

Exercises Singularity Theory

1. (3 points) Let k be an integer greater than 3. Prove that every finitely determined germ in $x^2y + \mathfrak{m}_{\mathcal{R}_2}^k$ is equivalent to one of the germs $x^2y \pm y^l$, $l \geq k$.

(Hints: (1) The determinacy of $x^2y \pm y^l$ is l . (2) Write a germ in $x^2y + \mathfrak{m}_2^k$ in the form $x^2y + ay^k + bxy^{k-1} + x^2q + rest$ where q is a degree $k-2$ homogeneous polynomial and $rest$ means terms of degrees $> k$. Try to simplify the germ using the local diffeomorphism $\phi(x, y) = (x - \frac{b}{2}y^{k-2}, y - q)$.)

2. (3 points) Let k be an integer greater than 3. Prove that every finitely determined germ in $x^3 + \mathfrak{m}_2^k$ is equivalent to either one of the germs

$$\varepsilon^{l-1}(x^3 + y^l)$$

or a germ in

$$x^3 + \varepsilon^l xy^{l-1} + \mathfrak{m}_2^{l+1}$$

where $\varepsilon = \pm 1$ and $l \geq k$.

(Hints: As in Exercise 1, write a germ in $x^3 + \mathfrak{m}_2^k$ in the form $x^3 + ay^k + bxy^{k-1} + x^2q + rest$ where q is a degree $k-2$ homogeneous polynomial and $rest$ means terms of degrees $> k$. Try to simplify the germ using the local diffeomorphism $\phi(x, y) = (x - \frac{1}{3}q, y)$.)

3. (6 points) Extend the classification of germs of smooth real functions to the case $\mu = 6$, namely:

- Show that $\mu_f = 6$ implies $k := \text{corank}(f) \in \{1, 2\}$.
- If $k = 1$, conclude that f is stably equivalent to x^7 .
- If $k = 2$, show that f is stably equivalent to $g \in \mathfrak{m}_2^3$ that is 5-determined.
- Write g as $g = p + h$ with p is a cubic form and $h \in \mathfrak{m}_2^4$. You may assume that (see lecture 11)

$$p \in \{0, x^3, x^2y, x^3 - xy^2, x^3 + y^3\}.$$

Conclude that the only possibilities are $p \in \{x^3, x^2y\}$. (Hint: To show that $p \neq 0$ you may need Exercise 3 of Blatt 8).

- For $p = x^3$ use Exercise 2 above to show that g can only be equivalent to $\pm(x^3 + y^4)$. (Hint: All the other possibilities can be ruled out using the fact that g is 5-determined or by comparing the Milnor numbers.)
- For $p = x^2y$ use Exercise 1 to conclude that g can only be equivalent to $x^2y \pm y^5$.

To be handed in until Friday, July 3rd 2020 (by email to valeria.bertini@mathematik.tu-chemnitz.de)