Exercises Singularity Theory

1. *(1 point)* Let $U \subset \mathbb{R}^n$ be open, $0 \in U$ and $f : U \to \mathbb{R}$ smooth, and let $\psi : U \to U$ be a local diffeomorphism. Let moreover 0 be a critical point of f. Let $g : U \to \mathbb{R}$ be give as in the lectures by $g(y) := f(\psi(y))$. Show that $D^2g(0)$ und $D^2f(0)$ have equal rank and index.

2. *(3 points)*

   Consider the space $\mathcal{T}(n, \mathbb{R})$ of upper triangular matrices and the space of $\mathrm{Sym}(n, \mathbb{R})$ of symmetric matrices.

   (a) Show that both are $\frac{1}{2}n(n+1)$-dimensional real vector spaces.

   (b) Let D_0 be a fixed diagonal matrix and $\alpha : \mathcal{T}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ be given as $\alpha(T) := T^{tr} \cdot D_0 \cdot T$. Show that α is a local diffeomorphism at $\mathrm{Id} \in \mathcal{T}(n, \mathbb{R})$ in the following way.

      i. Show that α is smooth.

      ii. Show that $D\alpha(\mathrm{Id})$ is the linear map $\mathcal{T}(n, \mathbb{R}) \to \mathrm{Sym}(n, \mathbb{R})$ defined by $T \mapsto T^{tr} \cdot D_0 + D_0 \cdot T$.

      iii. Show that $D\alpha(\mathrm{Id})$ is injective (Hint: use the fact that T is upper triangular) and hence also surjective.

3. *(3 points)*

   Let $U := \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}$ und $f : U \to \mathbb{R}$ be defined by $f(x, y) := \sqrt{1 - x^2 - y^2}$.

   (a) What are the critical points of f?

   (b) Let (x_0, y_0) be any point in $U \backslash \mathrm{Crit}(f)$. Construct a local diffeomorphism near (x_0, y_0), which transforms f into a linear function.