Exercises Singularity Theory

1. *(2 points)* Show that for any $f \in \mathbf{m}_{\mathcal{R}_n}^k$ and any $l \in \mathbb{N}$

$$\mu(f) \geq \binom{n+k+l-1}{k+l-1} - n\binom{n+l}{l}.$$

(Hint: Start by proving that for all l one has $J_f + \mathbf{m}_{\mathcal{R}_n}^{k+l} = \mathcal{R}_n^{\leq l}\partial_{x_1} f + \ldots + \mathcal{R}_n^{\leq l}\partial_{x_n} f + \mathbf{m}_{\mathcal{R}_n}^{k+l}$ where $\mathcal{R}^{\leq l}$ denotes the space of polynomials of degree at most l.)

2. *(3 points)* Let k be an integer greater than 3. Prove that every finitely determined germ in $x^2y + \mathbf{m}_{\mathcal{R}_2}^k$ is equivalent to one of the germs $x^2y \pm y^l$, $l \geq k$.

(Hints: (1) The determinacy of $x^2y \pm y^l$ is l. (2) Write a germ in $x^2y + \mathbf{m}_2^k$ in the form $x^2y + ay^k + bxy^{k-1} + x^2q + \textit{rest}$ where q is a degree $k-2$ homogeneous polynomial and *rest* means terms of degrees $> k$. Try to simplify the germ using the local diffeomorphism $\phi(x,y) = (x - \frac{b}{2}y^{k-2}, y - q)$.)

3. *(3 points)* Let k be an integer greater than 3. Prove that every finitely determined germ in $x^3 + \mathbf{m}_2^k$ is equivalent to either one of the germs

$$\varepsilon^{l-1}(x^3 + y^l)$$

or a germ in

$$x^3 + \varepsilon^l xy^{l-1} + \mathbf{m}_2^{l+1}$$

where $\varepsilon = \pm 1$ and $l \geq k$.

(Hints: As in Exercise 2, write a germ in $x^3 + \mathbf{m}_2^k$ in the form $x^3 + ay^k + bxy^{k-1} + x^2q + \textit{rest}$ where q is a degree $k-2$ homogeneous polynomial and *rest* means terms of degrees $> k$. Try to simplify the germ using the local diffeomorphism $\phi(x,y) = (x - \frac{1}{3}q, y)$.)

4. *(6 points)* Extend the classification of germs of smooth real functions to the case $\mu = 6$, namely:

   (a) Show that $\mu_f = 6$ implies $k := \operatorname{corank}(f) \in \{1, 2\}$.

   (b) If $k = 1$, conclude that f is stably equivalent to x^7.

   (c) If $k = 2$, show that f is stably equivalent to $g \in \mathbf{m}_2^3$ that is 5-determined.

   (d) Write g as $g = p + h$ with p is a cubic form and $h \in \mathbf{m}_2^4$. You may assume that (see Vorlesung 11)
   $$p \in \{0, x^3, x^2y, x^3 - xy^2, x^3 + y^3\}.$$
   Conclude that the only possibilities are $p \in \{x^3, x^2y\}$. (Hint: To show that $p \neq 0$ you may need Exercise 1).

   (e) For $p = x^3$ use Exercise 3 above to show that g can only be equivalent to $\pm(x^3 + y^4)$. (Hint: All the other possibilities can be ruled out using the fact that g is 5-determined or by comparing the Milnor numbers.)

   (f) For $p = x^2y$ use Exercise 2 to conclude that g can only be equivalent to $x^2y \pm y^5$.

To be handed in until Wednesday, December 13th 2017