Exercises Singularity Theory

1. *(2 points)*
   (a) Let $A, B \subset \mathbb{R}^n$ open sets and (A, p) resp. (B, p) germs of sets. Show that the intersection $(A, p) \cap (B, p)$ is a well defined germ of a set.
   (b) Show analogously that the product of germs $(A, p) \times (B, q)$ is well-defined.
   (c) Show that a map germ $[f]_p = (f, p)$ has a well-defined set germ (A, p) on which it is defined.

2. *(4 points)*
   Construct a local diffeomorphism at $0 \in \mathbb{R}^2$ which transforms the germ
   $$f = x^2 + y^2 + x^2y + xy^2 \in \mathcal{E}_{\mathbb{R}^2,0}$$
   into the germ
   $$f = x^2 + y^2 \in \mathcal{E}_{\mathbb{R}^2,0}.$$
   Hint: Use the proof of the Morse lemma.

3. *(3 points)*
   (a) Show that if $(R, \mathbf{m})$ is local, then for any $x \in \mathbf{m}$, the element $1 + x$ is a unit in R.
   (b) Let R be a local ring and $I \subset R$ any ideal. Show that the factor ring R/I is also local.
   (c) Let $\mathbb{R}[[x_1, \ldots, x_n]]$ be the local ring of formal power series over $\mathbb{R}$. Give an explicit expression for the inverse of $1 + x$ for $x \in \mathbf{m}$.
   (d) Show that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is not local.
   (e) For any local ring, define $k := R/\mathbf{m}$. Then k is a field, called the residue class field of $(R, \mathbf{m})$. Show that the residue class field of $\mathcal{E}$ is isomorphic to $\mathbb{R}$.
   (f) Let $(R, \mathbf{m})$ be local and define by
   $$H_R(d) := \dim_k(\mathbf{m}^d/\mathbf{m}^{d+1})$$
   the Hilbert function of the local ring R. Calculate the Hilbert function for the following local rings
      i. $R = \mathcal{E}_n$, $R = \mathbb{R}[[x_1, \ldots, x_n]]$,
      ii. $R = \mathbb{R}[[x, y]]/(xy)$,
      iii. $R = \mathbb{R}[[x, y]]/(x^2 - y^3)$.

4. *(2 points)* Consider the local ring $(\mathcal{E}_n, \mathbf{m})$ and let $\Psi := (\Psi_1, \ldots, \Psi_n) \in \mathbf{m}^{\oplus n} \subset (\mathcal{E}_n)^n = \mathcal{E}_{n,n}$ (caution: $\mathbf{m}^{\oplus n}$ denotes the direct sum $\mathbf{m} \oplus \ldots \oplus \mathbf{m}$, do not confuse this with the n-th power $\mathbf{m}^n$ of $\mathbf{m}$).
   (a) Show that the substitution map (also called pull-back or inverse image)
   $$\begin{aligned} \Psi^* : R &\longrightarrow R \\ f &\longmapsto f \circ \Psi \end{aligned}$$
   is an algebra homomorphism preserving the identity. Show further that $\Psi^*(\mathbf{m}^k) \subset \mathbf{m}^k$.
   (b) Deduce from (a) that Ψ induces linear maps
   $$(\Psi^*)_k : \mathbf{m}^k/\mathbf{m}^{k+1} \longrightarrow \mathbf{m}^k/\mathbf{m}^{k+1}.$$
   Show that Ψ is an automorphism iff $(\Psi^*)_1$ is invertible.