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Exercises Singularity Theory

1. *(2 points)* Let $g(x, y, z) = z^p + f(x, y) \in \mathcal{R}_3$ with $f \in \mathcal{R}_2$, $p \in \mathbb{N}$. Determine a formula for the Milnor number of g in terms of the Milnor number of f.

2. *(3 points)* Let $\mathcal{A}$ be an ideal of $\mathcal{R}_n$.

   (a) Using Nakayama's Lemma, prove that $\mathcal{R}_n/\mathcal{A}$ is finite dimensional over k if and only if $\exists k \geq 1$ such that $\mathbf{m}^k_{\mathcal{R}_n} \subseteq \mathcal{A}$.

   (b) Assume that $0 < \dim(\mathcal{R}_n/\mathcal{A}) < \infty$. Show that $\exists k, l \geq 1$ such that $\mathbf{m}^k_{\mathcal{R}_n} \subseteq \mathcal{A} \subseteq \mathbf{m}^l_{\mathcal{R}_n}$ and $\mathbf{m}^{k-1}_{\mathcal{R}_n} \nsubseteq \mathcal{A} \nsubseteq \mathbf{m}^{l+1}_{\mathcal{R}_n}$.

   (c) Let be $l \geq 0$. Show that $\mathbf{m}^l_{\mathcal{R}_1} \subset \mathcal{R}_1$ is the unique ideal $\mathcal{A} \subset \mathcal{R}_1$ satisfying $\dim(\mathcal{R}_1/\mathcal{A}) = l$.

3. *(1+2 points)*

   (a) Show that for any $f \in \mathbf{m}^2_{\mathcal{R}_n}$ with determinacy d

   $$\mu(f) \leq \binom{n+d}{n} - 1$$

   (b) Show that for any $f \in \mathbf{m}^k_{\mathcal{R}_n}$ and any $l \in \mathbb{N}$

   $$\mu(f) \geq \binom{n+k+l-1}{k+l-1} - n\binom{n+l}{l}.$$

   (Hint: Start by proving that for all l one has $J_f + \mathbf{m}^{k+l}_{\mathcal{R}_n} = \mathcal{R}^{\leq l}_n \partial_{x_1} f + \ldots + \mathcal{R}^{\leq l}_n \partial_{x_n} f + \mathbf{m}^{k+l}_{\mathcal{R}_n}$ where $\mathcal{R}^{\leq l}$ denotes the space of polynomials of degree at most l.)

To be handed in until Friday, 25th June 2020, (by email to `valeria.bertini@mathematik.tu-chemnitz.de`**)**