LIFSHITZ ASYMPTOTICS AND LOCALIZATION FOR RANDOM QUANTUM WAVEGUIDES

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1991 Mathematics Subject Classifications: 81Q10, 47B80, 81Q15

Received 20 March 1998

We investigate a family of Dirichlet Laplacians on randomly dented or bulged strips in $\mathbb{R}^2$; for this random quantum waveguide model, dense point spectrum with exponentially localized eigenfunctions near its fluctuation boundary at the bottom of the spectrum and Lifshitz asymptotics of the integrated density of states are established. For this purpose, multi-scale analysis in the quite abstract form of [21] is applied, and domain perturbations of the Laplacian are studied.

Keywords: Quantum Waveguides, Domain Perturbations, Random Schrödinger Operators, Localization, Lifshitz Tails.

1. Introduction

The notion of "quantum waveguide" has been coined for the investigation of two or three-dimensional motion of electrons in small channels, tubes or layers of crystalline matter of high purity; one should think of possibly several thin films or lines of semiconductor materials deposited on a wafer of insulating substance by epitaxial techniques. Experiments with such mesoscopic structures, i.e. structures shapeable by an experimentalist, but open for quantum effects, reveal a dependence of their conductivity properties on their form, on bendings or varying cross sections. For references, see the physical literature cited in [5]. On the other hand, from a purely mathematical point of view, a rigorous analysis of effects of this kind appears attractive: the influence of changes of the geometry of the semiconductor structures on the spectral properties of the model, which represent conductivity properties, is to be inquired into, an intuitively clear and challenging task.

The models to be considered for this purpose are usually constituted by (minus) the Laplacian operator with Dirichlet boundary conditions on a domain of $\mathbb{R}^2$ or $\mathbb{R}^3$, for example a curved or bulged strip or tube or two parallel strips or layers coupled through a window [1, 5–8]. Now, a common feature observed in all mentioned

Reviews in Mathematical Physics, Vol. 12, No. 10 (2000) 1345–1365

geometries is the occurrence of bound states with eigenvalues below the essential spectrum, which appear successively, if the originally straight strip or layer is slowly deformed respectively a window between two of them is opened in a finite region. However, up to now, serious deformations of the given region on its full length have not been investigated, although one might readily conjecture that a deformation in infinitely many places will cause infinitely many eigenvalues or dense point spectrum with exponentially localized eigenfunctions at the bottom of the spectrum. It is a major goal of the present article to establish this conjecture for almost all elements of a family of strips in $\mathbb{R}^2$ bulged or dented randomly all over their (infinite) length. As a second point, we regard the coupling of quantum waveguide theory with the theory of random operators, which is necessary for achieving this aim, as a desirable extension of quantum waveguide theory, because our random model can be looked at as representing an epitaxial line of semiconducting material with an irregularly rough boundary. Thus, in the framework of quantum waveguide theory, we are naturally invited to the analysis of the spectral properties of such a model.

Let us now present our model in detail:

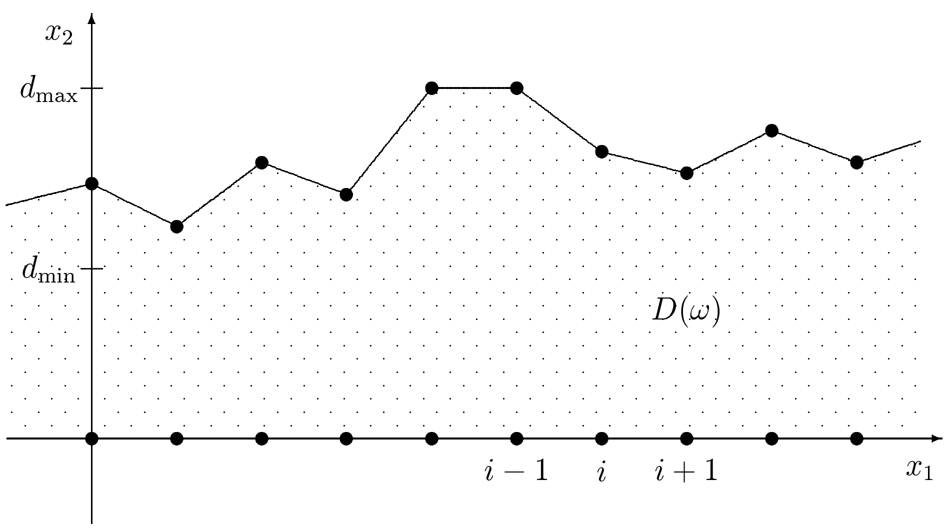
It consists of a collection of randomly dented versions of a parallel strip $\mathbb{R} \times (0, d_{\max}) = D_{\max}$. More precisely, let $d_{\max} > 0$, $0 < d < d_{\max}$, and consider $\Omega = [0, d]^{\mathbb{Z}}$. The ith coordinate $\omega(i)$ of $\omega \in \Omega$ gives the deviation of the width of the random strip from $d_{\max}$, i.e.

$$d_i(\omega) := d_{\max} - \omega(i) \,,$$

which lies between $d_{\min} = d_{\max} - d$ and $d_{\max}$. Define $\gamma(\omega) : \mathbb{R} \to [d_{\min}, d_{\max}]$ as the polygon in $\mathbb{R}^2$ joining the points $\{(i, d_i(\omega))\}_{i\in\mathbb{Z}}$ and

$$D(\omega) = \{(x_1, x_2) \in \mathbb{R}^2 | 0 < x_2 < \gamma(\omega)(x_1)\} \,.$$

The following picture will help in visualizing this domain:



We fix a probability measure μ on $[0,d]$ with $0 \in \operatorname{supp}\mu \neq \{0\}$ and introduce $\mathbb{P} = \mu^{\mathbb{Z}}$, a probability measure on Ω. Consider $H(\omega) = -\Delta_{D(\omega)}$, the Laplacian on $D(\omega)$ with Dirichlet boundary conditions, which is a self-adjoint operator in $L^2(D(\omega))$. Although the operators of the family $(H(\omega))_{\omega\in\Omega}$ act in different Hilbert-spaces, we can pull them back to $L^2(D_{\max})$ by a family of piecewise differentiable maps $\Phi_\omega : \overline{D_{\max}} \to \overline{D(\omega)}$, proceeding as in the proof of Proposition 3.2 then the resulting family of operators is ergodic, which implies that their spectra and spectral parts are deterministic in the sense that they coincide on a set of full measure in Ω. Thus, the same is true for $(H(\omega))_{\omega\in\Omega}$.

For further reference, let us record two assumptions which will be needed in the sequel:

For the proof of Lifshitz asymptotics we need:

(M1) There exist $a, \delta > 0$ such that $\mu[0,\epsilon] \geq a \cdot \epsilon^\delta$.

In our proof of localization (more precisely for the Wegner estimate) we use

(M2) μ is Hölder continuous, i.e. there exist $b, \alpha > 0$ such that $\mu(I) \leq b|I|^\alpha$ for every interval $I \subset [0,d]$; here $|I|$ denotes the length of I.

From $0 \in \operatorname{supp}\mu$ we immediately conclude that $D(\omega)$ contains rectangular boxes of length (x_1-direction) L and width $d_{\max} - \epsilon$ for small $\epsilon > 0$ and L arbitrarily large $\mathbb{P}$-almost surely. As $\inf\sigma(H(\omega))$ lies below the first eigenvalue of the Dirichlet Laplacian of such a box, this implies

$$\inf\sigma(H(\omega)) = E_0 := \frac{\pi^2}{d_{\max}^2} \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega\,.$$

In case $\operatorname{supp}\mu = [0,d]$ one could even deduce $\sigma(H(\omega)) = [E_0, \infty)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. As this does not matter in the sequel, we shall not go into details.

Our main results read as follows:

Theorem 1.1. (Lifshitz tales) *Assume* (M1). *Then the integrated density of states $N(t)$ for $H(\omega)$ satisfies*

$$\lim_{t\searrow 0} \frac{\log(-\log N(E_0+t))}{\log t} = -\frac{1}{2}\,. \tag{1.1}$$

Roughly speaking the preceeding result says that the number of electrons per unit volume (which is given by the integrated density of states) decreases very rapidly, as the bottom energy is approached. In Theorem 4.1 below we show a little bit more. Namely, the upper estimate holds without the requirement that (M1) is satisfied. The number $-\frac{1}{2}$ appearing on the rhs of inequality (1.1) is the Lifshitz exponent. Usually such an exponent is of the form $-\frac{\nu}{2}$, where ν is the dimension of the random medium in question. That fits perfectly well with our model, which is essentially one-dimensional (at least as far as the randomness is concerned). The next result contributes to one of the central topics in disordered systems, the occurence of pure point spectrum with exponentially decreasing eigenfunctions, usually called *localization*. According to the general philosophy this should happen near so-called *fluctuation boundaries*. Our model has E_0 as fluctuation boundary, and we can in fact prove:

Theorem 1.2. *Assume* (M2). *Then there exists* $\delta > 0$ *such that* $\mathbb{P}$*-a.s. the spectrum of* $H(\omega)$ *in* $[E_0, E_0 + \delta]$ *is pure point with exponentially decreasing eigenfunctions.*

By exponential decay of eigenfunctions we mean exponential decay in x_1-direction as stated explicitly in Theorem 2.1.

Let us now briefly describe the organization of the paper and comment on the techniques we use. In Sec. 2 we outline multi-scale analysis, by which the proof of Theorem 1.2 is reduced to two basic inequalities: the Wegner estimate and the initial length scale estimate. We take advantage of the abstract multi-scale analysis presented in [21], which is based on the variable energy method of von Dreifus and Klein [4]. In Sec. 3 we provide the necessary prerequisites for the proofs of the Wegner and initial length scale estimates in form of a thorough study of domain perturbations. In particular, in the framework of analytic perturbation theory, estimates on derivatives of eigenvalues will be given. Section 4 is devoted to the proof of the initial length scale and Lifshitz tail estimates. There we combine a new technique from [21] to deduce low probability for low lying eigenvalues from large deviation results with the results from Sec. 3: in particular, the derivative of the first eigenvalue, calculated in the "Hadamard–Rayleigh formula", Proposition 3.2, will play an important role. The major technical step is contained in Proposition 4.1, which is the key to the upper estimate for the integrated density of states as well as for the initial length scale estimate. The Wegner estimate is deduced in Sec. 5, where we combine the technique from [20] with estimates from Sec. 3 to prove "spreading of eigenvalues" for different ω. In the last section we comment on some possible extensions and modifications of the model presented here.

2. Outline of Multi-Scale Analysis

In this section we briefly present the variable energy multi-scale analysis, which can be used to prove localization for our random quantum wave guide model $H(\omega)$. It is based on the method developed by von Dreifus and Klein [4]. In the abstract form needed here it is taken from [21].

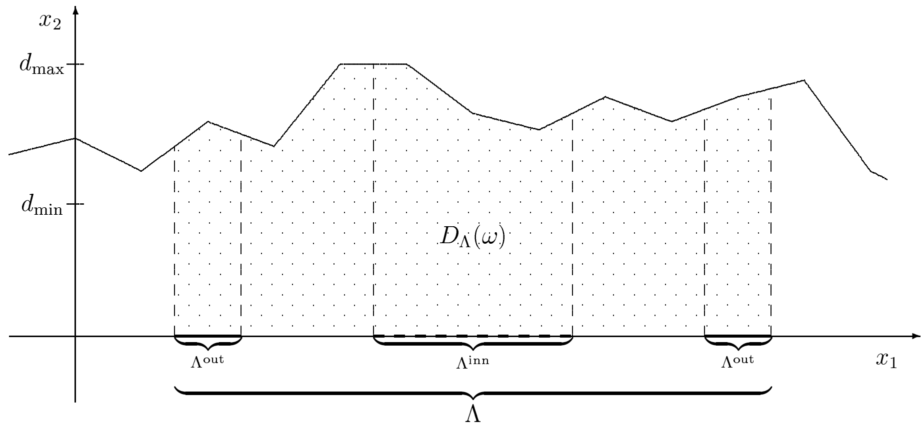
Starting point is the observation that the nature of the spectrum of $H(\omega)$ is determined by the behaviour of generalized eigenfunctions (in the sense of [19]), which are polynomially bounded in x_1-direction; this can be seen by Hilbert–Schmidt estimates for sandwiched resolvent powers as in the case of Schrödinger operators.

Assume that we can prove that for some fixed energy interval $I_0 = [E_0, E_0 + \delta]$ with $\delta > 0$ (E_0 is the bottom of the spectrum as defined in the introduction) there is a subset $\Omega_0 \subset \Omega$ of full measure such that for all $\omega \in \Omega_0$ and $E \in \sigma(H(\omega)) \cap I_0$ every generalized eigenfunction u of $H(\omega)$ is in L^2. Then it readily follows that the spectrum of $H(\omega)$ in I_0 is pure point. In principle, this is the strategy of the variable energy method. The necessary decay estimates for u will follow from exponential decay of the resolvents. To put this in precise terms, we introduce some notation.

Let $\Lambda = \Lambda_l(i) = (i - \frac{l}{2}, i + \frac{l}{2})$ be an interval centered at $i \in \mathbb{Z}$ with sidelength $l \in 2\mathbb{N}+1$. We consider

$$D_\Lambda(\omega) := D(\omega) \cap (\Lambda \times \mathbb{R})$$

for $\omega \in \Omega$ and denote by $H_\Lambda(\omega)$ the Laplacian $-\Delta$ on $D_\Lambda(\omega)$ with Dirichlet boundary conditions. See the next figure.



We let $\Lambda^{\text{inn}} := \Lambda_{\frac{l}{3}}(i)$, $\Lambda^{\text{out}} := \Lambda_l(i) \setminus \Lambda_{l-2}(i)$ and denote by $\chi_\Lambda^{\text{inn}}$, $\chi_\Lambda^{\text{out}}$ the characteristic functions of $\Lambda^{\text{inn}} \times \mathbb{R}$, $\Lambda^{\text{out}} \times \mathbb{R}$ or their restrictions to $D(\omega)$. Thus, multiplication by $\chi_\Lambda^{\text{inn}}$ and $\chi_\Lambda^{\text{out}}$ localize to the "inner third" respectively a region "near the boundary" of $D_\Lambda(\omega)$. (Notice that "inner" and "outer" only refer to the x_1-direction.) The connection between decay estimates for resolvents and decay estimates for eigenfunctions is achieved in the following lemma:

Lemma 2.1. (eigenfunction decay inequality) *There is a constant $C = C(d_{\max}, d_{\min})$ such that for every $\omega \in \Omega$ and every generalized eigenfunction u of $H(\omega)$ to $E \in [E_0, E_0+1] \cap \rho(H_\Lambda(\omega))$ we have*

$$\|\chi_\Lambda^{\text{inn}} u\| \le C \cdot \|\chi_\Lambda^{\text{out}} R_\Lambda(E) \chi_\Lambda^{\text{inn}}\| \, \|\chi_\Lambda^{\text{out}} u\|, \tag{2.1}$$

where $R_\Lambda(E) = (H_\Lambda(\omega) - E)^{-1}$.

This is proven by standard commutator estimates and estimates for weak solutions of second order pde in the fashion of [9], Lemma 26, Lemma 27. By inequality (2.1) it is quite clear that exponential estimates for $\|\chi_\Lambda^{\text{out}} R_\Lambda(E) \chi_\Lambda^{\text{inn}}\|$ can be turned into exponential estimates (in x_1-direction) for generalized eigenfunctions. The workhorse result is the following Theorem, which follows from the more abstract and general results of [21]. We denote by χ_x the characteristic function of $(x - \frac{1}{2}, x + \frac{1}{2}) \times \mathbb{R}$.

Theorem 2.1. *Let $H(\omega)$ be as in the introduction. Assume*

(i) *the Wegner estimate: there exist $\alpha > 0$, $C > 0$ such that for all intervals $I \subset [E_0, E_0+1]$, $\Lambda = \Lambda_l(i)$:*

$$\mathbb{P}\{\sigma(H_\Lambda(\omega)) \cap I \neq \emptyset\} \le C \cdot |\Lambda|^2 \cdot |I|^\alpha$$

and

(ii) *the initial length scale estimate*: *there exist* $\beta \in (0,1)$, $\xi > 0$ *such that for all* $\Lambda = \Lambda_l(i)$:

$$\mathbb{P}\{E_1(H_\Lambda(\omega)) \le E_0 + l^{\beta-1}\} \le l^{-\xi} .$$

Then there exists $\delta > 0$ *such that* $\mathbb{P}$*-a.s. the spectrum of* $H(\omega)$ *is pure point in* $I_0 := [E_0, E_0 + \delta]$. *Moreover, there exists* $\gamma > 0$ *such that for* $\mathbb{P}$*-a.e.* ω *and every* $E \in I_0 \cap \sigma(H(\omega))$ *there is a* $C = C(\omega, E)$ *with*

$$\|u\chi_x\| \le C \cdot \exp(-\gamma \cdot |x|) \tag{2.2}$$

for every generalized eigenfunction u *of* $H(\omega)$ *to* E.

Let us add a few words concerning the proof: in [21], Sec. 11, it is shown how to construct I_0, a sequence (l_k) of rapidly increasing length scales and γ such that for all $x, y \in \mathbb{Z}$ with $\mathrm{dist}(\Lambda_{l_k}(x), \Lambda_{l_k}(y)) \ge 2$

$$\mathbb{P}\{\omega | \text{ for all } E \in I_0 \text{ either } \Lambda_{l_k}(x) \text{ or } \Lambda_{l_k}(y) \text{ is } (\gamma, E)\text{-good for } \omega\} \ge 1 - l_k^{-2\xi} . \tag{2.3}$$

Here Λ is called (γ, E)-good for ω provided

$$\|\chi_\Lambda^{\mathrm{inn}}(H_\Lambda(\omega) - E)^{-1}\chi_\Lambda^{\mathrm{out}}\| \le \exp(-\gamma \cdot |\Lambda|) .$$

Apart from the Wegner estimate and the initial length scale estimate one only needs to know some basic properties, which are obviously satisfied for $H_\Lambda(\omega)$:

- A Weyl type estimate for the number of eigenvalues below a fixed energy; in our case, it can be calculated directly.
- Independence of H_{Λ_1}, H_{Λ_2} for $\mathrm{dist}(\Lambda_1, \Lambda_2) \ge 2$.
- A geometric resolvent identity relating $(H_{\Lambda'} - E)^{-1}$ to $(H_\Lambda - E)^{-1}$ for different cubes of the following form: there exists C such that for cubes $\Lambda \subset \Lambda'$, $A \subset \Lambda^{\mathrm{inn}} \times \mathbb{R}$, $B \subset (\Lambda' \setminus \Lambda) \times \mathbb{R}$ and $E \in [E_0, E_0 + 1]$

$$\|\chi_B(H_{\Lambda'}(\omega) - E)^{-1}\chi_A\| \le C\|\chi_B(H_{\Lambda'}(\omega) - E)^{-1}\chi_\Lambda^{\mathrm{out}}\| \cdot \|\chi_\Lambda^{\mathrm{out}}(H_\Lambda(\omega) - E)^{-1}\chi_A\| .$$

From (2.3) and (2.1) one can then deduce (2.2), which in turn implies that all polynomially bounded generalized eigenfunctions corresponding to energies near E_0 are in L^2. This is carried out in Sec. 11 of [21]. See also [4] for the case of discrete Schrödinger operators as well as [14], Secs. 4 and 5, for a discussion of these steps in the context of continuum Schrödinger operators.

The proof of our main result will thus be complete once we have established a Wegner estimate and an initial length scale estimate, i.e. verified assumptions (i) and (ii) of Theorem 2.1. This will be done in Secs. 3 and 4, after having treated some preparatory results concerning domain perturbations in the following section.

3. Domain Perturbations

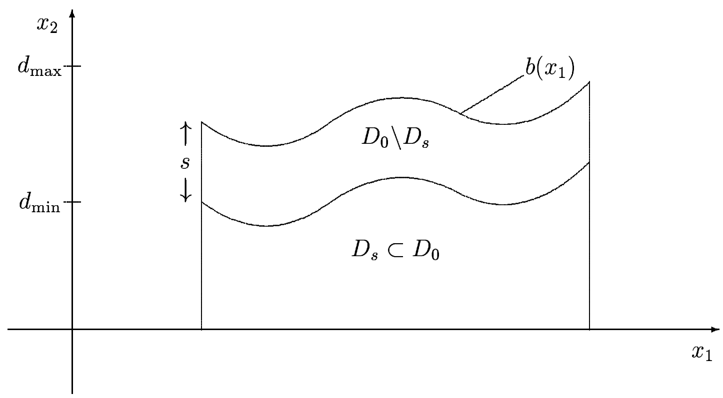
In this section, the necessary domain perturbation theory is developed, quite in the spirit of [11, 10, 18] and [12], VII.§6. As this theory is not too widely known, we will give the proofs in some detail, yet adapted to the cases which we are interested in.

For $l \in 2\mathbb{N}+1$, a measurable function $b : \Lambda = \Lambda_l(i) = (i-\frac{l}{2}, i+\frac{l}{2}) \to [d_{\min}, d_{\max}]$, $i \in \mathbb{Z}$, $0 < d_{\min} < d_{\max}$, and $0 \leq s < \min_{x_1 \in \Lambda} b(x_1)$ we consider the bounded domain

$$D_s := \{(x_1, x_2) \,|\, x_1 \in \Lambda, 0 < x_2 < b(x_1) - s\} \subset \mathbb{R}^2$$

and the Dirichlet Laplacian $-\Delta_{D_s}$ in $L^2(D_s)$, the self-adjoint operator associated with the closure h_s of the form

$$h_s^0[f, g] = \int_{D_s} \nabla f \nabla \bar{g} \, dx \text{ with domain } \mathcal{C}_0^\infty(D_s) \, .$$



It is well known that $-\Delta_{D_s}$ has purely discrete spectrum, and

$$\cdots \geq E_n(s) \geq E_{n-1}(s) \geq \cdots \geq E_1(s) > E_0(s) > 0$$

will denote its eigenvalues. As $\mathcal{C}_0^\infty(D_s) \subset \mathcal{C}_0^\infty(D_0)$ and therefore $h_s \geq h_0$ as forms in $L^2(D_0)$, $E_n(s) \geq E_n(0)$ is clear for all $n \in \mathbb{N} \cup \{0\}$ and s. We can turn this into a quantitative statement:

Proposition 3.1.

$$E_n(s) - E_n(0) \geq \frac{2}{d_{\max}^2} \cdot s \, .$$

Proof. Instead of tackling the eigenvalue problems on the domains D_s directly, we will pass to a family $(\tilde{D}_t)_{|t|<1}$ of domains arising from D_0 by dilations in direction of x_2, on which the problem can be handled more easily. So for $|t| < 1$, let us introduce the family $\Phi_t : \bar{D}_0 \to \bar{\tilde{D}}_t$, where

$$\tilde{D}_t = \{(x_1, x_2) \,|\, x_1 \in \Lambda, 0 < x_2 < (1-t) b(x_1)\} \, ,$$

of dilations, $\Phi_t(x_1, x_2) = (x_1, (1-t)x_2)$; now pull back the eigenvalue problem for the Dirichlet Laplacian in $L^2(\tilde{D}_t)$

$$-\Delta_{\tilde{D}_t} \tilde{u}_{n,t} = \tilde{E}_n \tilde{u}_{n,t} \, , \quad \tilde{u}_{n,t} \in H_0^1(\tilde{D}_t) \, ,$$

to $L^2(D_0)$; the form

$$\begin{aligned} a_t[f,g] &= \int_{\tilde{D}_t} f(\Phi_t^{-1}(x))\bar{g}(\Phi_t^{-1}(x))dx \\ &= \int_{D_0} f(x)\bar{g}(x)(1-t)dx \\ &= (1-t)(f|g) \text{ with domain } L^2(D_0) \end{aligned}$$

corresponds to the scalar product in $L^2(\tilde{D}_t)$, while

$$\begin{aligned} \tilde{h}_t[f,g] &= \int_{\tilde{D}_t} \nabla(f\circ\Phi_t^{-1})(x)\nabla(\bar{g}\circ\Phi_t^{-1})(x)dx \\ &= \int_{D_0} \left(\partial_1 f \partial_1 \bar{g}(x) + \frac{1}{(1-t)^2}\partial_2 f\partial_2\bar{g}(x)\right)(1-t)dx \text{ with domain } H_0^1(D_0) \end{aligned}$$

corresponds to the form of $-\Delta_{\tilde{D}_t}$. The operators defined by these forms are $A_t = M_{1-t}$ with $L^2(D_0)$ as domain of definition and $\tilde{H}_t = -(1-t)(\partial_1^2 + \frac{1}{(1-t)^2}\partial_2^2)$ with the same domain of definition as $-\Delta_{D_0}$; thus, the eigenvalue problem has been transformed into

$$\tilde{H}_t u_{n,t} = \tilde{E}_n(t) A_t u_{n,t}$$

or explicitly

$$-(1-t)\left(\partial_1^2 + \frac{1}{(1-t)^2}\partial_2^2\right)u_{n,t} = \tilde{E}_n(t)(1-t)u_{n,t}\,,$$

an eigenvalue problem in generalized form, which, however, at once reduces to an ordinary one:

$$-\left(\partial_1^2 + \frac{1}{(1-t)^2}\partial_2^2\right)u_{n,t} = \tilde{E}_n(t)u_{n,t}\,.$$

This means that we have to deal with the holomorphic family of operators

$$\tilde{\tilde{H}}_t = \frac{1}{1-t}\tilde{H}_t = -\left(\partial_1^2+\partial_2^2\right) + \left(\frac{1}{(1-t)^2} - 1\right)\partial_2^2\,, \quad |t|<1\,,$$

with the same domain of definition as $-\Delta_{D_0}$.

According to perturbation theory, derivatives of eigenvalues with respect to t can be calculated as

$$\begin{aligned} \tilde{E}_n'(t) &= (\tilde{\tilde{H}}_t'\, u_{n,t}|u_{n,t}) \\ &= -\frac{2}{(1-t)^3}(\partial_2^2 u_{n,t}|u_{n,t}) \\ &= \frac{2}{(1-t)^3}\|\partial_2 u_{n,t}\|^2\,, \end{aligned}$$

if $u_{n,t}$ is chosen continuous and piecewise holomorphic in t and fulfils $\|u_{n,t}\| = 1$ for real t. But for any $f \in H_0^1(D_0)$

$$\|\partial_2 f\|^2 = \int_{D_0} |\partial_2 f|^2 dx_2\, dx_1 \geq \frac{2}{d_{\max}^2} \int_{D_0} |f|^2 dx_2\, dx_1$$

by the one-dimensional Poincaré inequality ([3], IV.§7), applied in direction of x_2, so

$$\tilde{E}_n'(t) \geq \frac{4}{d_{\max}^2(1-t)^3} \quad \text{for } |t| < 1\,.$$

Now $D_s \subset \tilde{D}_{\frac{s}{d_{\max}}}$, and consequently

$$\begin{aligned} E_n(s) - E_n(0) &\geq \tilde{E}_n\left(\frac{s}{d_{\max}}\right) - E_n(0) \\ &= \int_0^{\frac{s}{d_{\max}}} \tilde{E}_n'(t)dt \\ &\geq \int_0^{\frac{s}{d_{\max}}} \frac{4}{d_{\max}^2(1-t)^3} dt \\ &= \frac{2}{(d_{\max}-s)^2} - \frac{2}{d_{\max}^2} \\ &\geq \frac{2}{d_{\max}^2} \cdot s\,. \end{aligned}$$

□

Our next goal will be the calculation of the derivative of the lowest eigenvalue of the Laplacian on a rectangle, if one of its sides is dented. For this purpose, an appropriate Hadamard–Rayleigh formula is established.

So this time, for a twice continuously differentiable function $p : \Lambda \to [0, d]$, $d = d_{\max} - d_{\min}$ with $\operatorname{supp} p \subset \Lambda$ and $t \in [0, 1]$, consider

$$D_t^p = \{(x_1, x_2); x_1 \in \Lambda, 0 < x_2 < d_{\max} - tp(x_1)\}$$

where $\Lambda = \Lambda_l(i)$ is an open interval centered at i with sidelength l. We consider $H_t^p = -\Delta$ on D_t^p, with Neumann boundary conditions on the vertical part $\partial\Lambda \times [0, d_{\max}]$ and Dirichlet boundary conditions on the rest of the boundary of D_t^p. In the following we will often suppress the superscript p, considering this function as fixed. Pulling back everything to $D_0 = \Lambda \times (0, d_{\max})$ we get a family of operators in $L^2(D_0)$ with purely discrete spectra, for whose lowest eigenvalue $E_1(t)$ we prove

Proposition 3.2. **("Hadamard–Rayleigh formula")**

$$(E_1^p)'(0) = \frac{2\pi^2}{d_{\max}^3} \frac{1}{|\Lambda|} \int_\Lambda p(x_1) dx_1\,.$$

Proof. There is, of course, some freedom in choosing a family of maps $\Phi_t : \overline{D_0} \to \overline{D_t}$. We choose

$$\Phi_t(x_1, x_2) = \left(x_1, \frac{d_{\max} - tp(x_1)}{d_{\max}} x_2\right) .$$

Pulling back the scalar products of the spaces $L^2(D_t)$ to $L^2(D_0)$ yields the forms

$$\begin{aligned} a_t[f,g] &= \int_{D_t} f(\Phi_t^{-1}(x))\bar{g}(\Phi_t^{-1}(x))dx \\ &= \int_{D_0} f(x)\bar{g}(x)\frac{d_{\max} - tp(x_1)}{d_{\max}} dx \end{aligned}$$

on $L^2(D_0)$, and pulling back the forms of the Laplacians yields

$$\begin{aligned} \tilde{h}_t[f,g] &= \int_{D_t} \nabla(f \circ \Phi_t^{-1})(x)\nabla(\bar{g} \circ \Phi_t^{-1})(x)dx \\ &= \int_{D_0} \Bigg(\partial_1 f \partial_1 \bar{g}(x) + \frac{tp'(x_1)x_2}{d_{\max} - tp(x_1)}(\partial_1 f \partial_2 \bar{g} + \partial_2 f \partial_1 \bar{g})(x) \\ &\quad + \frac{d_{\max}^2 + (tp'(x_1)x_2)^2}{(d_{\max} - tp(x_1))^2}\partial_2 f \partial_2 \bar{g}(x)\Bigg) \frac{d_{\max} - tp(x_1)}{d_{\max}} dx \end{aligned}$$

with domain $H^1(\Lambda) \otimes H_0^1(0, d_{\max})$. The associated operator $\tilde{H}_t$ has compact resolvent and a unique ground state $\tilde{u}_t$, which satisfies

$$\tilde{H}_t \tilde{u}_t = E_1(t) A_t \tilde{u}_t ,$$

where A_t is the operator associated with the pullback a_t of the scalar product in $L^2(D_t)$ above. Hence A_t is simply multiplication by $(d_{\max} - tp(x_1))/d_{\max}$. To reduce the above problem to an ordinary eigenvalue problem, we have to perform the substitution

$$u_t = A_t^{1/2} \tilde{u}_t ,$$

and get

$$A_t^{-1/2} \tilde{H}_t A_t^{-1/2} u_t = E_1(t) u_t ,$$

which is the eigenvalue problem for $\hat{H}_t := A_t^{-1/2} \tilde{H}_t A_t^{-1/2}$. The latter operator is associated with the form

$$\hat{h}_t[f,g] = \tilde{h}_t(A_t^{-1/2} f, A_t^{-1/2} g) ,$$

which can be calculated as

$$\begin{aligned} \hat{h}_t[f,g] &= \int_{D_0} \Bigg(\partial_1 f \partial_1 \bar{g}(x) + \frac{tp'(x_1)}{2(d_{\max} - tp(x_1))}(f \partial_1 \bar{g} + \partial_1 f \bar{g})(x) \\ &\quad + \left(\frac{tp'(x_1)}{2(d_{\max} - tp(x_1))}\right)^2 f \bar{g}(x) + \frac{tp'(x_1)x_2}{d_{\max} - tp(x_1)}(\partial_1 f \partial_2 \bar{g} + \partial_2 f \partial_1 \bar{g})(x) \end{aligned}$$

$$+ \frac{tp'(x_1)}{2(d_{\max} - tp(x_1))}(f\partial_2\bar{g} + \partial_2 f\bar{g})(x)$$
$$+ \frac{d_{\max}^2 + (tp'(x_1)x_2)^2}{(d_{\max} - tp(x_1))^2}\partial_2 f\partial_2\bar{g}(x)\Bigg)dx\,. \tag{3.1}$$

According to first order perturbation theory one gets

$$E_1'(0) = (\hat{h}_0)'[u,u]\,,$$

where $(\hat{h}_0)'$ is the derivative of $\hat{h}_t$ with respect to t at $t = 0$ (which can be read off 3.1), and u is the unique normalized ground state of $\hat{h}_0$. The latter is just the form associated with the Laplacian on the rectangular box D_0, so u is given by

$$u(x_1, x_2) = \sqrt{\frac{2}{d_{\max}|\Lambda|}}\sin\left(\frac{\pi}{d_{\max}}x_2\right)\,.$$

Thus we can proceed

$$\begin{aligned} E_1'(0) &= \int_{D_0}\left(\frac{p'(x_1)}{d_{\max}}u\partial_1 u(x) + \frac{2p'(x_1)x_2}{d_{\max}}(\partial_1 u\partial_2 u)(x)\right.\\ &\qquad \left. + \frac{p'(x_1)}{d_{\max}}u\partial_2 u(x) + \frac{2p(x_1)}{d_{\max}}(\partial_2 u)^2(x)\right)dx\\ &= \frac{1}{d_{\max}}\int_{D_0}(2p(x_1)(\partial_2 u)^2(x))dx\\ &= \frac{2\pi^2}{d_{\max}^3}\frac{1}{|\Lambda|}\int_\Lambda p(x_1)dx_1\,, \end{aligned}$$

where we used partial integration with respect to x_1 in the third summand, $p(x_1) = 0$ near $\partial\Lambda$ and $\partial_1 u = 0$. □

In the above proof the calculation was extraordinarily simple, as the domain is a rectangle, u is known explicitly and x_1-independent. It is quite reasonable that the derivative should be a boundary integral in general, too, because the result should be independent of the choice of Φ_t as long as the boundary is transformed in the desired way. For a somewhat different context, this is stated in [18], p. 88, where, however, the tricky part of the calculation (a clever application of Stokes' theorem) is missing. This can be found in [10].

We will later on consider the situation, where p takes the form

$$p(x) = \sum_{i\in\Lambda}\omega(i)\varphi(x - i)\,.$$

The Hadamard–Rayleigh formula will then enable us to relate questions about the bottom eigenvalue to the mean

$$\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\omega(i)\,,$$

which is a particularly well studied object in probability theory. In this connection we will need an estimate for the error of the linear approximation to $E_1(t)$.

Proposition 3.3. *In the situation above there exists a constant* $\tau = \tau(d_{\max}, |p|_\infty, |p'|_\infty)$ *such that for all* $\Lambda = \Lambda_l$ *with* $l \geq d_{\max}/\sqrt{3}$ *and* $0 \leq t \leq \tau l^{-2}$ *we have*

$$|E_1^p(t) - E_0 - t(E_1^p)'(0)| \leq \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2 \,.$$

Proof. The unperturbed operator is the Laplacian on the rectangular box D_0 with Dirichlet boundary conditions on $\Lambda \times \{0, d_{\max}\}$, Neumann boundary conditions on $\partial\Lambda \times [0, d_{\max}]$ and ground state energy $E_1(0) = E_0 = \pi^2/d_{\max}^2$. To estimate the remainder term in the Taylor expansion we want to use [12], formula II.(3.6), applied to the forms $\hat{h}_t$ defined in the preceding proof. The isolation distance ϑ, defined as the distance of E_0 to the rest of the spectrum of H_0, is given by

$$\vartheta = \min\left\{\frac{3\pi^2}{d_{\max}^2}, \frac{\pi^2}{|\Lambda|^2}\right\} = \frac{\pi^2}{l^2}\,. \tag{3.2}$$

As Γ we choose a circle around E_0 with radius $\vartheta/2$. We need an estimate for the r_0 appearing in [12], II.(3.3). To this end we appeal to [12], VII.(4.47) and are thus left to estimate the kth Taylor coefficient $(\hat{h}_0)^{(k)}$ of $\hat{h}_t$ at $t = 0$. To do so, we have to look at (3.1) above. Since the t-dependent coefficients can be expanded into series directly, we can easily deduce

$$\begin{aligned}
(\hat{h}_0)^{(k)}[f,f] = \int_{D_0} \Bigg(& \frac{p'(x_1)p^{k-1}(x_1)}{2d_{\max}^k}(f\partial_1\bar{f} + \partial_1 f\bar{f}\,)(x) \\
& + \frac{(p'(x_1))^2 p^{k-2}(x_1)}{4d_{\max}^k}(k-1)f\bar{f}(x) \\
& + \frac{p'(x_1)x_2 p^{k-1}(x_1)}{d_{\max}^k}(\partial_1 f\partial_2\bar{f} + \partial_2 f\partial_1\bar{f}\,)(x) \\
& + \frac{p'(x_1)p^{k-1}(x_1)}{2d_{\max}^k}(f\partial_2\bar{f} + \partial_2 f\bar{f}\,)(x) \\
& + \left(\frac{p^k(x_1)}{d_{\max}^k}(k+1) + \frac{(p'(x_1))^2 x_2^2 p^{k-2}(x_1)}{d_{\max}^k}(k-1)\right)\partial_2 f\partial_2\bar{f}(x)\Bigg)\,dx \\
\leq \left(\frac{2|p|_\infty}{d_{\max}}\right)^{k-1} & \Bigg(\left(\frac{|p'|_\infty}{d_{\max}} + \frac{|p'|_\infty^2}{4d_{\max}|p|_\infty}\right)\|f\|^2 \\
& + \left(\frac{|p'|_\infty}{2d_{\max}} + |p'|_\infty + \frac{2|p|_\infty}{d_{\max}} + \frac{|p'|_\infty^2 d_{\max}}{|p|_\infty}\right)\int_{D_0}|\nabla f|^2(x)dx\Bigg)
\end{aligned}$$

for $k \geq 1$. Now [12], VII.(4.47) yields $r_0 \geq c \cdot \vartheta$, where c only depends upon the relative bounds above, which in turn only depend upon $d_{\max}, |p|_\infty, |p'|_\infty$. Define

$\tau = c\pi^2/2$. Then for $0 \leq t \leq \tau \cdot l^{-2}$, the estimate [12], II.(4.47) gives

$$|E_1^p(t) - E_0 - t(E_1^p)'(0)| \leq \frac{\vartheta/2}{r_0 \cdot r_0/2} t^2 \,.$$

With τ as above and ϑ from (3.2) we get the assertion. □

4. Lifshitz Tails and Initial Length Scale Estimates

As the title suggests, the purpose of this section is twofold: in Corollary 4.1 we give an initial length scale estimate, which is one of the main ingredients of the multi-scale analysis we outlined in Sec. 2. In Theorem 4.1 we prove that our model exhibits Lifshitz asymptotics of the integrated density of states. Both results are based essentially upon the following Proposition. Our strategy of proof follows the ideas of [21] and is remarkably easy even in the case of Schrödinger operators with an Anderson potential (see [13, 2] for earlier results in this case). The other new, equally important point, which enters here, is the Hadamard–Rayleigh formula and the estimate on the remainder obtained in Propositions 3.2 and 3.3 above.

Recall that $H_\Lambda(\omega)$ was defined as the Dirichlet Laplacian on $D_\Lambda(\omega) = D(\omega) \cap (\Lambda \times (0, d_{\max}))$ in Sec. 1. To define and control the integrated density of states, we introduce $H_\Lambda^N(\omega)$ as the Laplacian on the same domain with Neumann boundary conditions on the vertical parts $D(\omega) \cap (\partial\Lambda \times (0, d_{\max}))$ and Dirichlet boundary conditions elsewhere on $\partial D_\Lambda(\omega)$. By stationarity, most statistical properties do not depend upon the center of Λ, but only on the sidelength l, so we will often write $H_l^N(\omega)$ instead of $H_\Lambda^N(\omega)$. Note that this operator is dominated by $H_\Lambda(\omega)$, a fact we shall use in order to compare the eigenvalues. Here comes the main technical result of this section:

Proposition 4.1. *Let* $m = \mathbb{E}\{\omega(0)\} = \int x\, d\mu(x)$, $v = \mathbb{E}\{\omega^2(0)\} = \int x^2\, d\mu(x)$. *Then there exist a universal constant* $K > 0$ *and a constant* $a = a(d_{\max}, d_{\min}) > 0$, *such that for all* $l \in 2\mathbb{N} + 1$ *and every*

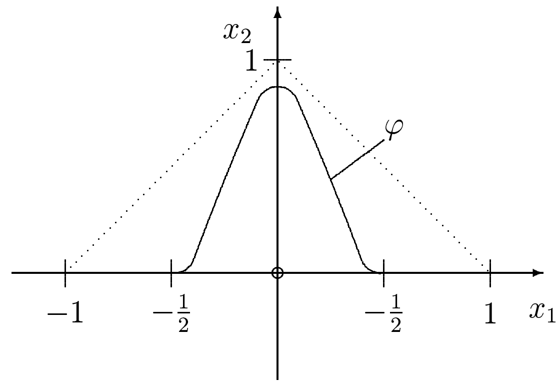
$$b \leq \min\left\{\frac{\pi^2}{4}, \frac{m^2}{a^2}\right\}$$

we have

$$\mathbb{P}\{E_1(H_l^N(\omega)) \leq E_0 + b \cdot l^{-2}\} \leq K \cdot \exp\left(-l \frac{m - a\sqrt{b}}{K d_{\max}} \log\left(1 + \frac{\left(m - a\sqrt{b}\right) d_{\max}}{v}\right)\right) . \tag{4.1}$$

Proof. The main idea is to define a deformation $D(\omega, t)$ of the maximal rectangle $\Lambda \times (0, d_{\max})$, which satisfies $D(\omega, t) \supset D_\Lambda(\omega)$, and to analyze the bottom eigenvalue by means of Propositions 3.2 and 3.3. To do so, we have to smoothen out the corners appearing in $D_\Lambda(\omega)$. So let $\varphi \in \mathcal{C}_c^2(-\frac{1}{2}, \frac{1}{2})$ be a bump function satisfying

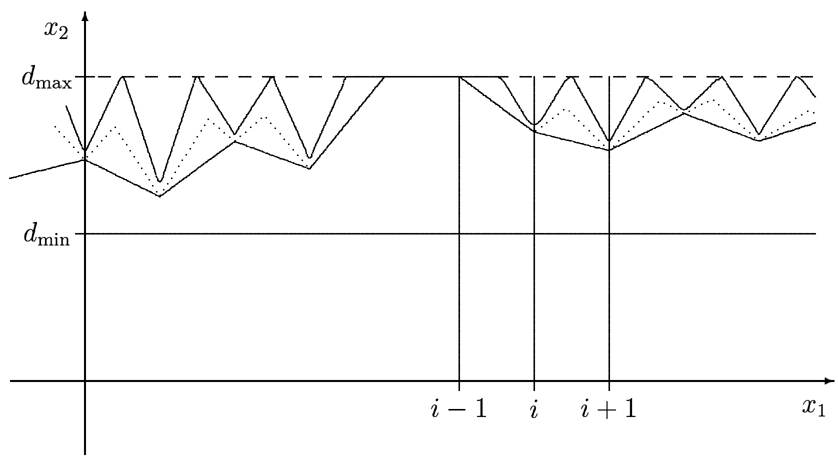
$$0 \leq \varphi(x) \leq 1 - |x| \quad \text{for all } x \,. \tag{4.2}$$



Define

$$p(\omega, x_1) = \sum_{i\in\Lambda} \omega(i)\varphi(x-i)\,,$$

$$D(\omega,t) = \{(x_1,x_2)\,|\, x_1 \in \Lambda, 0 < x_2 < d_{\max} - tp(\omega,x_1)\}\,.$$



Note that $D(\omega,t) = D_t^{p(\omega)}$ in the notation of Sec. 3. There we defined $H_t^{p(\omega)}$ as the Laplacian with Neumann boundary conditions on the vertical part of the boundary and Dirichlet boundary conditions elsewhere. We write $H_t(\omega)$ for this operator and $E_1(\omega,t)$ for its bottom eigenvalue. By construction,

$$E_1(H_l^N(\omega)) \geq E_1(\omega,t)$$

for all $t \in (0,1)$. Now by Proposition 3.2 we see that the rhs can not be too small for many ω, as the derivative obeys

$$\frac{d}{dt}E_1(\omega,t)|_{t=0} = c\cdot\left(\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\omega(i)\right)\,, \tag{4.3}$$

where

$$c = \frac{2\pi^3}{d_{\max}^3}\int_{-\frac{1}{2}}^{-\frac{1}{2}}\varphi(x)dx$$

only depends upon $d_{\max}$. The basic idea of the proof is that the sum on the rhs takes values near $m := \mathbb{E}\{\omega(0)\} > 0$ with overwhelming probability. This will imply that $E_1(\omega, t)$ is shifted away from $E_0 = E_1(\omega, 0)$ with very high probability.

More precisely, from the remainder estimate in Proposition 3.3 we have

$$|E_1^p(t) - E_0 - t(E_1^p)'(0)| \leq \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2 \quad (0 \leq t \leq \tau l^{-2}),$$

where τ depends upon $|p(\omega, \cdot)|_\infty$, $|p'(\omega, \cdot)|_\infty$, which in turn only depend upon $d_{\max}$. Assume that

$$E_1(\omega) \leq E_0 + b \cdot l^{-2}$$

for $b \leq \pi^2/4$. Then the above inequality yields

$$t \cdot E_1'(\omega, 0) \leq \frac{\pi^2}{4\tau^2} \cdot l^2 \cdot t^2 + b \cdot l^{-2} \quad \text{for all } 0 \leq t \leq \tau l^{-2}.$$

Inserting $t = s\tau l^{-2}$ we get

$$E_1'(\omega, 0) \leq \frac{\pi^2 s}{4\tau} + \frac{b}{\tau s} \quad \text{for all } 0 \leq s \leq 1.$$

Optimizing w.r.t. s we get $s = \frac{2}{\pi}\sqrt{b}$ and

$$E_1'(\omega, 0) \leq \frac{\pi}{\tau}\sqrt{b},$$

which implies

$$\frac{1}{|\Lambda|}\sum_{i \in \Lambda} \omega(i) \leq \frac{\pi}{c\tau}\sqrt{b}.$$

Define $a = \frac{\pi}{c\tau}$. Now, if $0 \leq b \leq \frac{m^2}{a^2}$, it follows that

$$\begin{aligned}\mathbb{P}\{E_1(\omega) \leq E_0 + b \cdot l^{-2}\} &\leq \mathbb{P}\left\{\frac{1}{|\Lambda|}\sum_{i \in \Lambda}\omega(i) \leq a\sqrt{b}\right\} \\ &\leq \mathbb{P}\left\{\left|\frac{1}{|\Lambda|}\sum_{i \in \Lambda}\omega(i) - m\right| \geq m - a\sqrt{b}\right\}.\end{aligned}$$

By [22], Theorem 1.4, this latter probability can be estimated by

$$K \cdot \exp\left(-l\frac{m - a\sqrt{b}}{K d_{\max}} \log\left(1 + \frac{\left(m - a\sqrt{b}\right) d_{\max}}{v}\right)\right),$$

the assertion. □

Let us now proceed to determine the asymptotics of the *integrated density of states* for our model as the energy approaches E_0. To begin with, let us recall the relevant notions. By $N(H,t) = \#\{n; E_n(H) \le t\}$ we denote the spectral counting function for a given semibounded operator H with compact resolvent.

For the operators $H_\Lambda(\omega)$ and $H_\Lambda^N(\omega)$ it is clear by usual min-max and Dirichlet–Neumann bracketing arguments that

$$N(H_\Lambda(\omega), t) \le N(H_\Lambda^N(\omega), t)$$

and that $N(H_\Lambda^N(\omega))$ is subadditive with respect to Λ. By the subadditive ergodic theorem,

$$N(t) := \inf_\Lambda \frac{1}{|\Lambda|}\mathbb{E}\{N(H_\Lambda^N(\omega), t)\}$$

exists, and the convergence

$$\lim_{l\to\infty} \frac{1}{l} N(H_l^N(\omega), t) = N(t)$$

holds $\mathbb{P}$-a.s. This means that $N(t)$ gives the number of energy levels per unit volume for the operator $H(\omega)$. For further reference we note that

$$\frac{1}{|\Lambda|}\mathbb{E}\{N(H_\Lambda(\omega), t)\} \le N(t) \le \frac{1}{|\Lambda|}\mathbb{E}\{N(H_\Lambda^N(\omega), t)\}\,. \tag{4.4}$$

The asymptotic behavior of $N(t)$ as $t \searrow E_0$ contained in the following Theorem is usually referred to as *Lifshitz asymptotics*. It is a central feature of disordered systems and has been established for various types of random Schrödinger operators. We refer to [2, 16], where one can also find more details of the definition of the integrated density of states than sketched above.

Theorem 4.1. *Let $H(\omega), E_0, m$ and $N(t)$ be as above. Then we have*:

(1) *There exists $C = C(d_{\max}, d_{\min}, m, v) > 0$ such that*

$$\limsup_{t\searrow 0} \frac{\log N(E_0+t)}{t^{-\frac{1}{2}}} \le -C\,.$$

(2) *Assume* (M1). *Then*

$$\liminf_{t\searrow 0} \frac{\log(-\log N(E_0+t))}{\log t} \ge -\frac{1}{2}\,,$$

consequently,

$$\lim_{t\searrow 0} \frac{\log(-\log N(E_0+t))}{\log t} = -\frac{1}{2}\,.$$

Proof. (1) Let $t > 0$. Then

$$\begin{aligned} N(E_0+t) &\le \frac{1}{l}\mathbb{E}\{N(H_l^N(\omega), E_0+t)\} \\ &\le \int_{\{\omega \mid E_1(H_l^N(\omega)) \le E_0+t\}} d\mathbb{P}(\omega)\frac{1}{l}N(H_l^N(\omega), E_0+t) \\ &\le A\cdot \mathbb{P}\{E_1(H_l^N(\omega)) \le E_0+t\}\,, \end{aligned}$$

where in the last inequality we have used a Weyl type estimate, which follows by comparing $H_l^N(\omega)$ with the Laplacian on the rectangle $\Lambda \times (0, d_{\max})$, see Sec. 5. Now let

$$t = b \cdot l^{-2}\,.$$

By Proposition 4.1 we get that

$$\begin{aligned} N(E_0 + t) &\le N(E_0 + b \cdot l^{-2}) \\ &\le AK \cdot \exp\left(-l\frac{m - a\sqrt{b}}{Kd_{\max}} \log\left(1 + \frac{\left(m - a\sqrt{b}\right) d_{\max}}{v}\right)\right)\,. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{t \searrow 0} \frac{\log N(E_0 + t)}{t^{-1/2}} &= \limsup_{l \to \infty} \frac{\log N(E_0 + b \cdot l^{-2})}{b^{-1/2} l} \\ &\le -\frac{\left(m - a\sqrt{b}\right)\sqrt{b}}{Kd_{\max}} \log\left(1 + \frac{\left(m - a\sqrt{b}\right) d_{\max}}{v}\right) \end{aligned}$$

for every $b \le \frac{m^2}{a^2}$. Inserting an appropriate b gives the asserted estimate.

(2) In pretty much the same way as in [2] (for the case of discrete random Schrödinger operators), the lower estimate follows from $\log\log$ calculations and the following Proposition. □

Proposition 4.2. *Assume that* μ *satisfies* (M1). *Then there exist* $C_1, C_2 > 0$ *such that*

$$\mathbb{P}\{E_1(H_l(\omega) \le E_0 + C_1 l^{-2}\} \ge (C_2)^{l+2}(l^{-2})^{\delta(l+2)}\,.$$

Proof. For $l \in 2\mathbb{N}+1$ large enough let $\epsilon > 0$, $\epsilon < d_{\max}$. We know that $E_0 = \pi^2/d_{\max}^2$ equals $\inf \sigma(H_l(\omega))$, if all $\omega(i) = 0$ for $i \in (-\frac{l}{2} - 1, \frac{l}{2} + 1) \cap \mathbb{Z}$; the extra ∓ 1 are needed, as D_Λ depends upon $\omega(\mp(\frac{l}{2}+1))$ by construction. Note that we are dealing with $\Lambda = \Lambda_l(0)$.

If $\omega(i) \in [0, \epsilon]$ for all $i \in (-\frac{l}{2} - 1, \frac{l}{2} + 1) \cap \mathbb{Z}$, then

$$D(\omega) \supset \left(-\frac{l}{2}, \frac{l}{2}\right) \times (0, d_{\max} - \epsilon) =: D_\epsilon\,.$$

The ground state energy of the Dirichlet Laplacian on D_ϵ is explicitly given by

$$E_\epsilon = \frac{\pi^2}{(d_{\max} - \epsilon)^2} + \frac{\pi^2}{l^2}\,.$$

Setting $\epsilon = l^{-2}$ we see that

$$E_1(H_\Lambda) \le E_\epsilon \le E_0 + C_1 l^{-2}$$

for

$$C_1 = \frac{8\pi^2}{d_{\max}^3} + \pi^2 \,,$$

whenever $\epsilon \leq d_{\max}/2$. Since the probability that $\omega(i) \in [0, \epsilon]$ for all $i \in (-\frac{l}{2} - 1, \frac{l}{2} + 1) \cap \mathbb{Z}$ is estimated below by

$$(\mu[0, \epsilon])^{l+2} \geq C^{l+2}(l^{-2})^{\delta(l+2)} \,,$$

we arrive at the asserted estimate. □

As was pointed out by Klopp in [15], Remark, p. 558f, the above Lifshitz tail estimate implies

$$\lim_{t \searrow 0} N(E_0 + t) \cdot t^{-n} = 0$$

for every $n > 0$, so that

$$\mathbb{P}\{E_1(H_\Lambda(\omega)) \leq E_0 + t\} \leq |\Lambda| \cdot N(E_0 + t)$$

immediately gives the following result:

Corollary 4.1. **(initial length scale estimate)** *For $\beta \in (0, 1)$ and $\xi > 0$ there exists $l_0 = l_0(\beta, \xi)$ such that for all $l \geq l_0$, $\Lambda = \Lambda_l(i)$ we have*

$$\mathbb{P}\{E_1(H_\Lambda(\omega)) \leq E_0 + l^{\beta - 1}\} \leq l^{-\xi} \,.$$

5. Wegner Estimates

Two ingredients are needed for the proof of a Wegner estimate: the possible number of eigenvalues in the interval in question has to be bounded, which may be achieved in great generality using Weyl's asymptotic law or, as possible in our case, by an explicit calculation. Secondly, a certain spreading of the eigenvalues has to be established, which is a consequence of Proposition 3.1 for our model.

Proposition 5.1. **(Wegner estimate)** *There exist $\alpha > 0$, $C > 0$ such that for all intervals $I \subset [E_0, E_0 + 1]$*

$$\mathbb{P}\{\sigma(H_\Lambda(\omega)) \cap I \neq \emptyset\} \leq C \cdot |\Lambda|^2 \cdot |I|^\alpha \,.$$

Proof. If $E_n(H_\Lambda(\omega))$ denotes the nth eigenvalue of $H_\Lambda(\omega)$, obviously

$$\mathbb{P}\{\sigma(H_\Lambda(\omega)) \cap I \neq \emptyset\} \leq \sum_n \mathbb{P}\{E_n(H_\Lambda(\omega)) \in I\} \,;$$

now $H_\Lambda(\omega) \geq -\Delta_{\Lambda \times (0, d_{\max})}$, which has eigenvalues

$$\left\{ E_{n,m} = \pi^2 \left(\frac{n^2}{d_{\max}^2} + \frac{m^2}{|\Lambda|^2} \right) \Big| \, n \in \mathbb{N}, m \in \mathbb{N} \cup \{0\} \right\} \,,$$

so

$$n \le \sqrt{1+\frac{d_{\max}^2}{\pi^2}} \quad \text{and} \quad m \le \frac{|\Lambda|}{\pi}$$

is a necessary condition on n, m for $E_{n,m}$ to fall into $[E_0, E_0+1]$. Thus, the number of such eigenvalues is limited by a bound proportional to $|\Lambda|$, and we still have to prove

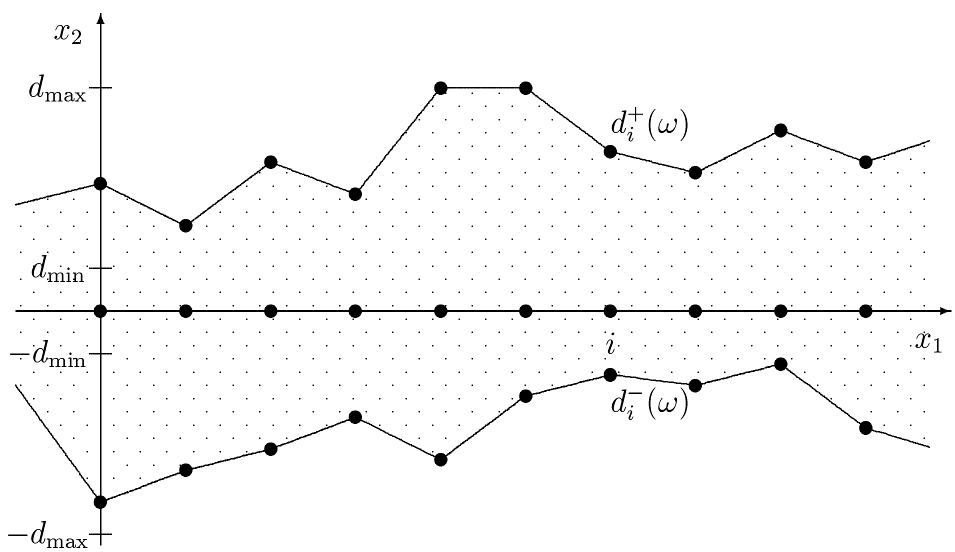
$$\mathbb{P}\{E_n(H_\Lambda(\omega)) \in I\} \le C'|\Lambda||I|^\alpha\,,$$

which follows from Proposition 3.1 for the case $D_0 = D(\omega)$, i.e. $b(x_1) = \gamma(\omega)(x_1)$, in the same way as in [20]. □

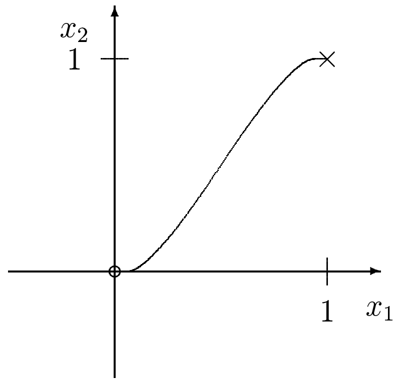
6. Concluding Remarks

The introduced model admits some modifications and extensions, for which the same results, localization and Lifshitz tails, can be proven with only minor modifications:

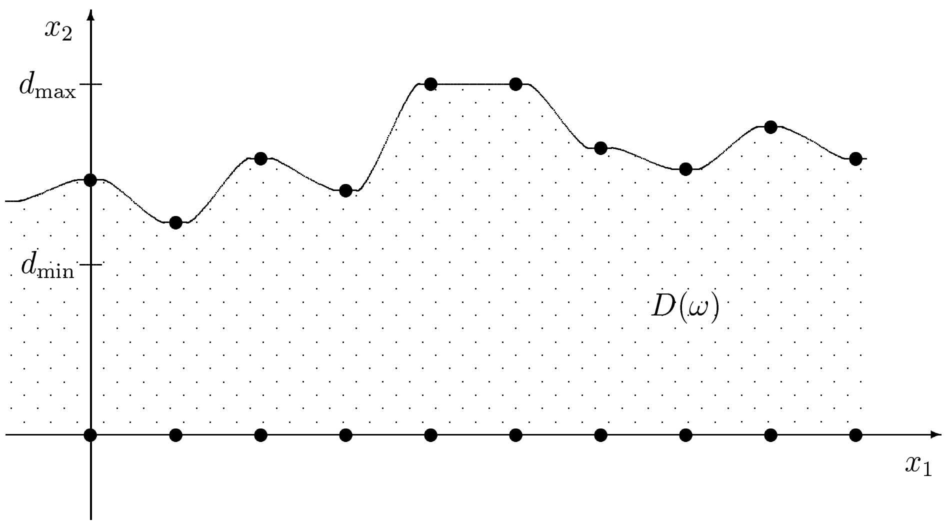
- It is possible to dent or bulge the strip on both sides, so that an element of the random family of domains might look as follows:



  We only have to "double" the number of coordinates of the probability space, i.e. introduce for $i \in \mathbb{Z}$ $d_i^+(\omega)$, which take the part of the $d_i(\omega)$ above, and $d_i^-(\omega)$, which are the vertices of a second polygon in the lower half plane of $\mathbb{R}^2$, the lower boundary of the random domain.
- It is not necessary to join the points $(i, d_i(\omega))$ by straight line segments, but one could use dilated (in x_2-direction) versions of an arbitrarily smooth curve joining (0, 0) and (1, 1), at best a monotone one, which is constant near its endpoints:



In this way, we could smoothen the boundary of $D(\omega)$:



- In order to obtain a 3- or n-dimensional model, one could rotate $D(\omega)$ about the x_1-axis resp. substitute $(n-1)$-spheres with radii $d_i(\omega)$ around $(i, 0, \ldots, 0)$ in $\mathbb{R}^n$ for the points $(i, d_i(\omega))$ and join them. However, the model obtained in this way is still essentially one-dimensional.
- To obtain a 3-dimensional model in the form of a thin layer, one could use a triangulation of $\mathbb{R}^2$ by equilateral triangles, prescribe the height of the layer at the vertices v_i of the triangulation by $d_{v_i}(\omega)$ and fill in the surface of the layer with flat triangles, whose vertices are the $(v_i, d_{v_i}(\omega))$.
- It is possible to choose slightly different probability measures μ_i for $i \in \mathbb{Z}$, according to which the $d_i(\omega)$ are picked.

Acknowledgment

We thank the referee for helpful comments on an earlier version of this paper.

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