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