ABSENCE OF ABSOLUTELY CONTINUOUS SPECTRA FOR MULTIDIMENSIONAL SCHRÖDINGER OPERATORS WITH HIGH BARRIERS

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ABSTRACT

We prove absence of absolutely continuous spectra for multidimensional Schrödinger operators with high barriers. The result is formulated in terms of a geometric condition on the barriers which entails singular spectrum. The proof combines probabilistic and functional analytic techniques.

1. *Introduction*

In [**10**], Simon and Spencer exhibited two typical situations in which Schrödinger operators have singular spectrum, namely when their potential has 'wide barriers' or 'high barriers'. For the case of wide barriers, their results were formulated for discrete Schrödinger operators. Similar results for continuum operators were proven in [**13**] for dimension $d = 1$, in [**1**] and [**12**] for arbitrary dimension; [**13**, **12**] contain somewhat more refined results in terms of a comparison criterion, allowing one to prove absence of absolute continuity also at high energies. In this note we treat multidimensional Schrödinger operators with high barriers, thus extending the $d = 1$ results of [**10**, **4**, **14**]. While our general method of proof is that of [**10**], there are some differences which we would like to point out. Since adding a Dirichlet boundary condition is no longer a finite rank perturbation in $d \geqslant 2$, the necessary trace class estimates become more involved. Our key to establishing such estimates (apart from a factorization technique introduced in [**11**] and developed in [**12**]) is Lemma 3 below, which deals with occupation times of Brownian motion and gives a quantitative version of the fact that Brownian particles which hit a set 'stay around for some time'. This will allow us to estimate in trace norm the effect of an additional Dirichlet boundary condition where the original potential is large. As the use of occupation times already indicates, we employ the Feynman–Kac formula for this purpose.

To give a flavour of our main result, let us formulate it for the special case where V has spherical barriers of radius R_n, width w_n and height h_n. If these barriers (not necessarily concentric) divide $\mathbb{R}^d$ into bounded components, and if

$$\sum_n R_n^{d-1} e^{-\eta w_n \sqrt{(h_n)}} < \infty$$

for some $\eta < 1/8\sqrt{2}$, then

$$\sigma_{\mathrm{ac}}(-\tfrac{1}{2}\Delta + V) = \varnothing.$$

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In this summability condition there is an additional feature, which, of course, does not occur in the one-dimensional case: besides height and width there is also a dependence on the surface area of the barrier. For general barriers we measure this by the generalized area, introduced at the beginning of the following section.

Although we prove absence of absolute continuity by comparing $-\frac{1}{2}\Delta+V$ with an operator with pure point spectrum, we cannot expect this kind of spectrum for the general potentials which are covered by our main theorem. This is known from several results in one dimension: in **[5]** it was shown for a class of potentials on the half line $(0,\infty)$ that dense pure point spectrum occurs for almost every boundary condition at 0. Nevertheless, a general result announced in **[2]** says that for some boundary conditions these potentials will yield purely continuous spectrum. Finally, a result in **[3]** states that for other special cases of the Simon–Spencer class, which are closely related to the examples given in **[6]**, the spectrum is singular continuous for every boundary condition. In these examples the singular continuity will be stable even under perturbation by a compactly supported potential.

2. *The results*

We shall consider the Schrödinger operator $H=-\frac{1}{2}\Delta+V$ in $L_2(\mathbb{R}^d)$, where $V_+\in L^1_{\text{loc}}(\mathbb{R}^d)$ and $V_-\in K_d$, the d-dimensional Kato class (compare **[9]** for the relevant definitions). In the sequel we say that the potential V has *barriers of form S_n, height h_n and width w_n*, if

(B.1) the S_n, for $n\in\mathbb{N}$, are compact subsets of $\mathbb{R}^d$ with Lebesgue measure zero and such that $\mathbb{R}^d\setminus\bigcup_{n\in\mathbb{N}}S_n$ is open and has only bounded connected components,

(B.2) $V(x)\geqslant h_n$ if $\operatorname{dist}(x,S_n)\leqslant w_n/2$, $n\in\mathbb{N}$.

We also use the *generalized area* $\sigma(S)$ for a compact subset S of $\mathbb{R}^d$:

$$\sigma(S)=\sup_{r\geqslant 0}\frac{\operatorname{meas}\{x;r\leqslant\operatorname{dist}(x,S)\leqslant r+1\}}{r^d+1}.$$

Essentially the same definition of $\sigma(S)$ and some discussion of this notion were given in **[12]**.

THEOREM. *Assume that V has barriers of form S_n, height h_n and width w_n. If*

$$\sup_n\frac{w_n}{\surd(h_n)}<\infty \tag{C.1}$$

and

$$\sum_n\sigma(S_n)\exp(-\eta w_n\surd(h_n))<\infty \tag{C.2}$$

for some $\eta<1/8\surd 2$, then

$$\sigma_{\text{ac}}(-\tfrac{1}{2}\Delta+V)=\varnothing.$$

Note that by the definition of $\sigma(S)$ we have $\sigma(S_n)\geqslant C>0$ uniformly in n. Therefore (C.2) implies $w_n\surd(h_n)\to\infty$ and, together with (C.1), $h_n\to\infty$, meaning the existence of high barriers. A typical special case of (C.1) is $h_n\to\infty$ and w_n bounded. In case of unbounded w_n, that is, wide barriers, the results of **[12]** will often apply.

Before going into the proof of the Theorem, we collect together some basic facts in three lemmas. The first gives a general trace class criterion for operators in L_2 over some σ-finite measure space. $\mathfrak{B}$ denotes the bounded operators.

LEMMA 1. *Let $T \in \mathfrak{B}(L_1, L_2)$, $S \in \mathfrak{B}(L_2, L_1)$ and $\Phi \in L_1$ such that*

$$|Sf(x)| \leqslant \Phi(x)$$

for a.e. x and all $f \in L_2$ with $\|f\|_2 \leqslant 1$. Then TS is trace class with trace norm

$$\|TS\|_{\mathrm{tr}} \leqslant \|\Phi\|_1 \|T\|.$$

Proof. See [**12**, Lemma 2.1].

In the sequel we shall use probabilistic methods, and thus we let $(\Omega^x, \mathbb{P}^x, (X_t)_{t \geqslant 0})$ denote Brownian motion starting in $x \in \mathbb{R}^d$ with expectation $\mathbb{E}^x$ (see [**8**, II.4] for background information). For $\omega \in \Omega^x$ and $S \subset \mathbb{R}^d$, the *first hitting time* of S is given by $\tau_S(\omega) = \inf\{t > 0; X_t(\omega) \in S\}$.

LEMMA 2. (a) *Given $t > 0$ and $\varepsilon > 0$, there exists $C = C(d, t, \varepsilon)$ such that for all $r > 0$*

$$\mathbb{P}^0[\tau_{\{y;|y|=r\}} \leqslant t] \leqslant C \exp\left(-\frac{r^2}{(2+\varepsilon)t}\right).$$

(b) *For $t > 0$ and $p > 0$, there exists $C = C(d, t, p)$ such that for all compact $S \subset \mathbb{R}^d$*

$$\int_{\mathbb{R}^d} (\mathbb{P}^x[\tau_S \leqslant t])^p \, dx \leqslant C\sigma(S).$$

Proof. (a) By the reflection principle (compare [**8**, Theorem 3.6.5, p. 25]),

$$\mathbb{P}^0[\tau_{\{y;|y|=r\}} \leqslant t] \leqslant 2\mathbb{P}^0[|X_t| \geqslant r] \leqslant 2(2\pi t)^{-d/2} \int_{|y| \geqslant r} \exp\left(-\frac{|y|^2}{2t}\right) dy,$$

which yields the required estimate.

(b) Since $\mathbb{P}^x[\tau_S \leqslant t] \leqslant \mathbb{P}^0[\tau_{\{y;|y|=\mathrm{dist}(x,S)\}} \leqslant t]$, part (a) with $\varepsilon = 1$ and the definition of $\sigma(S)$ imply

$$\begin{aligned}\int (\mathbb{P}^x[\tau_S \leqslant t])^p \, dx &= \sum_{n \geqslant 0} \int_{n \leqslant \mathrm{dist}(x,S) \leqslant n+1} (\mathbb{P}^x[\tau_S \leqslant t])^p \, dx \\ &\leqslant \sum_{n \geqslant 0} \sigma(S)(n^d + 1) C^p \exp\left(-\frac{n^2 p}{3t}\right) \\ &= C(d, t, p)\, \sigma(S).\end{aligned}$$

In the last of our three lemmas we estimate the probability that a particle hits S but spends only little time in a neighbourhood of S. To this end we introduce the *occupation time* of a δ-neighbourhood,

$$T_{S,\delta}(\omega) := \mathrm{meas}\{u \in (0, t); \mathrm{dist}(X_u(\omega), S) \leqslant \delta\}.$$

LEMMA 3. *Let $t > 0$, $\varepsilon > 0$. Then there exists $C = C(d, t, \varepsilon)$ such that for all $0 < \alpha < t$ and $S \subset \mathbb{R}^d$ compact,*

$$\mathbb{P}^x[\tau_S \leqslant t, T_{S,\delta} \leqslant \alpha] \leqslant C \exp\left(-\frac{\delta^2}{4(2+\varepsilon)\alpha}\right).$$

Proof. Let $B := \{x; \operatorname{dist}(x,S) \leqslant \delta\}$ and $B' := \{x; \operatorname{dist}(x,S) \leqslant \delta/2\}$. Denote by τ' the first hitting time of B', and let $T' := \inf\{s > 0; |X_{s+\tau'} - X_{\tau'}| \geqslant \delta/2\}$ on $\{\tau' < \infty\}$, and infinity otherwise. We have

$$\mathbb{P}^x[\tau_S \leqslant t, T_{S,\delta} \leqslant \alpha] \leqslant \mathbb{P}^x[\tau_S \leqslant t, \tau' \leqslant t-\alpha, T_{S,\delta} \leqslant \alpha] + \mathbb{P}^x[\tau_S \leqslant t, \tau' > t-\alpha].$$

The first summand can be estimated by

$$\mathbb{P}^x[\tau_S \leqslant t, T' \leqslant \alpha]$$

since $X_{s+\tau'} \in B$ as long as $|X_{s+\tau'} - X_{\tau'}| \leqslant \delta/2$. The second term is zero if $x \in B'$; for $x \notin B'$ it follows that $\operatorname{dist}(X_{\tau'}(\omega), S) = \delta/2$ for $\omega \in \Omega^x$, so that the second term can be majorized by

$$\mathbb{P}^x[T' \leqslant \alpha].$$

Put together, this gives

$$\mathbb{P}^x[\tau_S \leqslant t, T_{S,\delta} \leqslant \alpha] \leqslant 2\mathbb{P}^x[T' \leqslant \alpha].$$

By the strong Markov property and translation invariance of Brownian motion (compare [**8**, Theorem 7.9, p. 68]),

$$\mathbb{P}^x[T' \leqslant \alpha] \leqslant \mathbb{P}^0[\tau_{\{y;|y|=\delta/2\}} \leqslant \alpha] \leqslant C\exp\left(-\frac{\delta^2}{4(2+\varepsilon)\alpha}\right),$$

according to Lemma 2(a).

Proof of the Theorem. We define $H = -\frac{1}{2}\Delta + V$ on $L_2(\mathbb{R}^d)$ and $H_G = -\frac{1}{2}\Delta + V$ on $L_2(G)$ with Dirichlet boundary conditions, for arbitrary open $G \subset \mathbb{R}^d$, via the corresponding quadratic forms.

Fix S_n, h_n, w_n as in the assumption of the Theorem, and let

$$H_D := H_{\mathbb{R}^d \setminus \bigcup_n S_n} = \bigoplus_i H_{U_i},$$

where U_i are the connected components of $\mathbb{R}^d \setminus \bigcup_n S_n$. Since the U_i are bounded, H_{U_i} has discrete spectrum and consequently $\sigma_{ac}(H_D) = \varnothing$. Thus if we manage to construct wave operators for H, H_D, the assertion of the Theorem will be proved. By the well-known invariance principle (see, for example, [7, Corollary 4, p. 31]), it is sufficient to show that $\exp(-2tH) - \exp(-2tH_D)$ is trace class for some $t > 0$. Writing

$$H_0 := H, \quad H_n := H_{\mathbb{R}^d \setminus \bigcup_{k=1}^n S_k},$$

we have that

$$\exp(-2tH) - \exp(-2tH_D) = \sum_{n=1}^{\infty} (\exp(-2tH_{n-1}) - \exp(-2tH_n))$$

in the strong sense (by monotone form convergence). We shall now demonstrate that assumptions (C.1) and (C.2) imply that

$$\sum_{n=1}^{\infty} \|\exp(-2tH_{n-1}) - \exp(-2tH_n)\|_{\mathrm{tr}} < \infty,$$

which suffices for the desired trace class property. To estimate $\|\exp(-2tH_{n-1}) - \exp(-2tH_n)\|_{\mathrm{tr}}$, we shall apply Lemma 1, and therefore factorize

$$e^{-2tH_{n-1}} - e^{-2tH_n} = e^{-tH_{n-1}}(e^{-tH_{n-1}} - e^{-tH_n}) + (e^{-tH_{n-1}} - e^{-tH_n})\,e^{-tH_n}.$$

By (C.2) there is an $\varepsilon > 0$ such that $\sum_n \sigma(S_n) \exp((-1/8\sqrt{(2+\varepsilon)})\, w_n \sqrt{(h_n)}) < \infty$. Fix $t > \sup_n (1/8\sqrt{(2+\varepsilon)})\, w_n(h_n)^{-1/2}$ which, according to assumption (C.1), is possible. In order to apply Lemma 1, we let $f \in L_2$, $\|f\|_2 \leqslant 1$ and use the Feynman–Kac formula (compare [**8**, II.6]) to write

$$|(e^{-tH_{n-1}} - e^{-tH_n}) f(x)| = \left| \mathbb{E}^x \left[\exp\left(-\int_0^t V \circ X_s\, ds \right) f \circ X_t, \tau_{\bigcup_{k=1}^{n-1} S_k} > t, \tau_{S_n} \leqslant t \right] \right|.$$

Applying Cauchy–Schwarz twice, we obtain

$$\begin{aligned} \ldots &\leqslant \left(\mathbb{E}^x \left[\exp\left(-2\int_0^t V_+ \circ X_s\, ds \right), \tau_{S_n} \leqslant t \right] \right)^{1/2} \left(\mathbb{E}^x \left[\exp\left(2\int_0^t V_- \circ X_s\, ds \right) |f|^2 \circ X_t \right] \right)^{1/2} \\ &\leqslant (\ldots)^{1/2} \| e^{(-1/2\Delta - 2V_-)t} \colon L_1 \to L_\infty \|^{1/2} \\ &\leqslant C \cdot \left(\mathbb{E}^x \left[\exp\left(-\int_0^t 4V_+ \circ X_s\, ds \right), \tau_{S_n} \leqslant t \right] \right)^{1/4} (\mathbb{P}^x[\tau_{S_n} \leqslant t])^{1/4}. \end{aligned} \tag{3}$$

Here we have used that the norm occurring above is finite by [**9**, Theorem B.1.1, p. 460] and, again, Feynman–Kac. Lemma 2(b) gives us control on the last term on the right. To estimate the remaining factor uniformly in x, we use the occupation time estimate from Lemma 3 together with the fact that V_n is large in a $w_n/2$-neighbourboood of S_n: fix $n \in \mathbb{N}$ and let $T_n(\omega) := \operatorname{meas}\{u \in (0,t); \operatorname{dist}(X_u(\omega), S_n) \leqslant w_n/2\}$. Since $\int_0^t V_+ \circ X_s\, ds \geqslant h_n T_n$ we have, for $0 < \alpha < t$,

$$\mathbb{E}^x \left[\exp\left(-\int_0^t 4V_+ \circ X_s\, ds \right), \tau_{S_n} \leqslant t \right] \leqslant e^{-4h_n\alpha} \mathbb{P}^x[T_n > \alpha] + \mathbb{P}^x[\tau_{S_n} \leqslant t, T_n \leqslant \alpha].$$

Using Lemma 3 this can be estimated by

$$\exp(-4h_n \alpha) + C \exp\left(-\frac{(w_n/2)^2}{4(2+\varepsilon)\alpha} \right).$$

Setting $4h_n \alpha = (w_n/2)^2/4(2+\varepsilon)\alpha$, we arrive at $\alpha = (1/8\sqrt{(2+\varepsilon)})\, w_n(h_n)^{-1/2} < t$ by our assumption on t. With this choice of α, we obtain

$$\mathbb{E}^x \left[\exp\left(-\int_0^t 4V_+ \circ X_s\, ds \right), \tau_{S_n} \leqslant t \right] \leqslant C \exp\left(-\frac{w_n \sqrt{(h_n)}}{2\sqrt{(2+\varepsilon)}} \right).$$

Plugging this into (3), we find

$$|(e^{-tH_{n-1}} - e^{-tH_n}) f(x)| \leqslant C \exp\left(-\frac{w_n \sqrt{(h_n)}}{8\sqrt{(2+\varepsilon)}} \right) (\mathbb{P}^x[\tau_{S_n} \leqslant t])^{1/4} =: \Phi_n(x)$$

for all $f \in L_2$, $\|f\|_2 \leqslant 1$. Hence we can apply Lemma 2(b) to estimate

$$\|\Phi_n\|_1 \leqslant C \cdot \exp\left(-\frac{1}{8\sqrt{(2+\varepsilon)}} w_n \sqrt{(h_n)} \right) \cdot \sigma(S_n).$$

Since $\|e^{-tH_n} \colon L_1 \to L_2\| \leqslant \|e^{-tH} \colon L_1 \to L_2\|$ (monotonicity), it follows from Lemma 1 that

$$\|e^{-tH_{n-1}}(e^{-tH_{n-1}} - e^{-tH_n})\|_{\mathrm{tr}} \leqslant C \cdot \sigma(S_n) \exp\left(-\frac{1}{8\sqrt{(2+\varepsilon)}} w_n \sqrt{(h_n)} \right).$$

As $\|(e^{-tH_{n-1}}-e^{-tH_n})e^{-tH_n}\|_{tr} = \|e^{-tH_n}(e^{-tH_{n-1}}-e^{-tH_n})\|_{tr}$, we finally obtain

$$\|e^{-2tH_{n-1}}-e^{-2tH_n}\|_{tr} \leqslant C\sigma(S_n)\exp\left(-\frac{1}{8\sqrt{(2+\varepsilon)}}w_n\sqrt{(h_n)}\right),$$

which by (C.2) implies the summability of the trace norms, and hence the Theorem.

We conclude with some remarks.

(a) Assume that V has barriers of form S_n, height h_n and width w_n with the property that for each subsequence S_{n_k} the connected components of $\mathbb{R}^d/\bigcup_k S_{n_k}$ are bounded. Then (C.1) and

$$\lim_n \sigma(S_n)\exp(-\eta w_n\sqrt{(h_n)}) = 0$$

for some $\eta < (8\sqrt{2})^{-1}$ already implies that

$$\sigma_{ac}(-\tfrac{1}{2}\Delta+V) = \varnothing.$$

To see this one has only to apply the Theorem to a suitable subsequence S_{n_k}.

(b) For $d = 1$ and $S_n = \{x_n\}$ satisfying $(-1)^k x_k \to \infty$ for $k \to \infty$, we are in the situation of (a). As, moreover, $\sigma(S_n)$ is constant in this case, the summability condition of the Theorem can be weakened to

$$\lim_n w_n\sqrt{(h_n)} = \infty. \tag{4}$$

Thus we obtain as a special case of our Theorem the original result from **[10]**. (From $h_n \to \infty$ it follows that in (4) we can assume boundedness of w_n, and therefore also (C.1) is satisfied.)

(c) If the potential V is spherically symmetric, then by separation of variables the one-dimensional result of **[10]** applies directly. This gives a slightly stronger result than our Theorem, since only $h_n \to \infty$ and $w_n\sqrt{(h_n)} \to \infty$ are required, and no growth condition on the $\sigma(S_n)$.

(d) In order to obtain $\sigma_{ac} = \varnothing$, the barriers S_n need not be 'massive'. One may cut holes into the S_n (thus violating property (B.1) of barriers) as long as the holes get small quickly enough as $n \to \infty$ to allow another trace class perturbation argument. More interesting would be an answer to the following question. May the holes be big as long as the hole in S_n is far apart from that in S_{n+1}, $n = 1, \ldots$? Think of concentric S_n with holes in alternating directions.

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