Exercises Singularity Theory

1. Prove the Weierstraß division theorem for the formal power series, that is: for every formal power series $f \in \mathbb{C}[[x_1, ..., x_n]]$ which is regular of order b in x_n and for every $g \in \mathbb{C}[[x_1, ..., x_n]]$, there exist a uniquely determined $q \in \mathbb{C}[[x_1, ..., x_n]]$ and $r \in \mathbb{C}[[x_1, ..., x_{n-1}]][x_n]$, with $deg(r) < b$ such that:
$$g = qf + r$$
Hint: write f as
$$f = f_0 + x_n^b u_f$$
with:
   (a) f_0 a polynomial of degree $< b$ in x_n with coefficients in $(x_1 ..., x_{n-1})$
   (b) u_f a unit

   From the previous equation it follows that $x_n^b = \frac{f}{u_f} - \frac{f_0}{u_f}$ with $\frac{f_0}{u_f} \in (x_1, ..., x_{n-1})$. Construct r_i, q_i, g_i with the following properties:
   (a) $g = q_i \cdot f + r_i + g_i$
   (b) $g_i, q_i - q_{i+1}, r_{i+1} - r_i \in (x_1, ..., x_{n-1})^i$
   (c) r_i is a polynomial of degree $\leq b - 1$ in x_n

   Use as initialization $q_0 = r_0 = 0$ and $g_0 = g$, proceed by dividing g_i by x_n^b.

2. Using the Weierstraß preparation theorem, prove the implicit function theorem: let $f \in \mathbb{C}\{x_1, ..., x_n, y\}$ with $f(0) = 0$ and $\delta_y f(0) \neq 0$. Then there exists a unique $\varphi \in (x_1, ..., x_n)$ with:
$$f(x_1, ..., x_n, y) = 0 \Leftrightarrow y = \varphi(x_1, ..., x_n)$$

3. Consider $f(x, y) = 2y + y^2 - x$. Apply the Weierstraß preparation theorem to write $f(x, y) = u(y - \varphi(x)$ for some unit u. Write down the first steps of Exercise 1 in order to get a power series expressing φ and u. Verify that they coincide respectively with the power series of $-1 + \sqrt{x + 1}$ and $y + 1 + \sqrt{x + 1}$.